Integral Equations, Calculus of Variations and their Applications
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Lecture 16
Green's Function for Self Adjoint Linear Differential Equations

Hello friends let us utilize the theory discussed in previous lecture to construct a Green function for some boundary value problem. So let us start with the problem here, first before that let us take a self adjoint linear differential equation and try to see how we can construct our function for this given linear differential equation.

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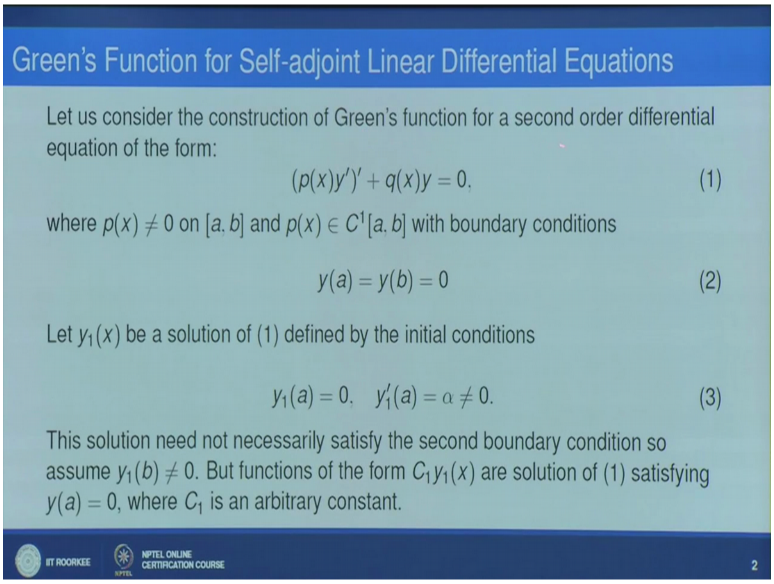

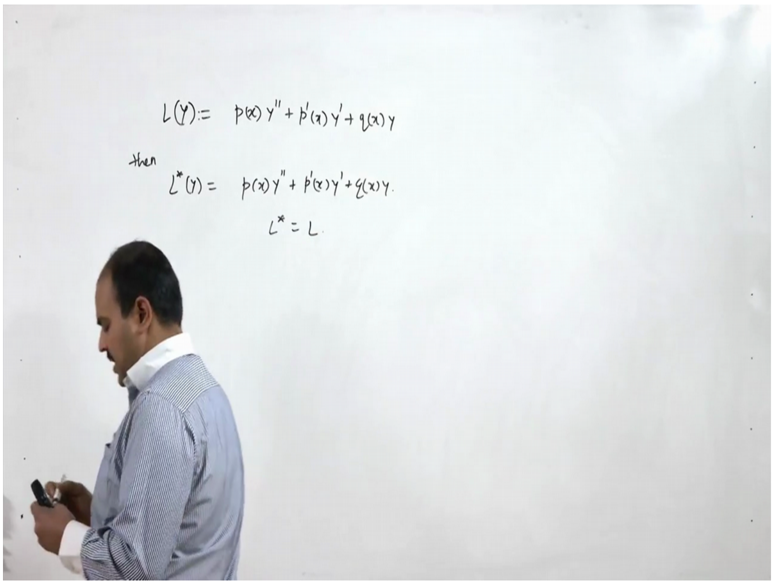

So here we have taken this we will consider here the Greens function for second order differential equation of the form p x y dash whole dash plus q x y equal to 0, where p x is not equal to 0 on this entire interval a, b and p x is a C 1 function C 1 function means continuously differentiable function first derivative is continues on this interval a, b with boundary condition y a equal to y b equal to 0 and we try to construct Green function for the given problem.

Here it is to be noted here that this is a self adjoint differential equation here it means that if we define this as if we define this operator L y is equal to this p x y double dash plus p dash x y dash plus q of x y then its adjoint operator L star y is also coming out to be same. So here your L star is nothing but same L. So here we try to see the self adjoint linear operator for this.

So now to begin with let us take a solution y 1 x and it is defined by the initial condition y 1 a equal to 0 and y 1 dash a equal to some nonzero constant alpha. Now it is to be noted here that if your coefficients are continuous function then by the existence of existence uniqueness of initial value problem we can make sure that such a solution exist. So we take one solution y 1 which satisfy the first condition that y 1 a equal to 0 and y 1 dash a is not equal to 0.

Now this solution need not necessarily satisfy the given boundary condition given at point b that y 1 b may not be equal to 0. So let us assume that y 1 b is not equal to 0, but since y 1 x is the solution so any constant multiple of y 1 x is going to be a solution of this problem. So it means that similar that any constant multiple of y 1 x is again a solution of 1.

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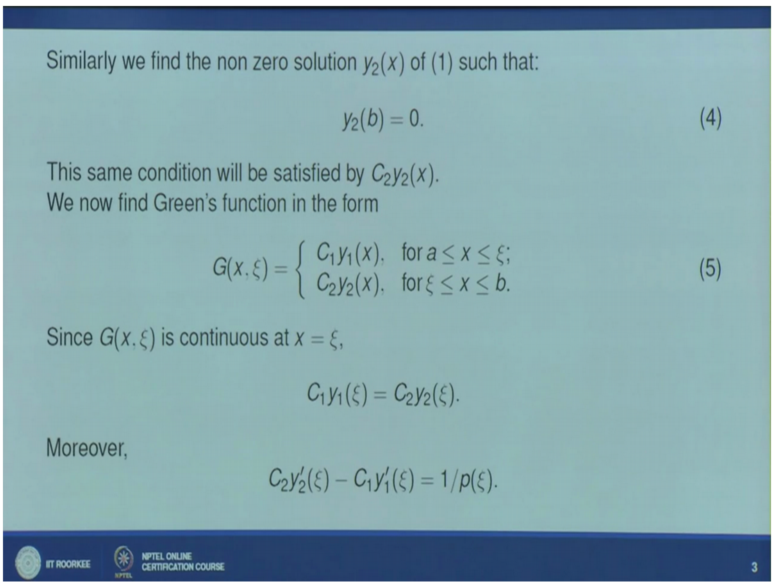

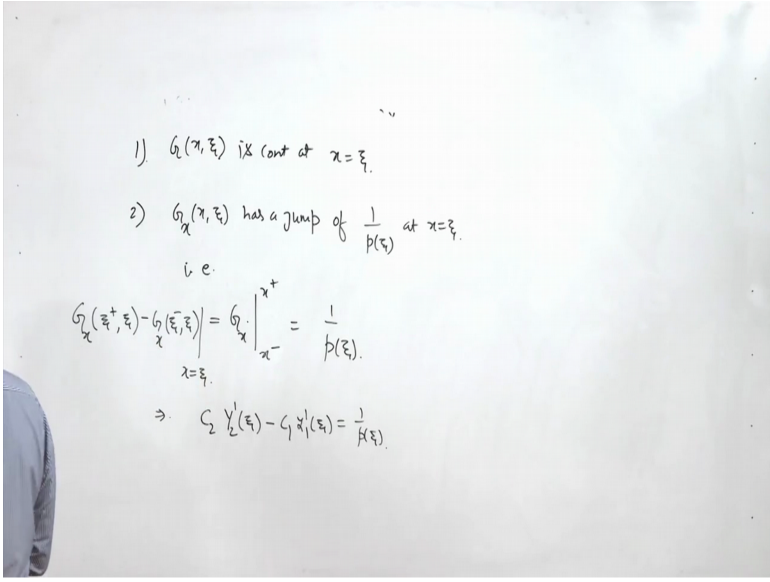

Now again in a same way we can find out another solution y 2 x which is 0 at the point b but its derivative is nonzero at the point b. Now with the help of these y 1 and y 2 let us consider Green function of this form. So here G x, xi is equal to C 1 y 1 x, where x is lying between a to xi and C 2 y 2 x when x is lying between xi to b. Now see we have already seen that such a Green function will satisfy certain condition so first condition is that G x, xi is continues at x equal to xi so it means that at x equal to xi the right hand limit and the left hand limit both are same, so it means that C 1 y 1 xi is equal to C 2 y 2 xi.

