Course on Integral Equations, Calculus of Variations and their Applications
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Lecture 17
Green's Function for Non-homogeneous Boundary Value Problem

Hello friends, so far we have discussed the construction of Green function for homogeneous boundary condition with homogeneous boundary value problem with homogeneous boundary conditions. Let us consider this non-homogeneous equation p x y double dash plus p dash x y dash plus q x y x equals to f of x.

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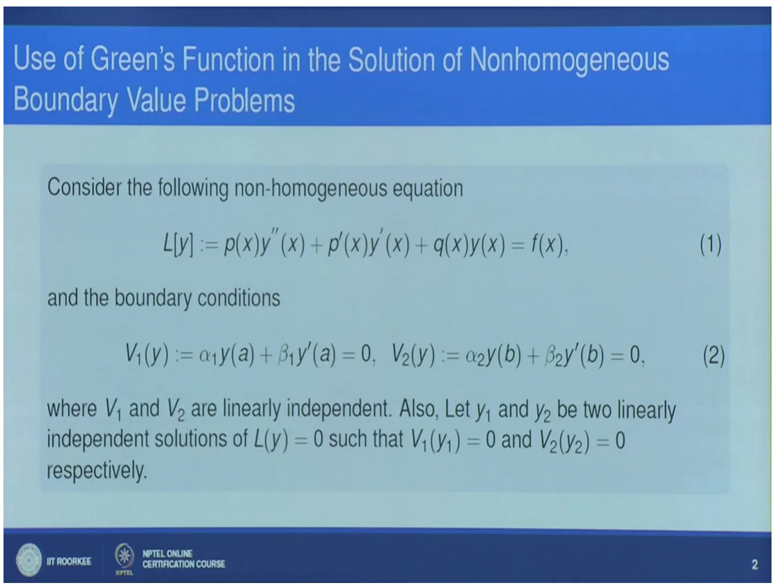

If you look at carefully this equation is given in terms of self adjoin form so this is a self adjoin equation and boundary condition also given as v 1 y and v 2 y and v 1 y is define at the point x equal to a and v 2 y is specified at the point x equal to b.

And here we have assumed that this v 1 and v 2 both are linearly independent, also we have assumed that we have 2 linearly independent solution of the homogeneous problem that is Ly equal to 0 such that the first solution y 1 satisfy the condition given at point x equal to a and the second solution y 2 satisfy the boundary condition define at x equal to b.

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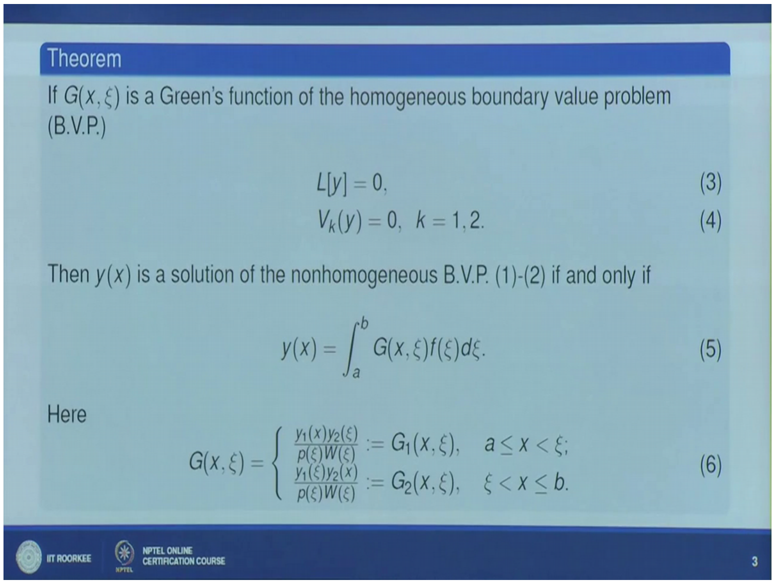

And now with the help of this now we have this result which says that if G x, xi is a green function of the homogeneous boundary value problem Ly equal to 0 with the boundary condition v 1 y equal to 0 and v 2 y equal to 0 then y x is a solution of the non-homogeneous boundary value problem 1 and 2 means this problem.

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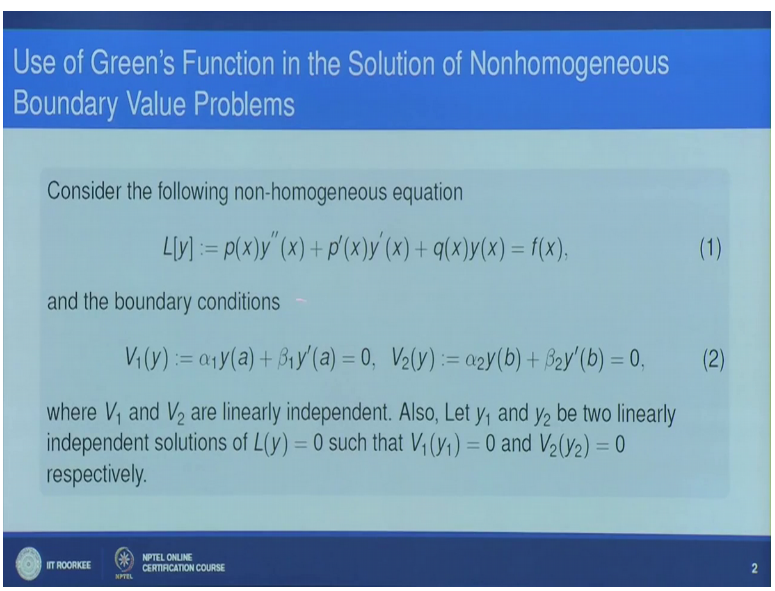

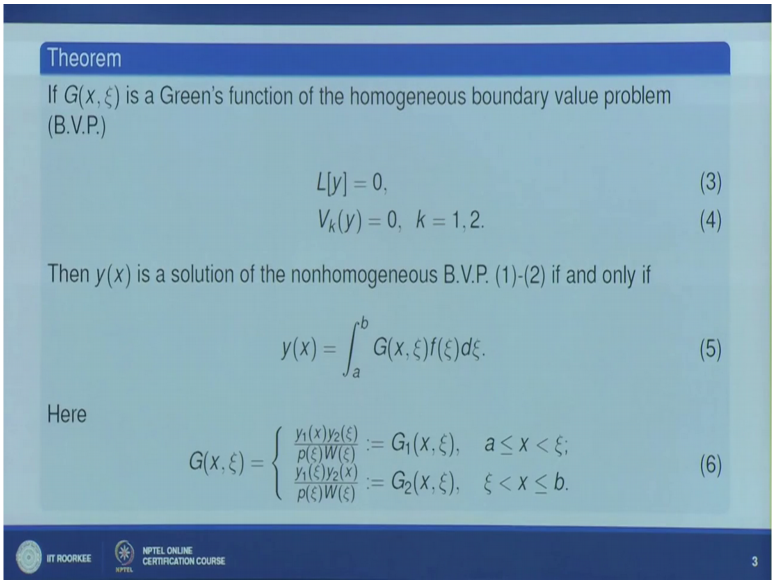

So this is Ly equal to f problem if and only if y x is given by this expression a to b G x, xi f of xi d xi where here your G x, xi is the green function of this homogeneous boundary value problem if you look at carefully this we have constructed in the previous lecture, so you remember here this y 1 x is a solution of the problem Ly equal to 0 and satisfying the boundary condition specified at the point x equal to a, similarly y 2 x is a solution of the problem Ly equal to 0 which satisfy the boundary condition specified at x equal to b.

And for our calculation purpose we have assumed that this is denoted as G 1 x, xi and this is denoted as G 2 x, xi and here the term in denominator that is p xi W xi this is a basically a constant and it is independent of xi and x that we can prove by this result.

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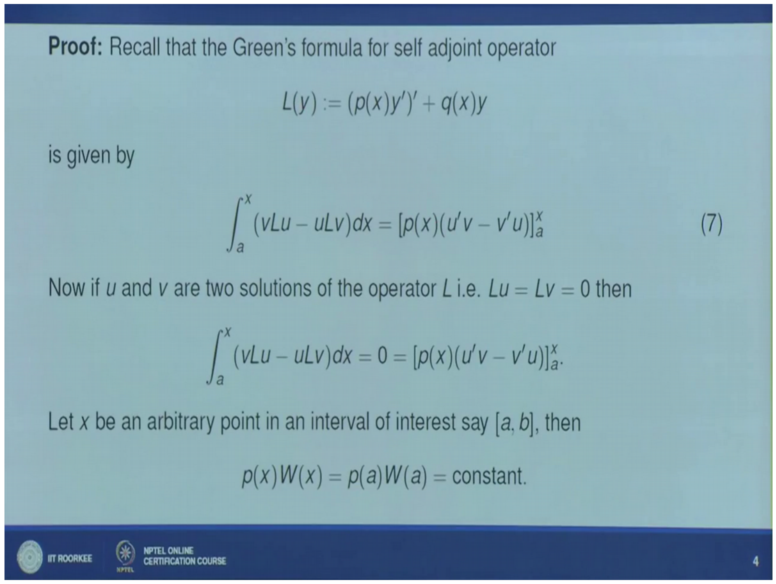

So if you remember we have already discussed the Green's formula for this self adjoin operator so if we take the operator Ly which is defined as this self-adjoin form then we can define our Green's formula which is between a to x and it is given by vLu minus uLv dx equal to this quantity.