Then we have a jump condition of its first derivative so that C 2 y 2 dash xi minus C 1 y 1 dash xi equal to 1 upon p xi. Here we have assumed that G x, xi is continues at x equal to xi this is our condition, second thing is that G x x, xi has a jump of 1 upon p xi at x equal to xi that is that G x here your x plus to x minus is equal to 1 upon p xi. So here if you look at here then this will be what sorry it is x minus to x plus, okay.

So if you differentiate here then you will get C 2 y 2 dash xi minus C 1 y 1 dash xi equal to 1 upon p xi. This you can write it like this, okay. So this now it is given by this. So now we have two relation one is C 1 y 1 xi equal to C 2 y 2 xi and C 2 y 2 dash xi minus C 1 y 1 dash xi equal to 1 upon p xi.

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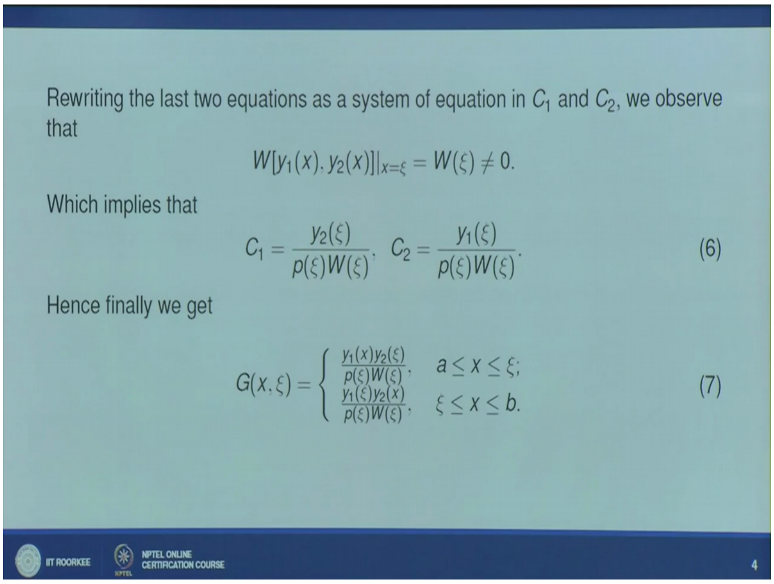

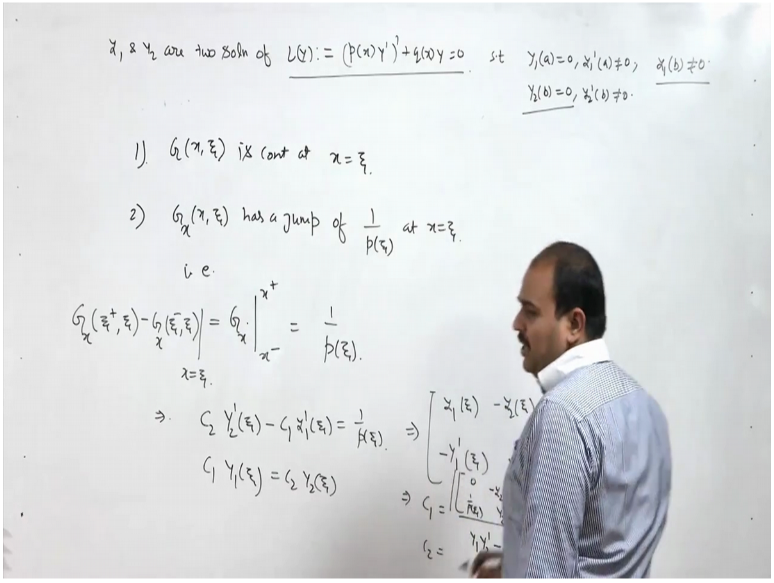

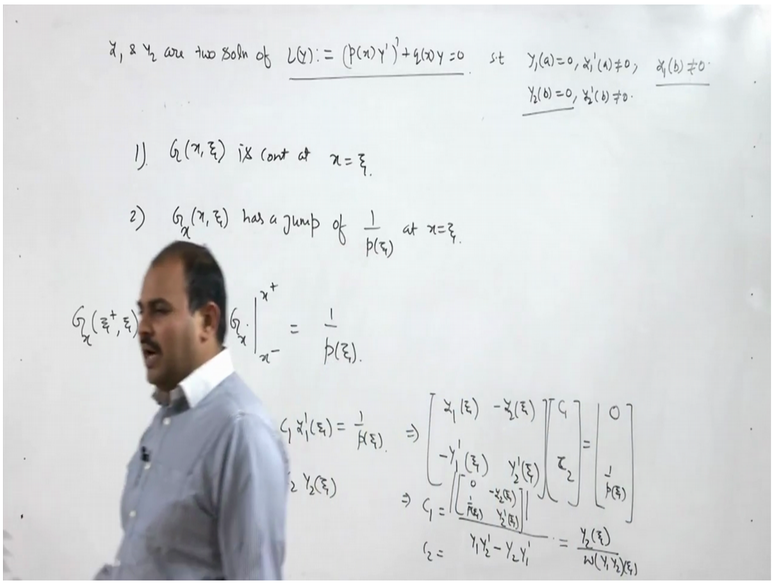

So now we can solve this and since we already know that this y 1 and y 2 are linearly independent solution of the system, so this wronskian is going to be 0 so wronskian of y 1 and y 2 at x equal to xi is going to be nonzero. So with the help of this we can solve for our C 1 and C 2, so C 1 and C 2 can be you can find out the solution C 1 and C 2 as this C 1 equal to y 2 xi upon (p xi plus) p xi W xi and C 2 equal to y 1 xi p xi W xi.

This you can get it here if you look at here what is this C 1 y 1 xi equal to C 2 y 2 xi, so we can write it like this this is nothing but y 1 xi, y 2 xi and here we have y 1 dash xi minus and here we have y 2 dash xi and C 1 and C 2 is equal to so here we have 0 and here we have 1 upon p xi. So this is an algebraic equation this we can solve and we can get the value of C 1, C 2 using Cramer's rule, so Cramer's rule says that this is 0, 1 upon p xi determinant of this and y 2 xi and y 2 dash xi determinant of this divided by determinant of this determinant of this is going to be y 1 y 2 dash minus y 2 y 1 dash.

So there is I think there is a small mistake here this is minus y 2 so this is minus thing. So here it is minus thing, okay and here we have now it is okay. So this is nothing but this you can get it as y 2 xi upon wronskian of y 1 and y 2 at xi. By Cramer's rule we can find out the value of C 1 and C 2 as follows C 1 equal to y 2 xi upon W xi, C 2 equal to y 1 xi divided by p xi W xi and we can get our Greens Function defined as this.

So here we have done this thing and if you remember we have assumed that our wronskian is nonzero. So it means that we have assumed that y 1 and y 2 are two linearly independent solution. So how we followed if you remember we have written what is y 1, so here we have assumed that y 1 and y 2 are two solutions of say L y which is defined as p x y dash whole dash plus q x y 0.