Now here if I if we take two u and v are two function of the homogeneous problem that is Lu equal to Lv equal to 0, then if you put it u and v here as a solution of L operator L then this term become 0 so v into 0 minus u into 0 minus that becomes 0. So this implies that this expression this expression is basically 0 and if you look at this expression we can say that px minus times W of x is equal to p of a minus times W of a or I can say this is that px Wx equal to p of a W of a and since a is fixed point we have we are taking so this is a constant value and x we can take any arbitrary point in this interval.

So we can say that this quantity px into Wx is always remain constant.

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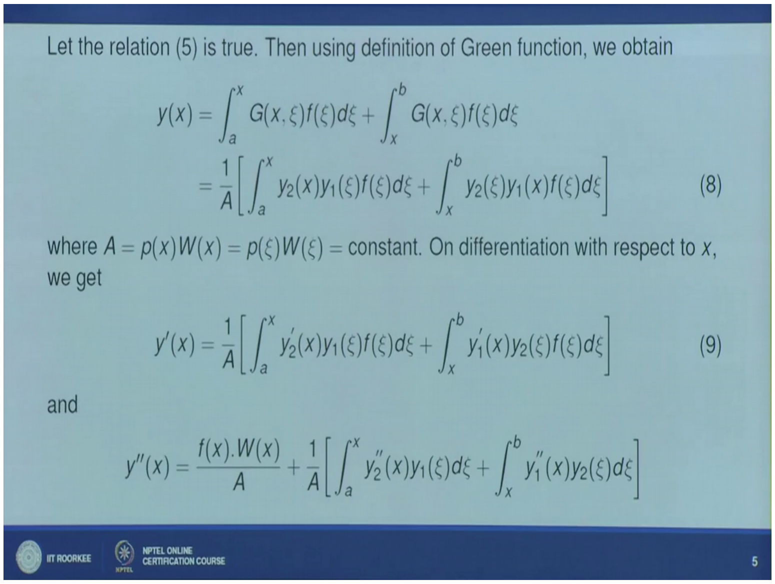

So using this expression that this result that px Wx is constant we can write it yx as this here y of x we have splitted into two interval a to x and x to b. So here in between a to x G x, xi f xi d xi is given by y 2 x y 1 xi divided by this a where it is defined as p xi W xi but since we already know that p xi W xi is a constant so we can write it some value in. So we are here we are assuming this a is denoting this constant value.

So between a to x we have written G x, xi equal to y 2 x y 1 xi and between x to b G x, xi is written as y 2 xi y 1 x. Now here we want to show that this actually this actually satisfy the differential equation non-homogeneous differential equation for that we need to say that this y x satisfy the non-homogeneous boundary value problem. So for that we simply calculate the derivative of this, so let us calculate the derivative here y dash x equal to this quantity here we are using the (())(5:50) formula for differentiation under this sign of integration I am just going to write that formula here.

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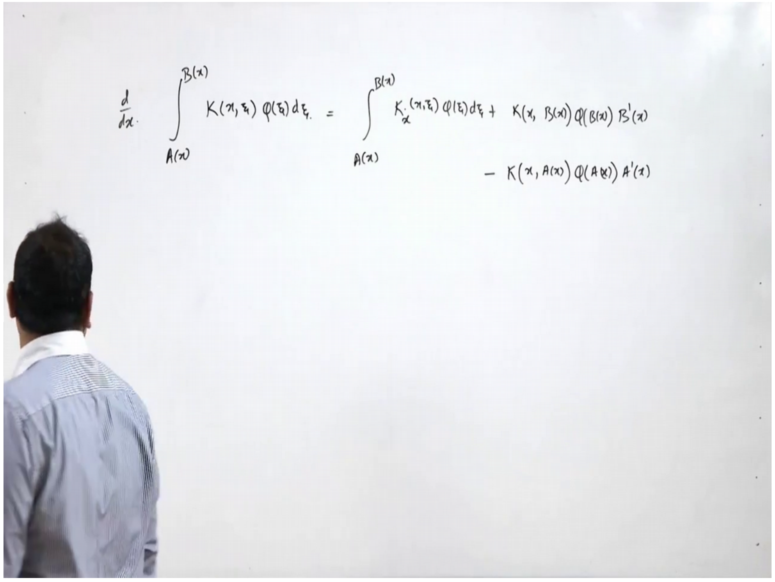

So if we write Ax Bx here we have K x, xi your phi xi d xi and if we differentiate with respect to say d of with respect to x I can write it here as Ax B of x and differentiation of this with respect to x x, xi phi of xi d xi plus now take the upper limit say Bx so you take x in place of xi in place of the variable integration variable you put this Bx so B of x here phi of B of x here and the differentiation of this B B dash x minus now in place of xi you can write out the lower limit x A of x phi of A of x and A dash x here we are assuming that this K x, xi phi xi and limit Ax and Bx all are continuous function with respect to x.

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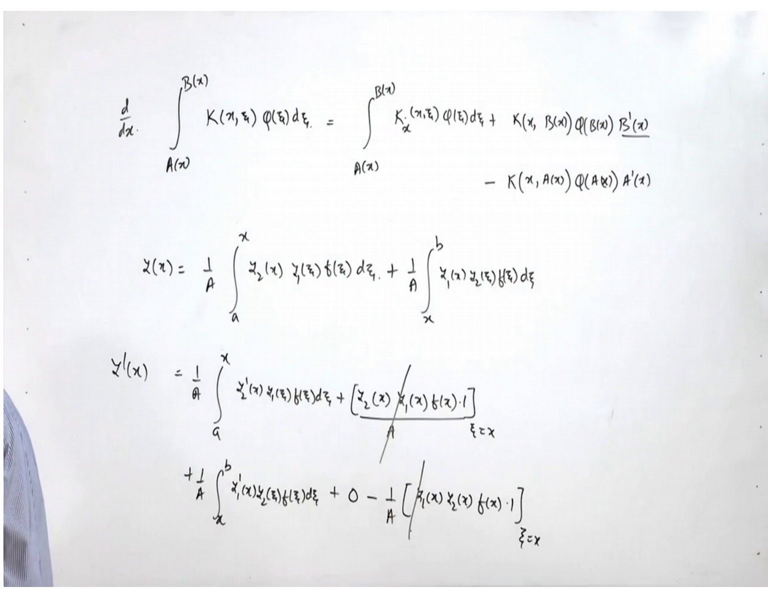

So this is a the formula which we are using to find out y dash x because if you look at using this formula let me write it here this is y of x just quickly I want to write it here so y of x is equal to 1 upon A, A to xi and here we have y 2 x and y 1 xi f xi d xi I will write only 4, okay plus 1 upon A similarly you can write it xi to B and y 1 x y 2 xi f xi d xi I am going to write only for this rest you can say so it is 1 upon A now if you look at this term simply say that between A to xi since we have only this term for x so we simply write it y 2 dash x y 1 xi f of xi d xi plus so there is no other term for x so we have differentiated only this y 2 x plus if you look at this term then here the upper limit is xi then if you can put here but when we take the derivative of upper limit that is going to be 0.

Sorry here I think it is x not xi so let me, sorry this is x not xi so if you take here this thing and if we take the derivative of this that is coming out to be y and if you look at this term then we have say y 2 x y 1 x sorry here lower limit is A so I am writing xi as A so we simply write y 1 A and f of A and if you find out say derivative of A with respect to x so that is going to be 0 here. So here this term is simply vanish and we have only this term left and similarly if we do it here we will get 1 upon A here it is x to b y 1 x now derivative of this means y 1 dash x y 2 xi f of xi d xi plus use this term if you take this term then upper limit is b constant so this upper limit will simply vanish so we will consider only this lower limit.

So that is 0 here minus I will write it here 1 upon A that thing I have left here A so this A here 1 upon A I am writing here y 1 x y 2 x and f of x and here it is x here. So derivative of x is simply 1 it is this this is please correct this also this is x, okay is it okay here? Yeah. So between A to x we differentiated this using this same formula and we are getting this. Now if you look at this quantity is same as this quantity here basically it is at xi equal to x so I am putting xi equal to x here.

So this quantity will cancel each other and we have y dash x defined as 9. So here we have seen that this y dash x is given by this thing so y dash x is given by this, this will cancel out and y dash x is given by this. And you can say that it is given by this formula 9 so y dash x is given by this. Using the same formula you can define y double dash x and this now here the boundary condition so when you put take this term here when you consider the upper limit and here the lower limit and using the jump discontinuity of Green's function it is coming out to be this quantity that y double dash x is equal to f x into Wx (())(11:39) of x divided by A plus this quantity.