So this is a linear differential equation we have assumed that y 1, y 2 are two solution here such that y 1 a equal to 0 and y 1 dash a is not equal to 0, y 2 b equal to 0 and y 2 dash b is not equal to 0. And also we have assumed one more condition on y 1 that y 1 b is also not equal to 0. So with the condition that y 2 b is equal to 0 and y 1 b is not equal to 0 and y 1, y 2 both are the solution of L y equal to 0, here we have this is possible only when y 1 and y 2 are two linearly independent solution.

So that is why we are able to write down the solution as this because if your W xi is going to be 0 then we cannot write it like this.

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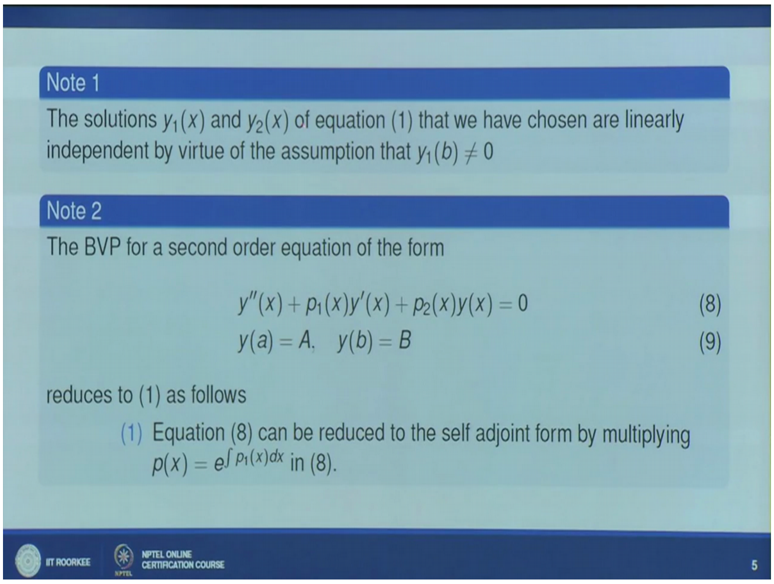

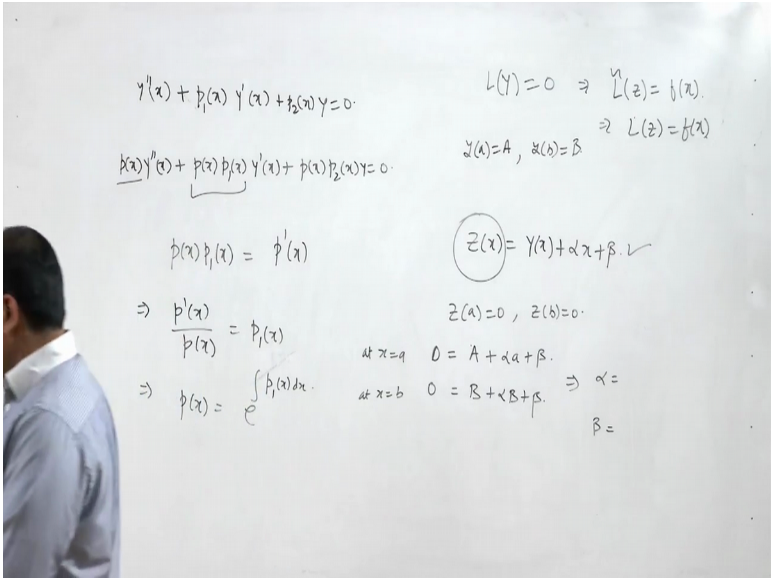

So now let us discuss here and so note 1 is (())(10:19) the same thing that the solution y 1, y 2 are of the equation 1 that we have chosen are linearly independent by the virtue that y 1 b is not equal to 0 but (y 1) y 2 b is equal to 0. Now it may happen that for a given second order equation of the form because what we have discussed is the particular case when we have a self adjoint that linear differential equation is given in self adjoint form plus boundary conditions are given in a homogeneous form.

Now it may happen that in general it may not be true, it means that your equation may not be a self adjoint equation. So it means that y double dash plus p 1 x y dash x plus p 2 x y x is equal to 0. So here the condition that the coefficient of y dash x is not the derivative of coefficient of y double dash x. So it is not in given in terms of homogeneous form, this is not given in terms of self adjoint form.

And if you look at the condition also here condition is given as y a equal to a and y b equal to b and (a and) capital A and capital B may not be equal to 0 (())(11:21). So what to do in this particular case so the first thing that this equation can be converted into self adjoint form by multiplying by this integrating factor. I can tell you just if you look at here y double dash x plus p 1 x y double dash x plus p 2 x (y this is y dash x) y equal to 0.

Now what we try to do is we just multiply by some say function let us say p x is integrating factor, so p x y double dash x plus p x p 1 x y dash x plus p x p 2 x y equal to 0 and we have already discussed that this is is in self adjoint form if coefficient of say y dash x that is p x p 1 x is equal to derivative of the coefficient of y double dash x. So it means that derivative means p dash x.

So if you look at this is a simple differential equation in terms of p of x p x is unknown p 1 x is already given to you because equation is given so p 1 x is given to you. So this this you can solve as p dash x divided by p of x is equal to p 1 x. So it means that this you can find out so p x is equal to you can take e to power integration of p 1 x dx. So by multiplying so it means that a equation which is not given in self adjoint form you can always make them self adjoint form by multiplying this factor p x which is e to power integration of p 1 dx. So you multiply by this factor and this can be converted into self adjoint form.

Now second thing that the condition 9 means this thing y of a equal to a and y b equal to b we want to make it 0 condition so for that we take a linear transformation from which take the point this capital A, capital B to 0, 0. So what we do here we simply take transform z x equal to y x plus this quantity in fact it is this, you take z of x is equal to y of x plus alpha x plus beta.

Now we choose alpha and beta in a way such that this condition is transformed to homogeneous condition if you look at condition are what y of a equal to capital A and y of b is equal to capital B. So what we want here that z of a is equal to 0 and z of b equal to 0, right. So this is basically what this is your 0 equal to y of small a is basically capital A, so this is your capital of A plus alpha a plus beta and here it is what it is 0 and at point b, so it is at x equal to a and at x equal to b it is what z of b is 0 that we want and y of b is capital B plus alpha B plus beta.

So with the help of this you can easily find out what is your alpha and beta, is it okay. So using this linear transformation you can convert our nonhomogeneous condition to in a homogeneous condition.

So now if you do this then we are using a new variable z x. So your equation will also change, so initially your equation is your L y equal to 0 but when you use z of x equal to y x plus alpha x plus beta then it will be reduced to something kind L tilde z equal to your some f of x here. Now here since L is a linear operator so L tilde is same as L here. So this will be reduced to L of z equal to f of x.