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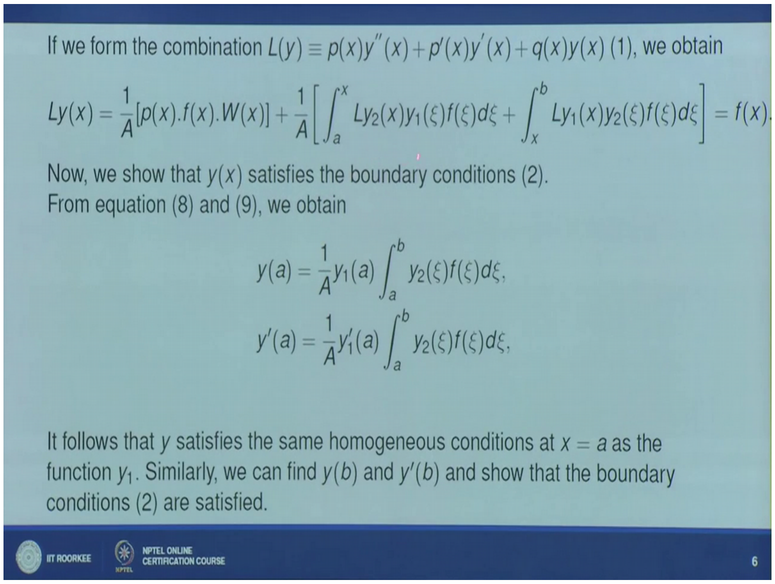

Now once we have your y of x y dash x y double dash x then we can form this expression Ly which is px y double dash x plus p dash x y dash x plus Q of x y of x and we can calculate and this quantity Ly x is coming out to be this and this is nothing but your f of x here we, yeah.

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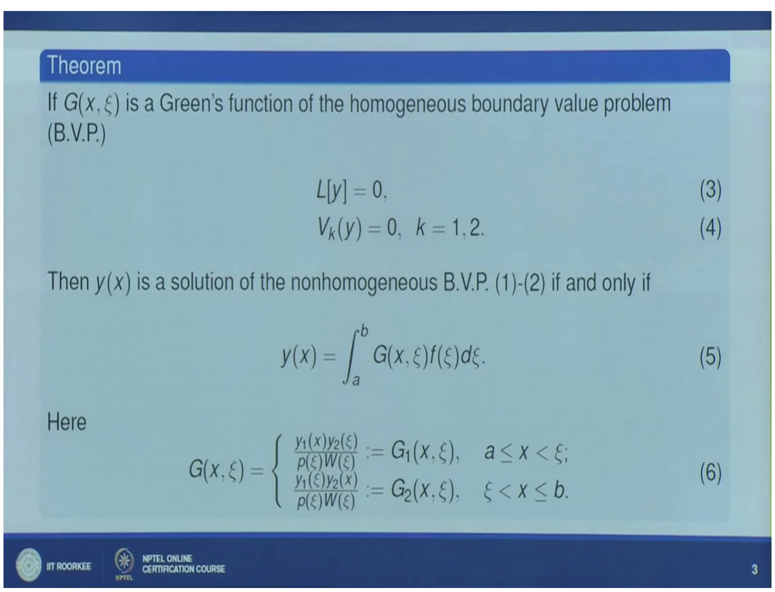

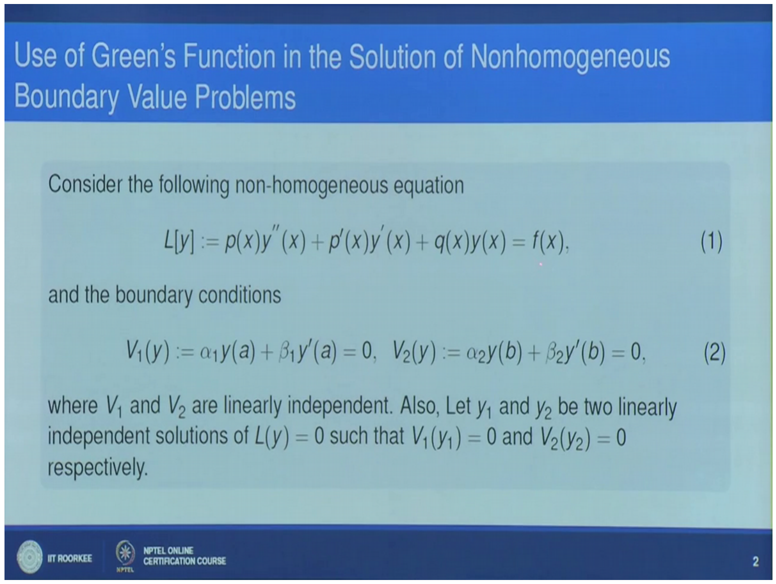

So it means that if your G is given by expression 6 then the y x equal to a to b G x, xi f of xi d xi will satisfy the non-homogeneous problem given by this Ly equal to f of x.

The only remaining thing left is that it also satisfy the boundary condition specified at point a and b.

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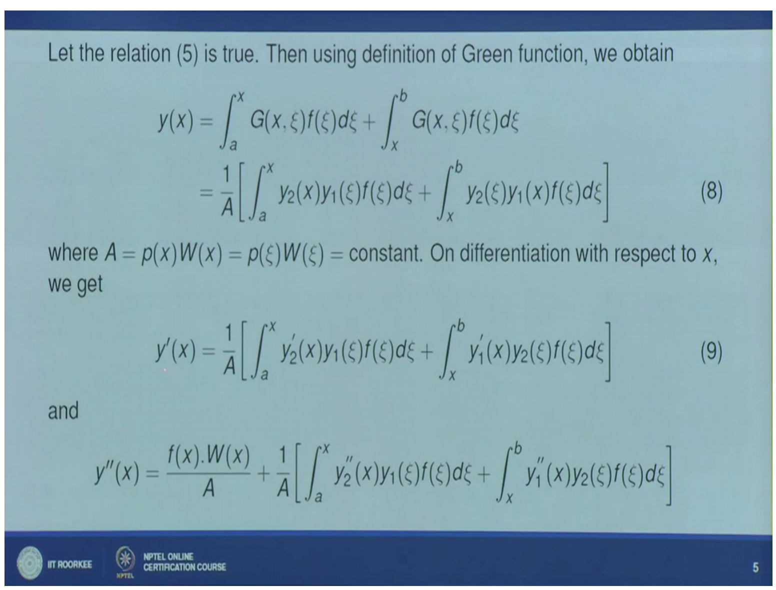

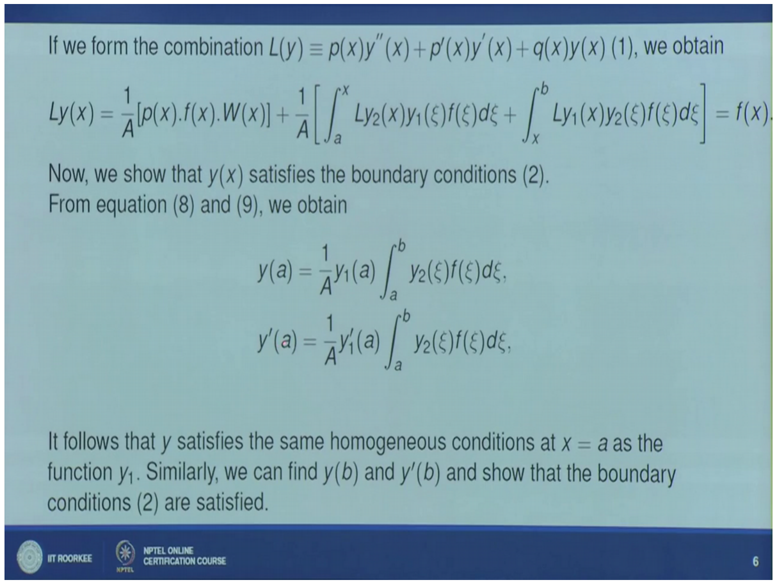

So for this you look at this equation, equation number 9 and if you put x equal to a then this integral will simply vanish and it is a to b y 1 dash a y 2 xi f of xi d xi. So I can take y 1 dash xi out and I can write expression as y dash a equal to 1 upon A y 1 dash a equal to y 1 dash a a to b y 2 xi f of xi d xi.

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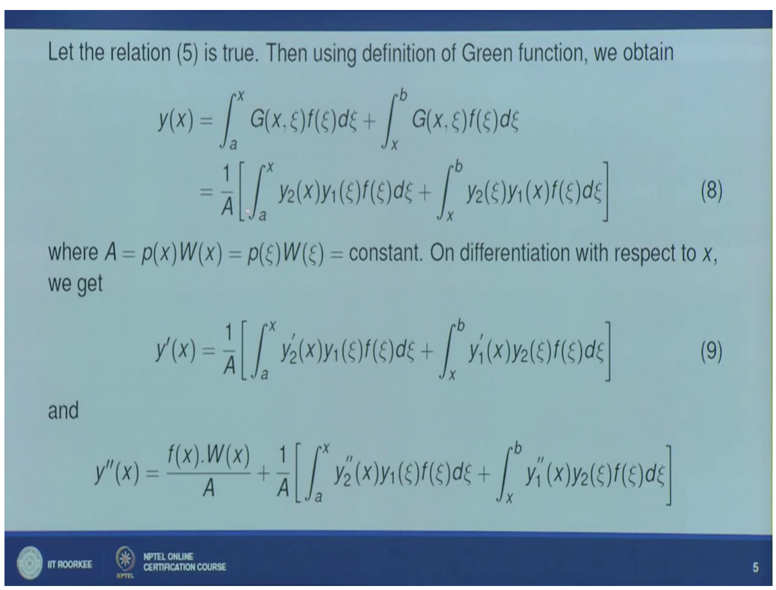

Similarly you can use equation number 8 and put x equal to a this term integral will vanish and here it is a to b y 1 x you can take it out and y 1 a basically will be taken out and a to b y 2 xi f of xi d xi.

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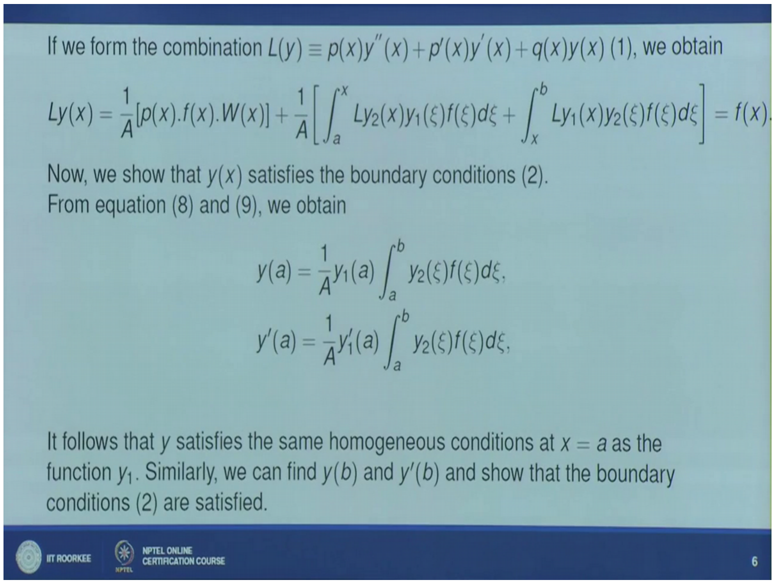

So it means we are getting this small a is equal to this and y dash a, so basically y a whatever condition y 1 a satisfy the homogeneous condition y 1 a satisfy the same boundary condition will be satisfy by this function y of a.

So this follows that y satisfy the same homogeneous condition at x equal to a as the function y 1, similarly we can do the things for point x equal to b.

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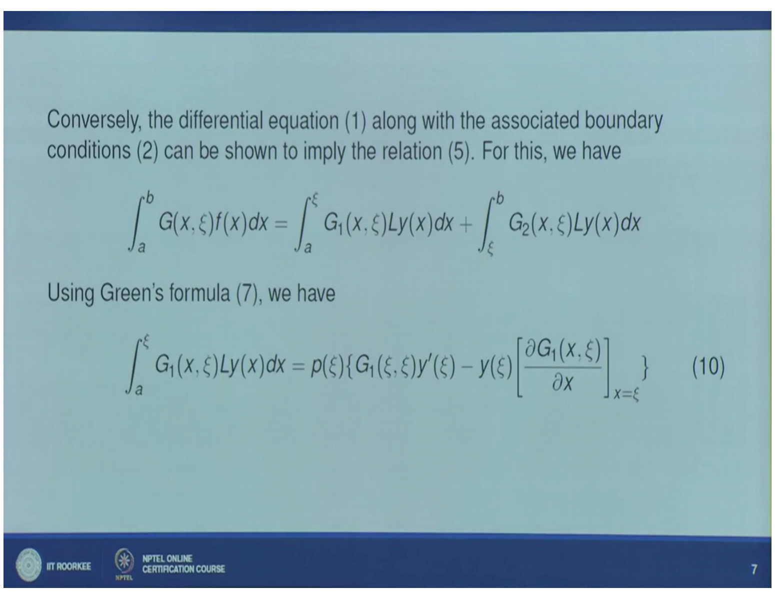

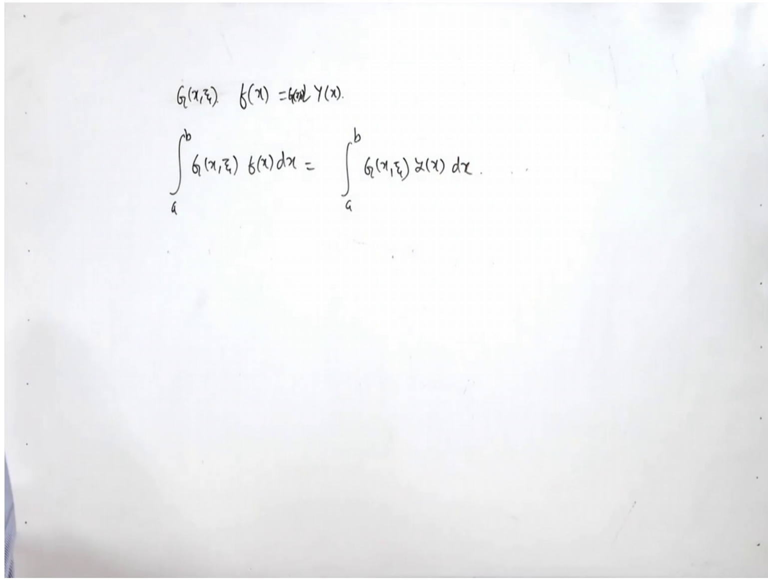

So now looking at the converse part we want to show that this differential equation 1 along with the associated boundary condition will give you the expression y x equal to a to b G x, xi f of xi d xi for that we consider simply L if you look at here f of x equal to Ly x, right? And then what we do? We simply multiply by G of x, xi here as G x, xi and then integrate between a to b so a to b G x, xi f of x dx is equal to a to b G x, xi y of x d of x.

So it means that we are getting this expression that a to b G x, xi f of x dx, now here we again truncate this interval a to b into a to xi and xi to b and between a to xi this G is given by let me go to this between a to xi your G is nothing but G 1 x, xi. Similarly between xi to b G x, xi is given as G 2 x, xi so using this expression 6 we can write it a to b G x, xi f x dx as sum of these two integral. Now with the now we want to show that this is nothing but your y of xi here.

So for that we use (())(15:31) Green's formula between a to xi, so a to xi G 1 x, xi Ly x d of x if you use this it is nothing but p xi G 1 xi xi xi y dash xi minus this p xi is outside the entire bracket and this thing. So we are just using the Green's formula for v equal to G 1 x, xi and u equal to y.

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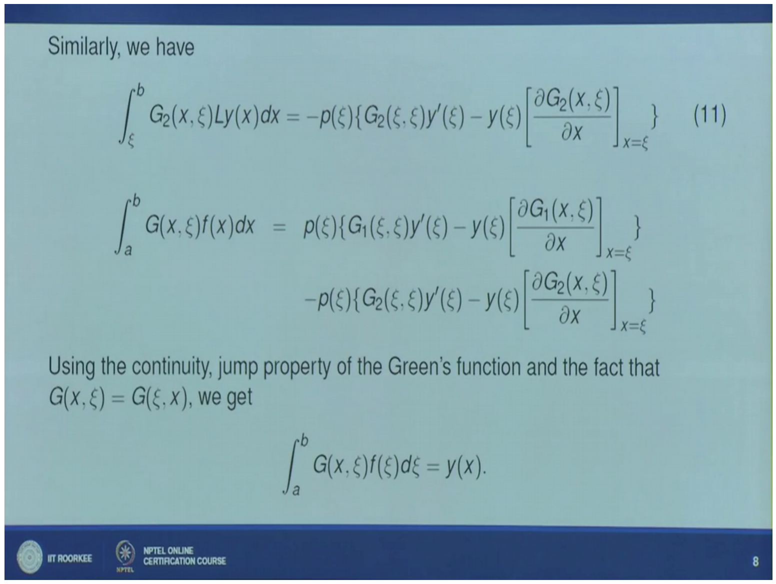

So the same formula we can apply for the second one for xi to b G 2 x, xi Ly x d of x. So here this will be given like this, so using Green's formula we can write it like this, if we sum these two if we use this expression given in 11 and the expression given in 10 and put it here then we can write a to b G x, xi f of x dx as this quantity. Now if you look at here this first term G 1 xi xi y dash xi and here G 2 xi xi y dash xi.