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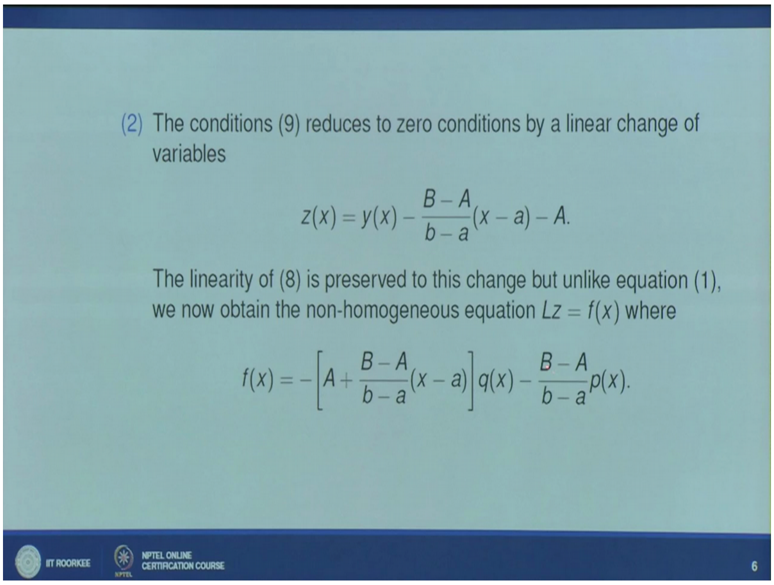

So what is f of x, f of x can be given in this particular form, so f of x is given by this expression that minus of A plus (B minus A) capital B minus A divided by small b minus a x minus small a q x minus capital B minus A upon b minus a p of x. So with these two things a second order linear ordinary differential equation can be converted into a self adjoint form with homogeneous boundary condition.

So it means once we have converted form then this procedure will give you the method to construct Green function for this. So with the help of this we can construct Green function for any second order linear differential equation with any type of condition whether it is homogeneous boundary condition or nonhomogeneous boundary condition you can always find out your Green function, okay.

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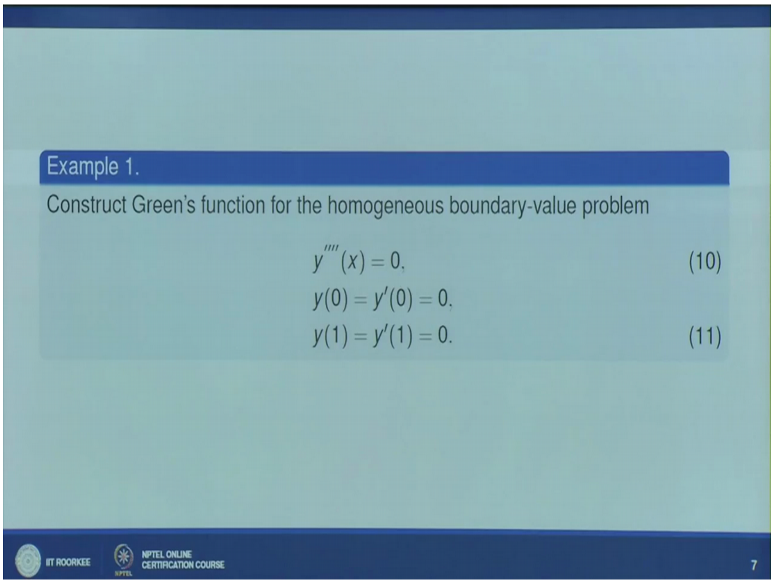

Now let us use this and the previous theory which we have discussed to find out some Green function. So first example let us consider here that y 4 x is equal to 0 with the boundary condition given in this particular form if you look at here this problem is also given in n terms of self adjoint form, so here L is your fourth derivative of y with respect to x. So its adjoint is same as this, okay. So now let us consider the construction of Green function.

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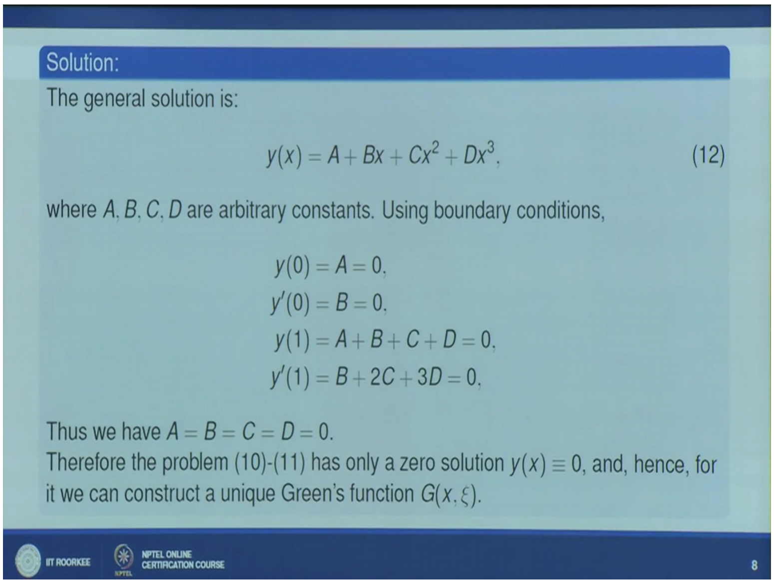

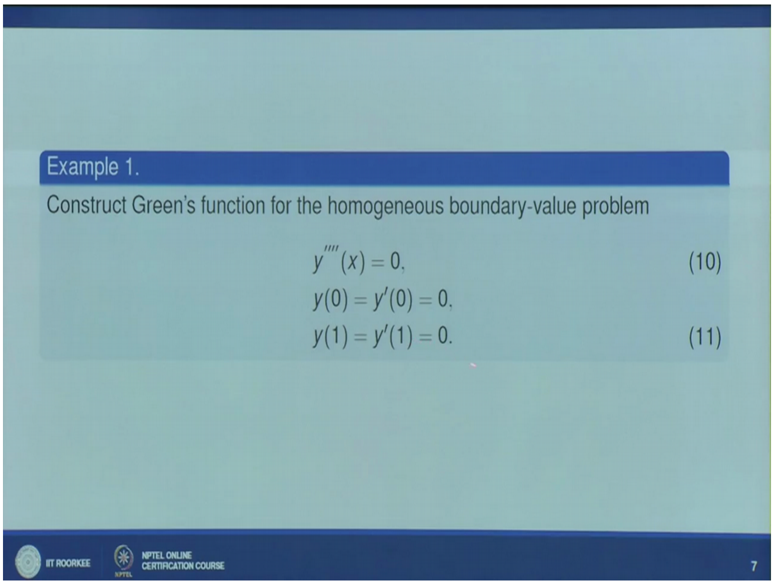

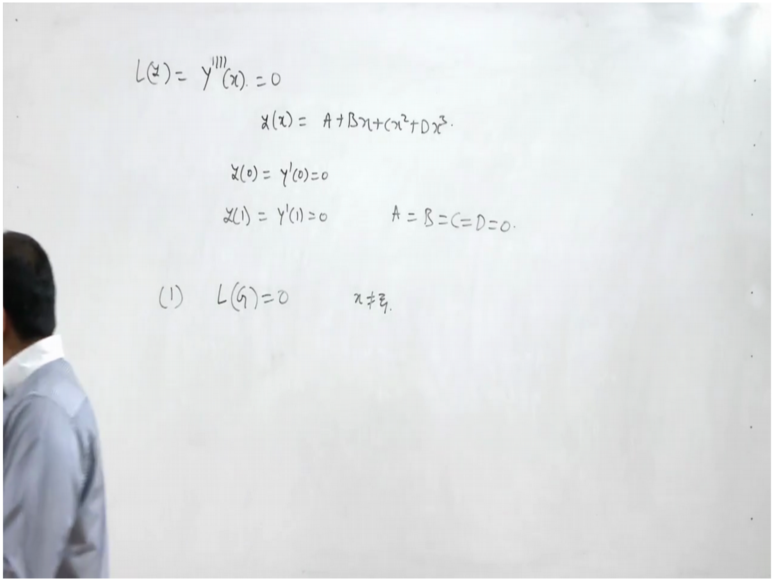

So here the general solution is given by this, so y x is equal to let me write it here. So equation is what L is basically L y is given as fourth derivative of x. So here your general solution we can find out y of x is given as A plus Bx plus Cx square plus D of x cube. So first condition that if you look at your forum the first condition here that this boundary value problem has a trivial solution. So just just verify that thing.

So if you use the boundary condition boundary condition is y 0 y dash 0 equal to 0, y 1 equal to y dash 1 equal to 0. So if I use this condition y of 0 equal to y dash 0 equal to 0 and y of 1 equal to y dash 1 equal to 0. If we use this you can see that all these condition A equal to B equal to C equal to D all are 0. So it means that this homogeneous boundary value problem will have a 0 solution. So here we can construct a Green function to solve this particular problem.

So now let us start with construction of Green function, so Green function is again as we know that first condition on Green function is that it is the solution of operator L here, right. Since it is self adjoint operator so we can write it that G is the solution of this L of G is equal to 0 for x not equal to xi.

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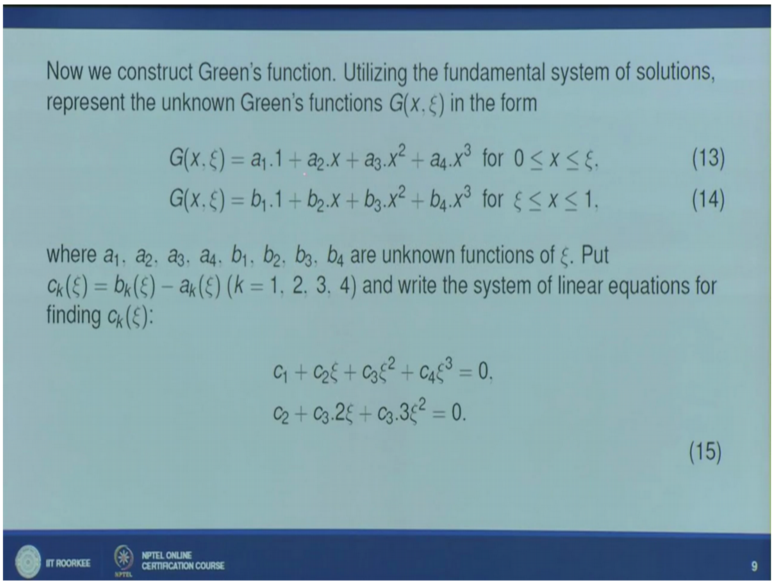

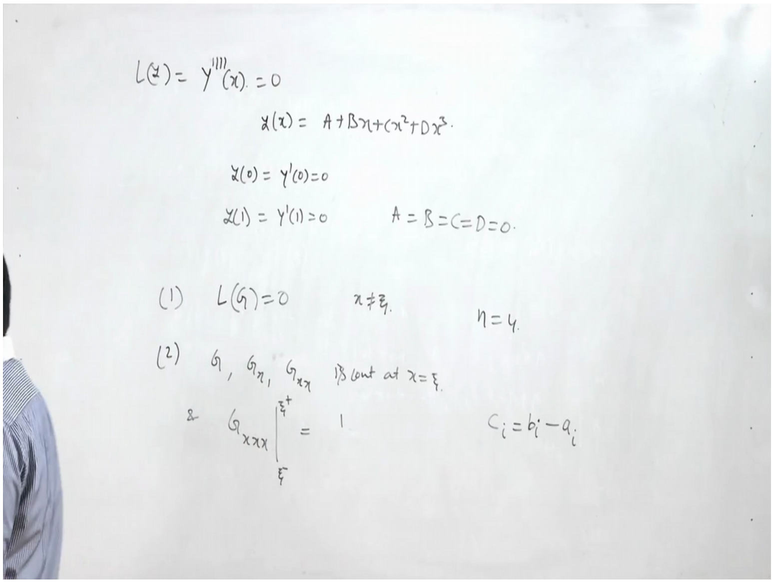

So let us take that G x, xi is equal to (a 1 plus a 2) a 1 plus a 2 x plus a 3 x square plus a 4 x cube and x is lying between 0 to xi. Similarly we can define for (xi) x lying between xi and 1. So and here we need to find out all these coefficient a 1, a 2, a 3, a 4 and b 1, b 2, b 3, b 4 such that it satisfy the certain condition. If you look at the condition on this that G is continuous not only G n minus 2th derivative of G is going to be continuous here, so n in this case it is 4.

So it means that G and G dash and G double dash is continuous let me write it in a proper form let G, G x, G xx is continuous at x equal to xi and your and G triple x and xi minus to xi plus is equal to 1 upon the coefficient here is p not is 1 here so it is 1 here. So we will have this thing. So condition is given by this, if you use this condition and the fact that I am just assuming that c i is equal to b i minus a of i.

Then we can write down these conditions in this particular format I can write it here that c 1 plus c 2 xi plus c 3 xi square plus c 4 xi cube equal to 0, here we are using that G x, xi is continuous at x equal to xi. Second is c 2 plus c 3 2 xi plus c 3 times 3 xi square equal to 0. Here I am assuming that G dash x is continuous at x equal to xi.

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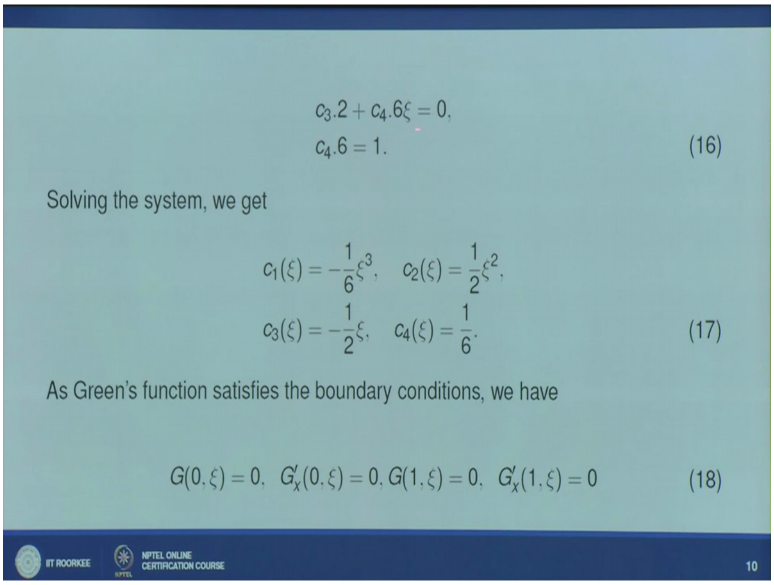

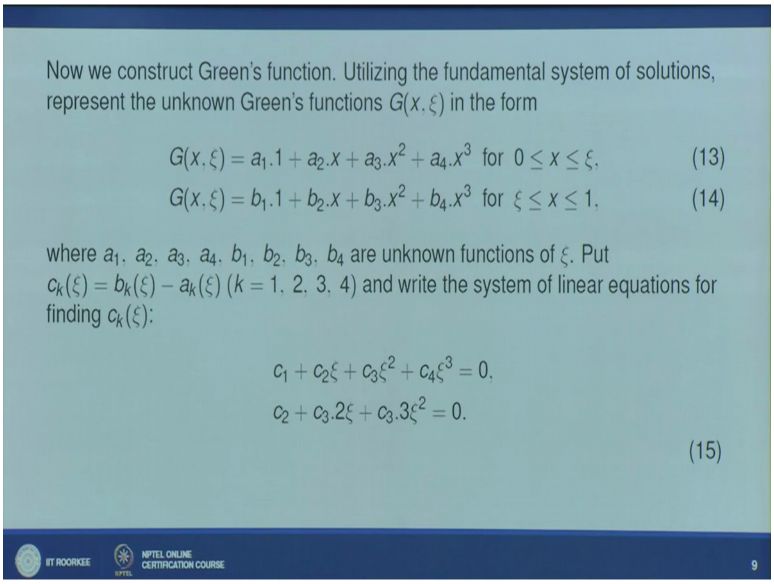

Similarly here I am using that G xx is continuous as x equal to xi and this last condition is showing the jump jump condition that third derivative of G with respect to x is having jump of quantity 1, so c 4 x equal to 1. So we have four conditions and we have four unknowns so we can find out c 1, c 2, c 3, c 4 if you solve this this is a simply triangular system, so you can easily find out c 1, c 2, c 3, c 4.