So here since we already know that function G x, xi is continuous at x equal to xi so these two will simply equal and since it is plus sign here minus sign so these two will cancel here. Now again using the jump condition for Green function G x, xi we can say that deba G 2 x, xi deba x minus deba G 1 x, xi deba x is going to give you 1 upon p x here it is x equal to xi, right? So when you take this this will be cancel out and what you will get is your y of xi here.

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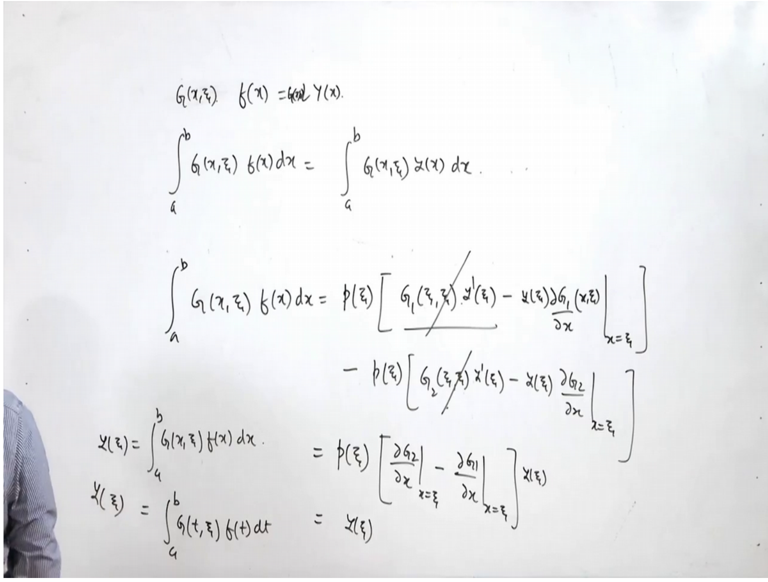

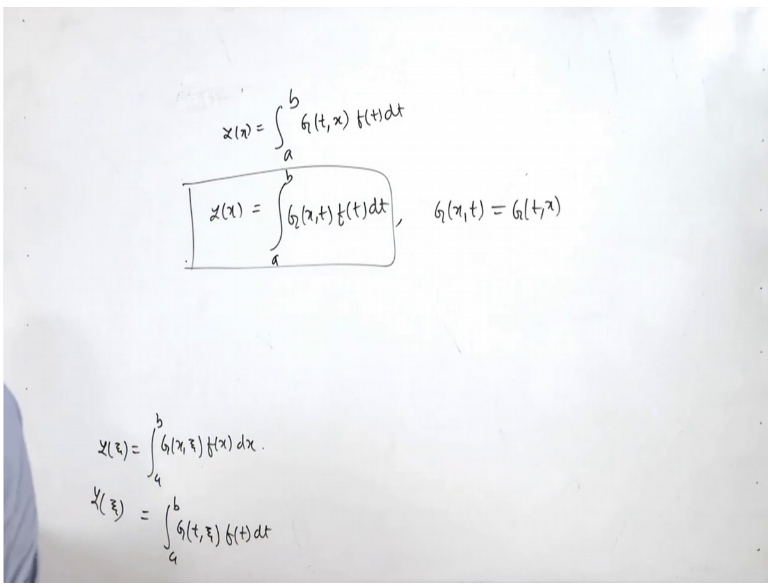

So what you will get here let me write it here you will get a to b G x, xi f of x dx here I am writing here p of xi and within bracket I am writing here G 1 xi, xi and y dash xi minus y xi G 1 with respect to x here so deba G 1 by deba x and x, xi and it is evaluated at x equal to xi here minus p of xi and G 2 xi, xi y dash xi minus y xi deba G 2 deba x evaluated at x equal to xi. Now since G is spontaneous so this will be canceled out here and using the jump condition it is p of xi and here it is written as deba G 2 deba x at x equal to xi minus deba G 1 deba x at x equal to xi and this y xi is also there so I am taking y xi out because it is common here.

So this is basically 1 upon p xi so I will get y of xi here. So y of xi is given by this formula, now what we try to do here we simply write y xi equal to a to b G x, xi f of x d of x so it means I can write y of say I can write down the same thing as a to b and I can write it here say I can write x as some t so I can write it t xi and f of t d of t. So I can write this as y xi, then I can change xi by x and I can write it here y of x is equal to a to b G of t x f of t dt.

Now we already know that by looking at the structure of G t, x G t, x as same as G x, t so I can write it here a to b G of x, t f of t dt so that is your y of x. So here we are using this fact that G x, t is equal to G t, x. So this says that we have that y, x is defined by this formula if we take our differential equation Ly equal to f and use the property of Green's function then it will also satisfy this expression that is given as y of x equal to a to b G x, xi f of xi d xi.

So that shows that if we know Green function for homogeneous boundary value problem then with the help of this we can actually solve the non-homogeneous boundary value problem.

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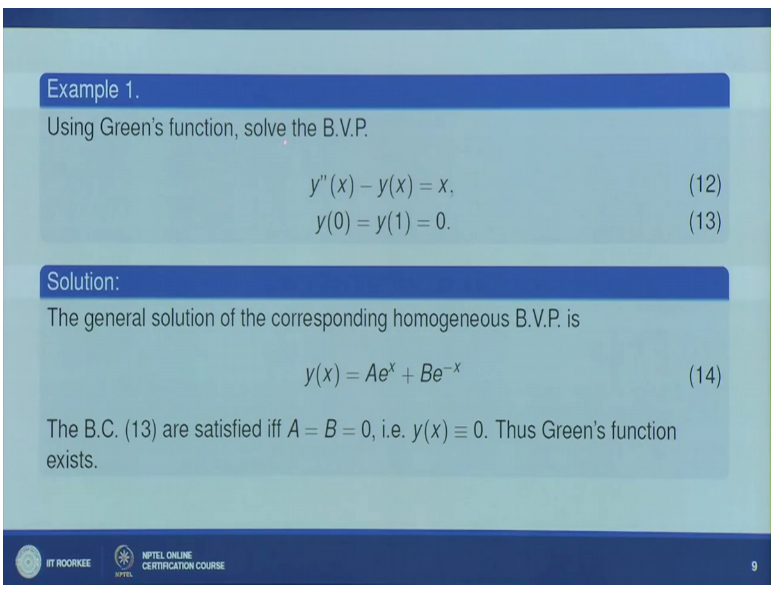

Now let us look at the example based on this so using Green's function solve this non-homogeneous boundary condition so we here we have y double dash x minus y x equal to x boundary conditions are defined at y 0 equal to y 1 equal to 0.

So if you look at this is the operator let us define this as a operator and we can write down the general solution like this and if you use the boundary condition here y 0 equal to y 1 equal to 0 then we can get your both the condition both the constants a and b equal to 0 it means that we have only (())(21:21) solution. So it means that we can now construct our Green function with the help of this general solution.

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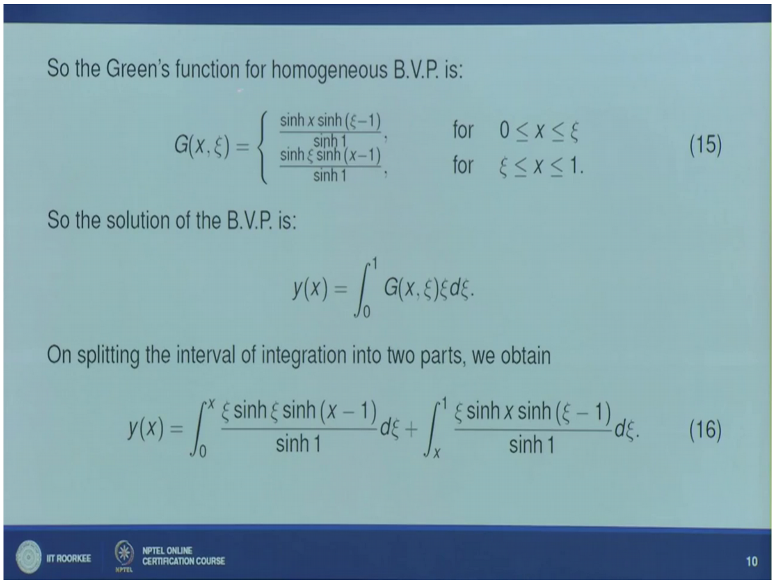

So for that using the method we have already discussed for construction of Green function for homogeneous boundary value problem we can construct our Green function for this homogeneous boundary value problem and it is given by this sin G x, xi equal to sin hyperbolic x sin hyperbolic xi minus 1 divided by sin hyperbolic 1 in this interval and similarly we can define for in this interval.

So our solution by the previous I can write y x equal to 0 to 1 G x, xi xi dx here if you look at your fx is nothing but x here so f of xi is going to be xi here. So now putting the values G x, xi here which we have already calculated we can write down like this 0 to x and x to 1 like this. Now if here everything you can calculate basically you can take out this sin hyperbolic x minus 1 upon sin hyperbolic 1 and in this interval you can take sin hyperbolic x divided by sin hyperbolic 1 out.