So now how to fix how to find out a 1, b 1 so now utilize the boundary conditions, so boundary condition is that at point 0 G 0, xi is equal to 0 and G x 0, xi is equal to 0, G 1, xi equal to 0, G x 1, xi is equal to 0.

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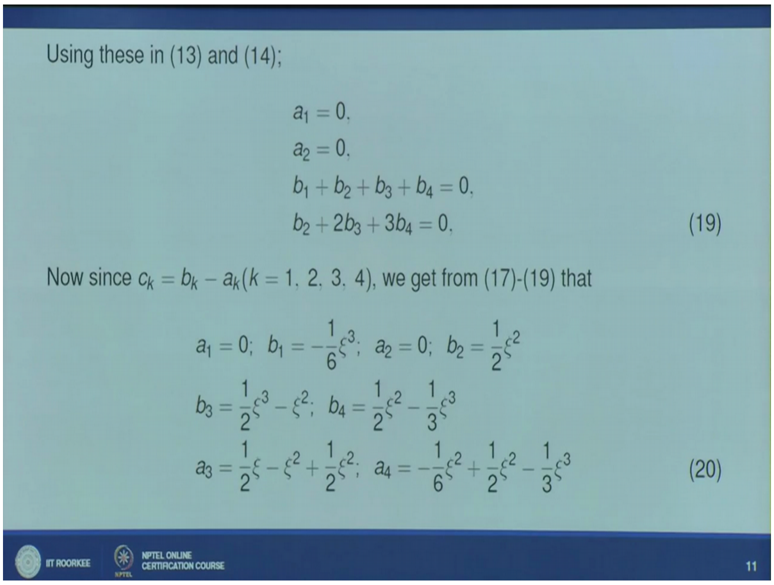

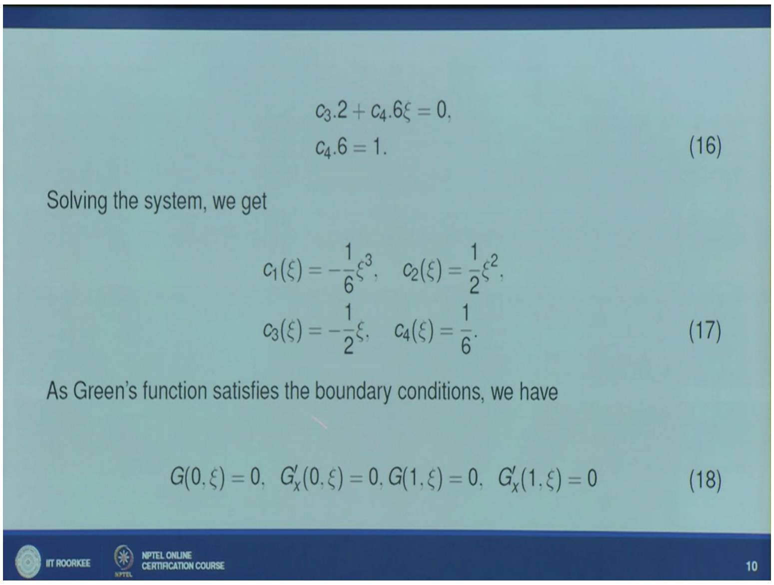

If you use this condition you will get equation condition on a 1, a 2, b 1, b 2, a 1, a 2, a 3, a 4 and b 1, b 2, b 3, b 4. So we have condition a 1 equal to 0, a 2 equal to 0, b 1 plus b 2 plus b 3 plus b 4 equal to 0 and b 2 plus 2b 3 plus 3b 4 equal to 0. So using these four condition and the value of c 1, c 2, c 3, c 4 you can easily find out all these condition a 1, b 1, a 2, b 2, b 3, b 4 and with the help of b 3, b 4 you have a 3, a 4.

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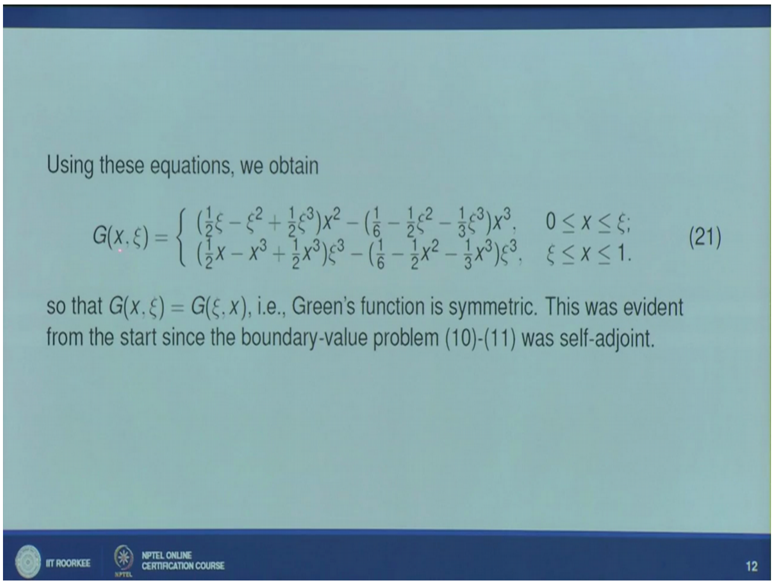

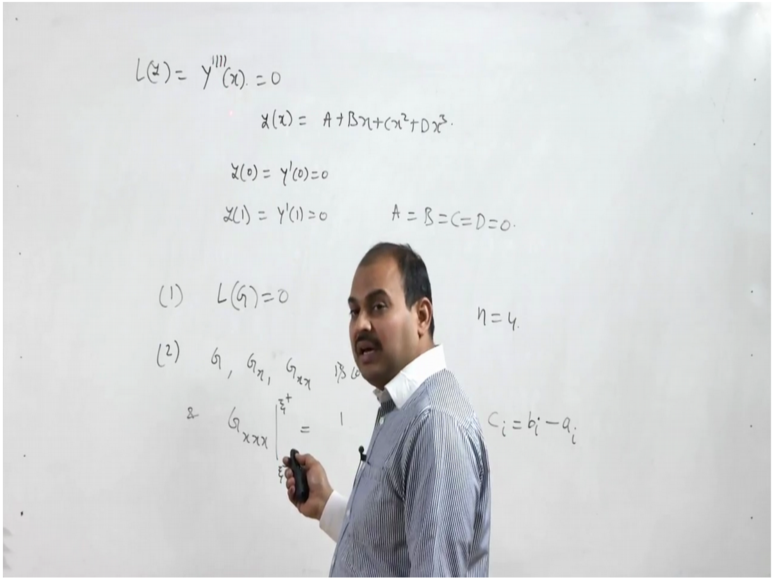

So now we have we are able to find out all these ai's and bi's for this and with the help of this now we can write down our Green function in this particular form G x, xi equal to this in interval x less than or equal to xi and this equal to when x is greater than equal to xi and here we can note this that G x, xi is same as G of xi, x. So it means that this Green function is a symmetric Green function.

And this is quite clear because we assumed we have taken this operator y fourth derivative of x d 4 by dx 4 and it is a self adjoint operator so Green function is going to be asymmetric Green function, okay.

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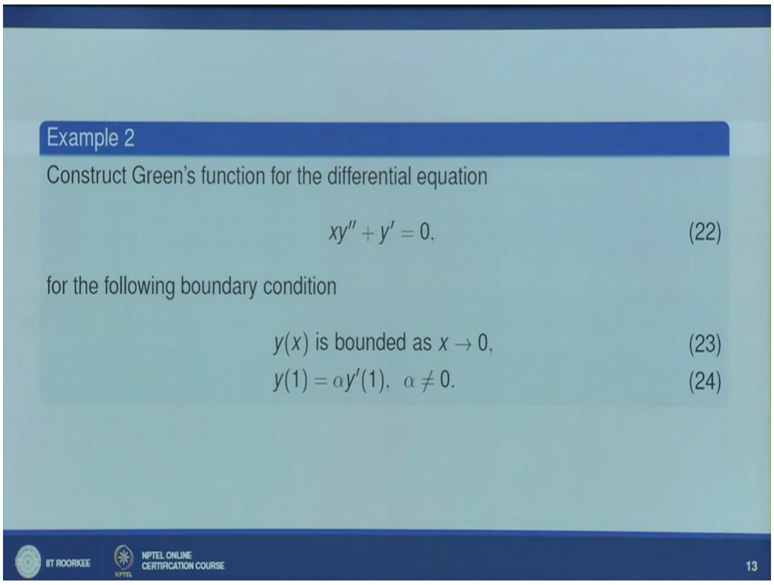

Now let us consider another example if you remember we have assumed one condition that the coefficient of the highest order derivative is nonzero in the entire interval a and b. But suppose it is not suppose your coefficient of the highest order derivative is zero at one of the end point. For example if you look at here xy double dash plus y dash is equal to 0, so this is a ordinary differential equation.

So if we take interval from 0 to 1, right so if we take this interval from 0 to 1 then the coefficient of y double dash which is x is going to be vanished at point x equal to 0. So it means that here we have to define our boundary condition in a way such that we can construct our Green function.

So here we have put another condition that is y x is going to be bounded function as x tending to 0. So this is a natural condition which is coming through coming from the equation itself. So if there is any coefficient of highest order derivative is vanishing at one of the end point then we have assumed that we will assume that your solution is going to be bounded at that particular end point and the other boundary condition is given here that y of 1 equal to alpha y dash 1, here alpha is some nonzero constant.

So here first of all to construct our Green function which is a solution of L of G equal to 0, if L is L y is defined as xy double dash plus y dash equal to 0. So first of all we try to find out the solution of this particular differential equations.

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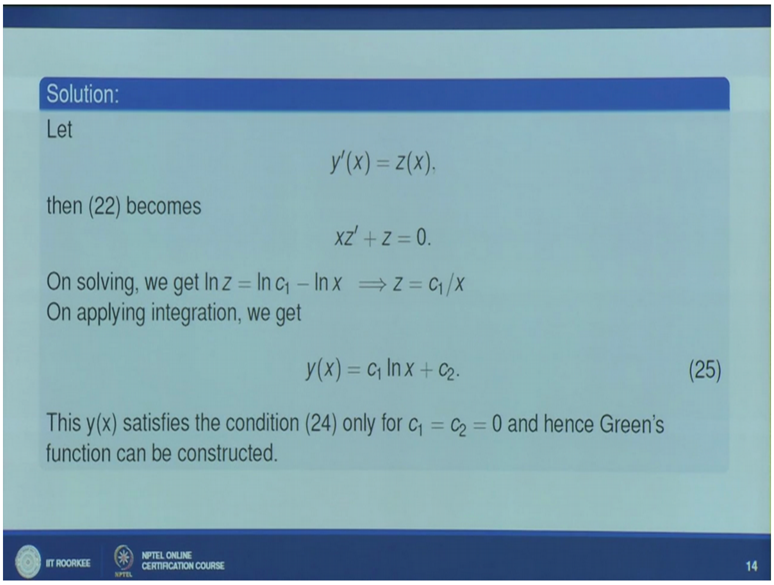

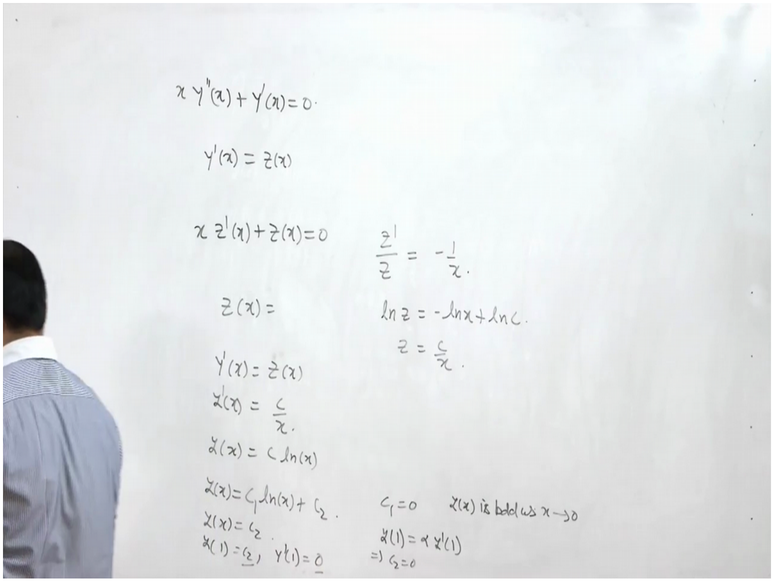

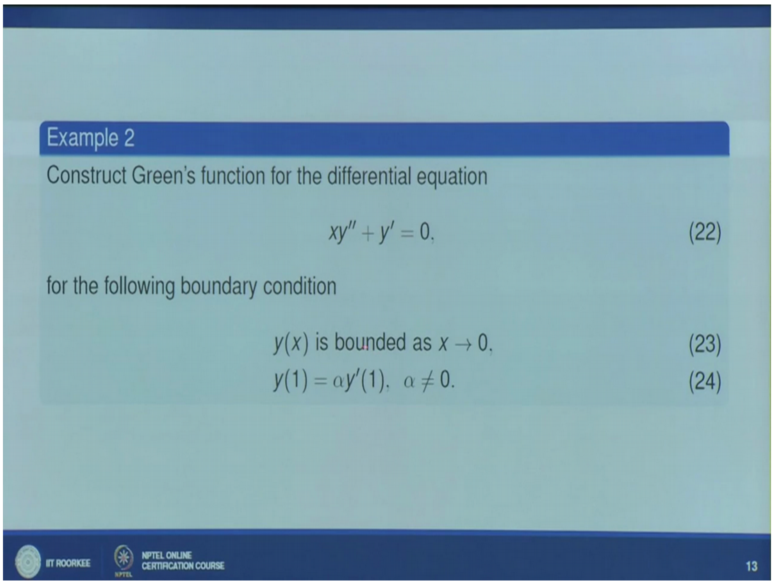

So here we have just assumed that y dash x equal to z of x and then this can be transformed into this xz dash plus z equal to 0. So here we have taken this, so here we have xy double dash x plus y dash x is equal to 0. So what we try to do here we take y dash x equal to z of x or you can simply say that it is a (())(26:08) equation in terms of y dash x. So here if we assume this then it is what x z dash x plus z of x is equal to 0, so you can find out what is z of x and then once we have z of x then you can find out y dash x as this condition z (dash) x, once we have z of x you can find out y of x.