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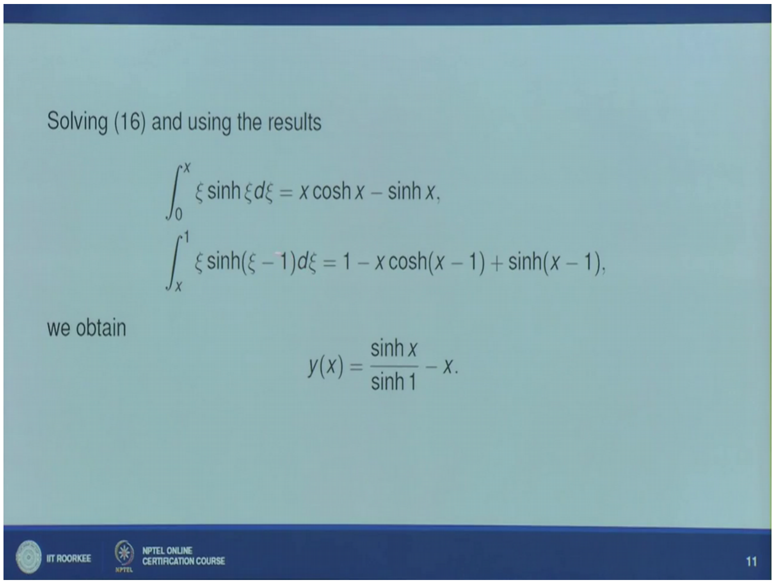

And these are simple integral you can find out by integration by part and you can say that this value is equal to this and this integral x to 1 xi sin hyperbolic xi minus 1 d xi is equal to this quantity and if we use this these two integral and the previous expression for y of x you can simplify this expression and you can say that the solution of this non-homogeneous boundary value problem is given by y of x is equal to sin hyperbolic x divided by sin hyperbolic 1 minus x you can actually verify this also.

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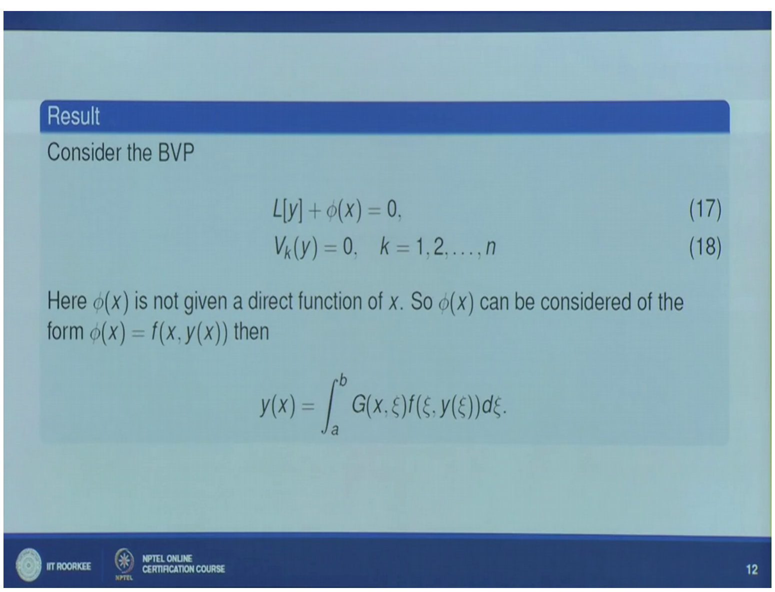

So now let us look at the next result which says that if this function phi of x is not a direct of x maybe it means that it is also involving the unknown function y of x then are you can say that phi of x is written as f of x y of x then the solution we can write it y of x equal to a to b G x, xi f of xi y of xi d xi. So basically in this case this is not a solution y of x which is given here but the solution y x is given in terms of integral equation or I can say that this differential equation along with this boundary condition is equivalent to the integral equation given here and this is a (()) (23:45) integral equation given here.

So it means that in this case when phi of x is not a function of x only it is involving y of x then this differential equation is converted into integral equation.

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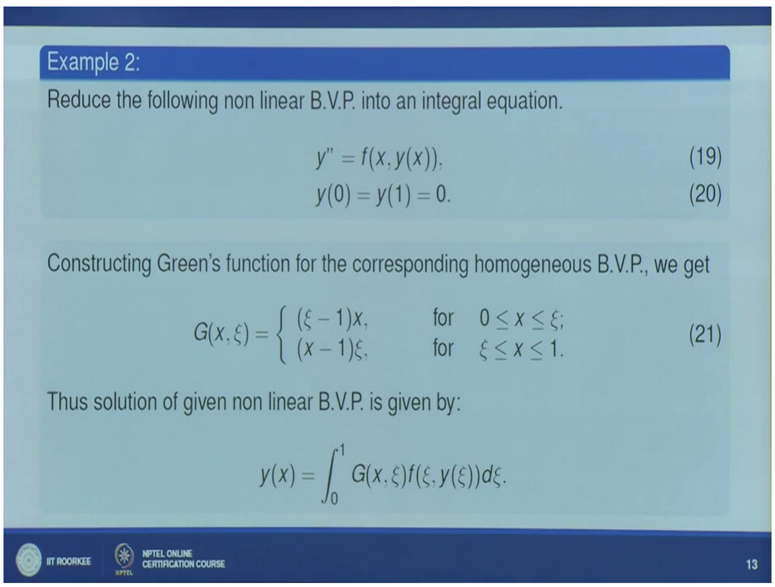

So just look at one example here take very simple problem Ly is y double dash fx yx y 0 equal to y 1 equal to 0 if you remember we have already constructed Green function for this problem homogeneous problem so for this y dash y double dash equal to 0 with boundary condition y 0 equal to y 1 equal to 0 we have G x, xi equal to this and with this we can write down y x equal to 0 to 1 G x, xi now here your phi x is fx y fx yx so I can write it yx equal to 0 to 1 G x, xi f of xi y of xi d xi.

So here the solution of this boundary value non-homogeneous boundary value problem given in terms of Fredholm integral equation, so this is another use of Green function.

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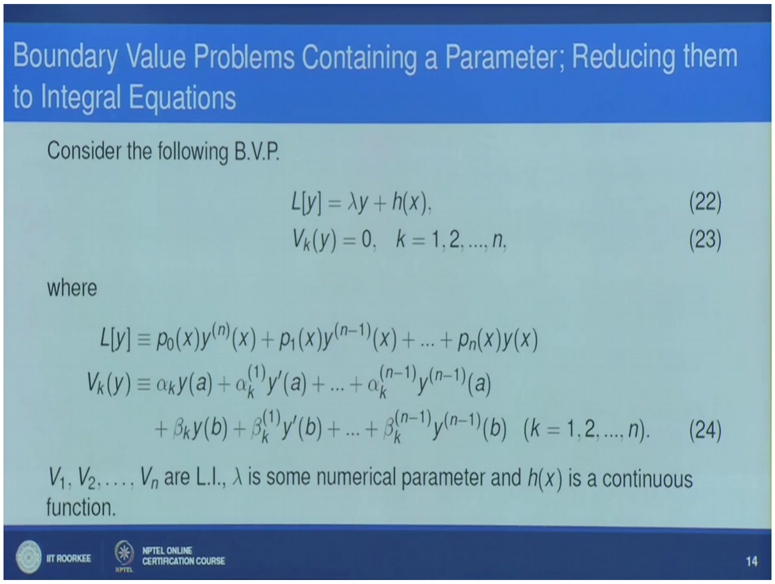

Now let us use the same thing but now we have f parameter involving that Ly is given as lambda y h of x or you can say that your phi x is defined as h of x plus lambda y.

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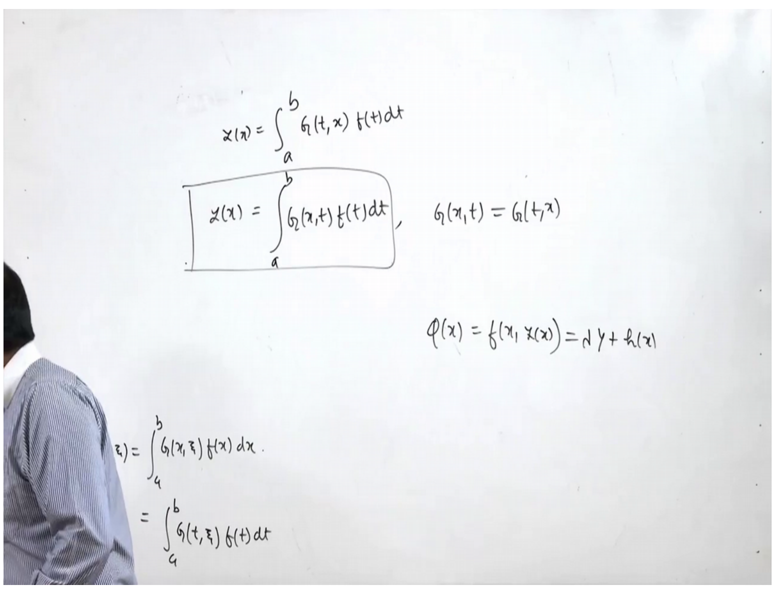

So here we have defined phi of x equal to f of x y of x and we can define this as lambda y plus h of x.

So we can use the result which we have already described and we can say that this problem along with boundary condition here Ly you can take it like this any linear operator (())(25:28) linear operator we have taken here and boundary condition this.

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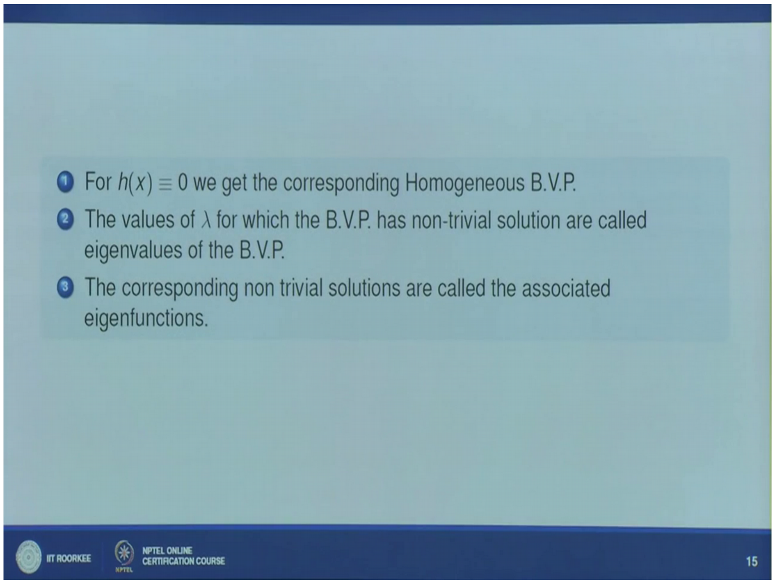

Then this also can be converted into integral equation. So here if we take h of x equal to 0 we have a homogeneous boundary value problem and the values for which the values of lambda for which we are getting a non-trivial non zero solution we call those values of lambda as eigen values of the boundary value problem and the corresponding non-trivial solution we call this as associated eigen functions.

For example, we can have this particular result that if we have Ly equal to 0 and Vk y equal to 0 and for this if we have constructed our Green function this then the previous problem this boundary value problem Ly equal to lambda y plus hx with boundary condition this the solution of that boundary value problem is given in terms of integral equation this Fredholm integral equation where f of x you can define as a to b G x, xi h of xi d xi.

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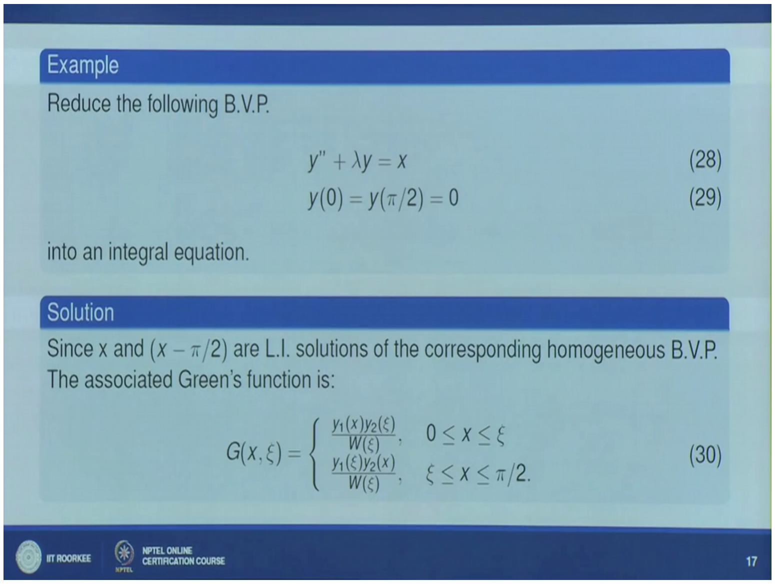

So for this let us consider a simple example, so here we have y double dash plus lambda y equal to x so here hx is equal to x here with the boundary condition y 0 equal to y pi by 2 equal to 0 and we try to use Green function to convert this differential equation along with boundary condition into to a Fredholm integral equation. So for this particular problem y double dash equal to 0 we have so we just look at this problem as this.

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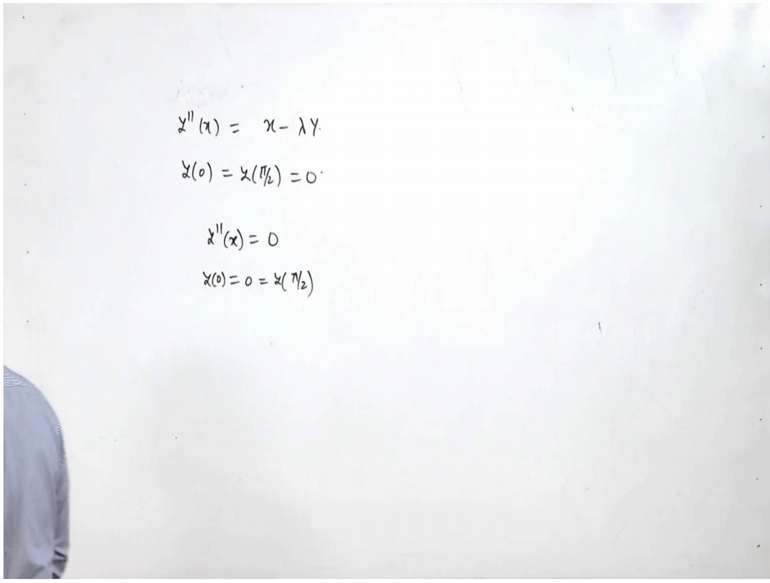

So here I will write it here y double dash x equal to I can write it x minus lambda y or you can write it like this and we have y of 0 equal to y of pi by 2 equal to 0.

So what we try to do is we have a corresponding homogeneous condition homogeneous boundary value problem y double dash 0 equal to 0, sorry y double dash x equal to 0 with the condition y 0 equal to 0 equal to y pi by 2 and here we can construct our Green function which is given here as if you remember we have formula for Green function like this. Now here coefficient of y double dash is simple 1 so I have not written coefficient of y double dash so it is given by this.

Now y 1 x y 2 x you can find out as x, x is the solution of y double dash equal to 0 which satisfy the boundary condition given at x equal to 0 that is y 0 equal to 0. Similarly, x minus phi by 2 is a solution of y double dash equal to 0 which satisfy the boundary condition given at the end x equal to pi by 2. So we can say that y 1 x is equal to x and y 2 x is equal to x minus pi by 2.

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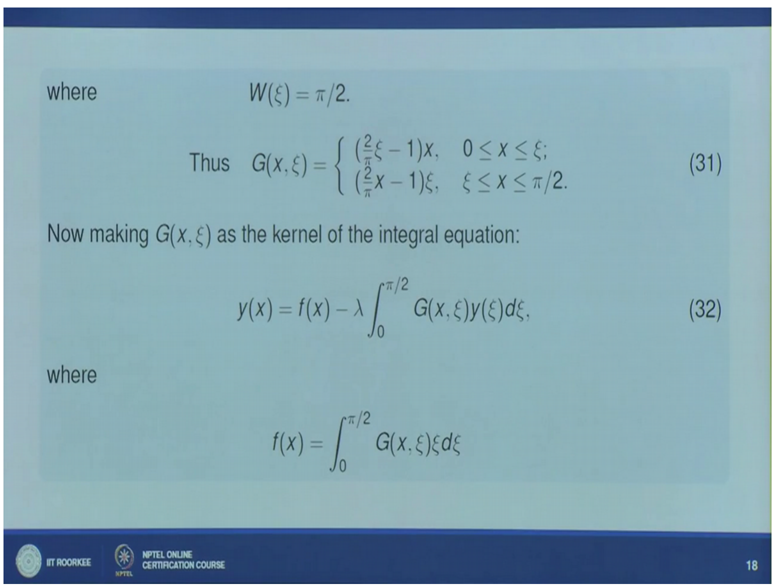

So with the help of this and the formula 30 we can write down the expression for G x, xi here.

Now here (())(28:40) we can calculate easily and we can see that it is nothing but pi by 2, so with the help of this we have G x, xi like this.

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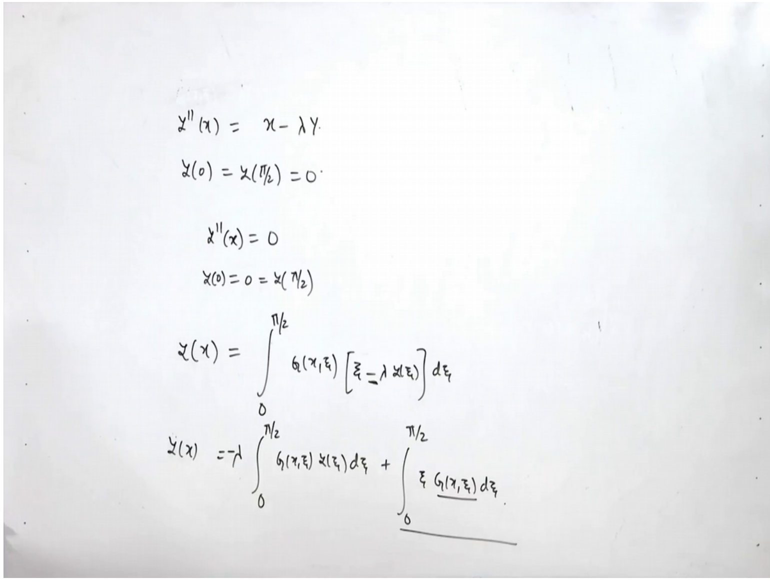

Now it means that you can write it here y of x equal to now this is between 0 to pi by 2 G x, xi and here we have x minus lambda y we can say now this is what this is f of xi here so f of xi here basically what it will be xi minus lambda y of xi d of xi. So if you simplify you will get lambda 0 to pi by 2 G x, xi y of xi d of xi plus 0 to pi by 2 and it is xi G of x, xi d of xi here so that is y of x.

And this once we know G of x, xi you can actually calculate this we have calculated here and we can write it this quantity as sum f of x and we can say our solution is given as y of x equal to fx minus, okay I have missed here minus 1 minus I have missed here, yeah minus this is minus which I have missed here so it is placed here.

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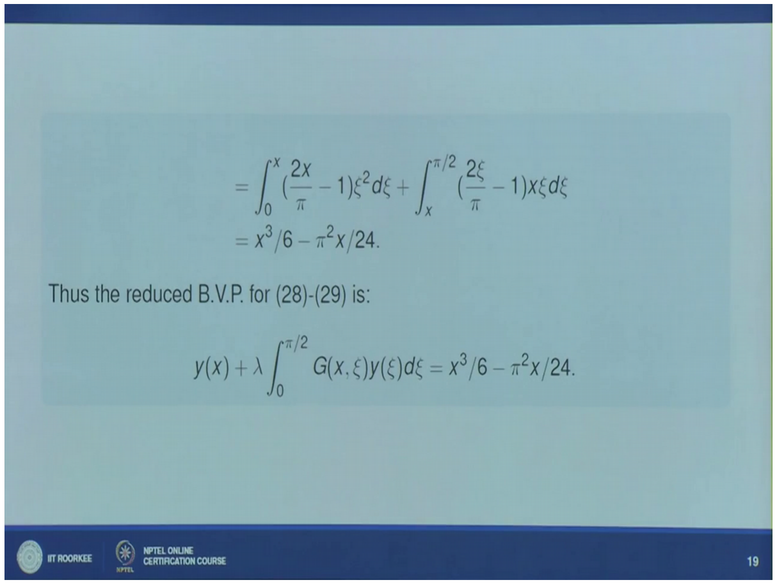

So here fx you can calculate as 0 to pi by 2 G x, xi xi d xi. So G x, xi is already given here so let us calculate this f of x here so between 0 to x xi is we have between, sorry between xi between 0 to x means xi less than x we have what xi less than x we have this 2 x pi by minus 1 into xi.

So use this here we have the expression and between xi greater than x we can use this thing 2 xi by pi minus 1 x so using this we can find out the value of this 2 integral and it is coming out to be x cube by 6 minus pi square x by 24 it is not very difficult thing it is simple integration and with this value you have the solution of this problem yx equal to lambda 0 to pi by 2 G x, xi y xi d xi and this is your f of x given here.

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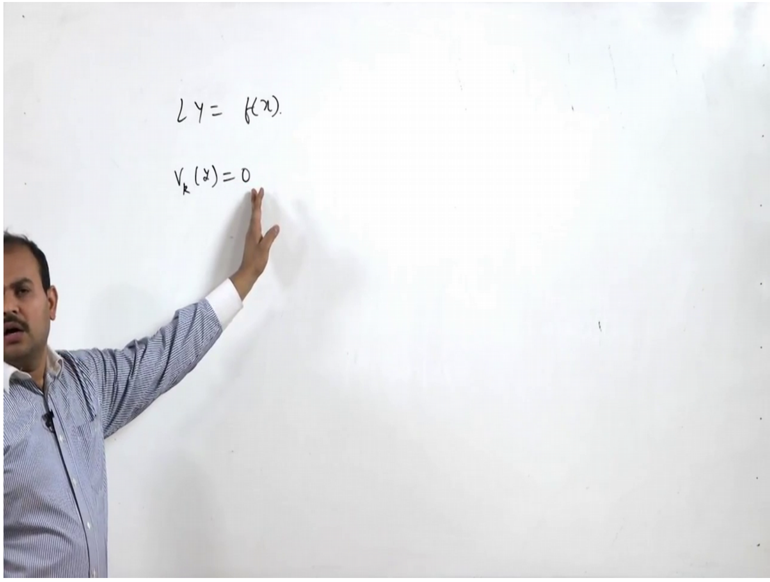

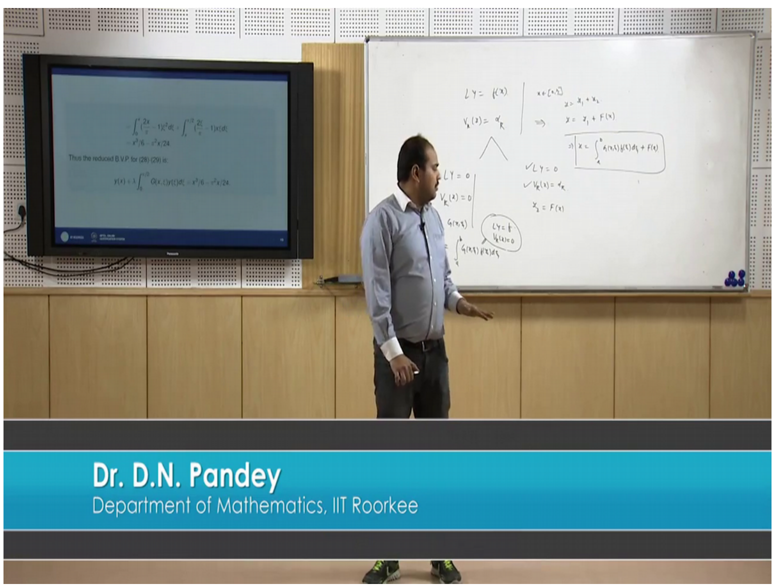

So it means that with the help of Green function you can actually show the equivalence of a non-homogeneous boundary value problem of this particular type into Fredholm integral equation that what happened I hope we have already discussed that we have Ly equal to f of x with the boundary condition that V 1 say V k y equal to 0 and we have discussed this but suppose in that suppose these boundary conditions are not homogeneous it is basically non-homogeneous then how we can write it.

So we can say that suppose your these are not homogeneous suppose we have these are some alpha k some values which is non-zero, then how we can handle here? So this we can handle in this two way so what we try to do is we construct Green function for this Ly equal to 0 with the condition that V k y is simply 0, right? And we say that for this we have a Green function G of x, xi and we can find out the solution of the corresponding non-homogeneous boundary value problem as y of x equal to simple a to b and assuming that x is lying between a to b x is in between a to b.

So here it is a to b G of x, xi f of xi d of xi, so if it is not alpha k or all 0 your solution is given in this particular form, but suppose alpha k is simply non-zero then what to do then what we try to do we simply convert this thing into two problem so let me write it here it is f of x here okay, so this is the solution of what? This is the solution of Ly equal to f with V k y equal to 0. Now what we try to do let us look at this particular problem we take Ly equal to 0 and V k y equal to say alpha k and we can find out the solution here.

And suppose the solution you can find out and say that f of x, then the solution of the original problem is solution of this problem is basically your y equal to solution of this problem call it y 1 and solution so it is given as y 1 plus f of x here. So solution of the non-homogeneous problem with non-homogeneous boundary condition can also be given with the help of this that solution of homogeneous problem you find out like this and the solution of homogeneous problem but non-homogeneous condition and say that solution is given by y equal to fx so we call it y 2 equal to fx.

And solution of this non-homogeneous solution of this original problem I can give it like this y equal to y 1 plus y of 2 where y 1 is given by this and y 2 is given by this. So this is y equal to a to b G x, xi f of xi d of xi plus f of x here you can verify this by simply operating your L on this, okay. So here we are using principle of super position to truncate this problem into 2 part one with non-homogeneous equation with homogeneous boundary condition and a homogeneous differential equation with non-homogeneous boundary condition and you can easily solve this problem and your solution on this problem is given by this you can do this problem maybe we can discuss some problem in next assignment here.

So now here we close this lecture here and we will again discuss some more problem, okay thank you very much.