So if you solve this your solution is going to be so here your you can solve it here z dash upon z equal to so minus 1 upon x. So here we can say that ln z equal to minus ln of x plus ln of c here, so z is equal to c by x here. So using this you can find out y x here, so (y of x is equal to c of x here), so y sorry y dash x equal to c by x so y x is going to be c of ln of x. So here we simply say that x is greater than 0. So here put this is just to define it, okay.

So our solution is going to be y of x is equal to c of ln of x plus if you look at one obvious solution of this particular problem is any constant solution. So you can write it here c 1 ln x plus c 2. So it means that the general solution of this equation is written as y x equal to c 1 ln x plus c 2.

Now if we assume the condition boundary condition which is given as y x is bounded as x tending to 0 then this c 1 ln x plus c 2 will remain bounded as x tending to 0 provided that this c 1 has to be 0. So it means that first condition is that your solution is bounded at x equal to 0 (())(28:13) to assume that c 1 is equal to 0 and then if you look at the other boundary condition that is so c 1 is equal to 0 because y x is bounded at x say at x tending to 0 bounded as x tending to 0, so c 1 is equal to 0.

Now if you look at the other boundary condition that is y of 1 is equal to alpha y dash 1 then you can say that that y of x is equal to c 2 then y of 1 is equal to c 2 but y dash 1 is going to be 0 here. So this implies that c 2 has to be 0, so c 2 has to be 0. So it means that this homogeneous boundary value problem has only trivial solution. So now once we have this condition then we can only construct our Green function.

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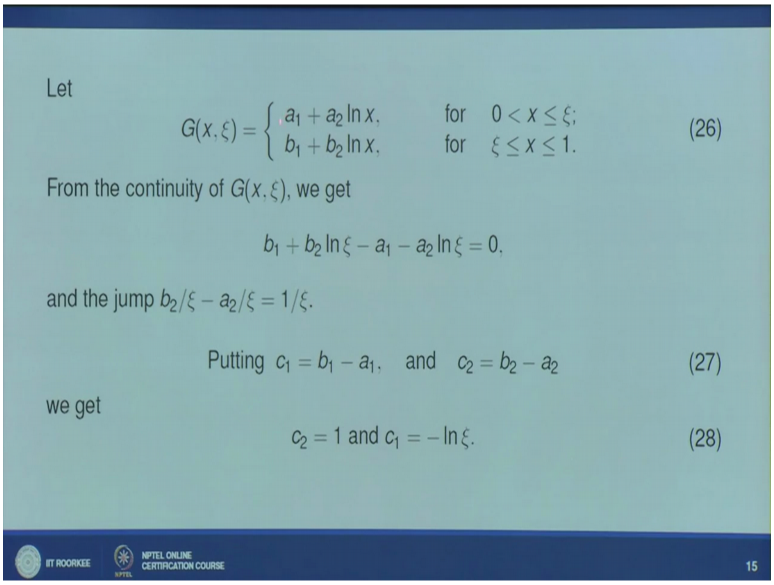

So here Green function is now constructed as G x, xi as linear combination of your solutions and between x to xi and b 1 plus b 2 ln x between xi to 1, okay. So here this a 1, a 2 we can find out that it will satisfy the boundary condition given at the point a and b 1 plus b 2 ln x is the we can find out using the condition that this is going to be satisfy this is going to satisfy the boundary condition given at the point 1, right.

So now so from the continuity of G x, xi we have b 1 plus b 2 ln xi is equal to a 1 plus a 2 ln xi. So we have this condition. Now jump condition if you look at then the derivative of this so that is b 2 by xi minus a 2 by xi is going to be 1 over xi here. Here p not xi is nothing but xi here. So now by using b i minus a i as c i I can write it c 1 equal to b 1 minus a 1 and c 2 equal to b 2 minus a 2.

So if you use this notation then this condition is going to be reduced to c 1 plus c 2 ln xi equal to 0 and here it is c 2 by xi equal to 1 by xi. So c 2 is coming out to be 1 and c 1 is minus ln of xi.

So using this now we put that condition that this a 1 plus a 2 ln x satisfy the boundary condition given at the point x equal to 0 and b 1 plus b 2 ln x satisfy the condition satisfy the boundary condition given at the end 1.

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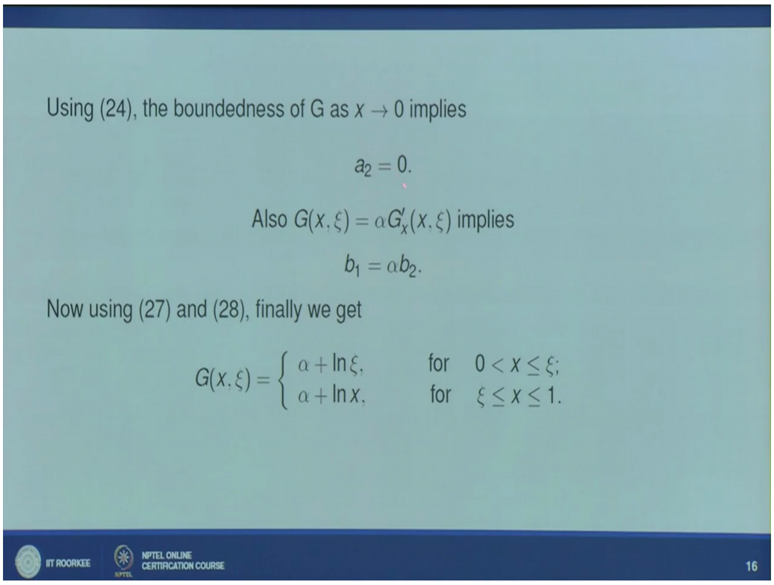

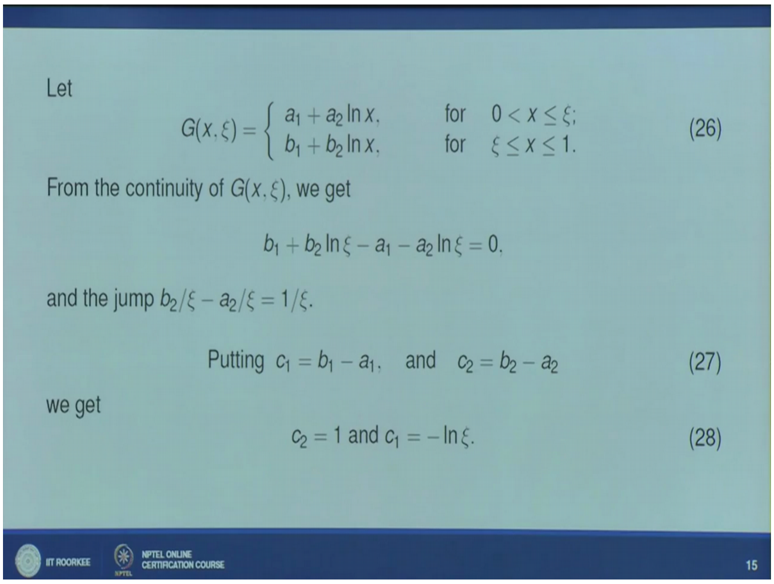

So if you use this the boundary condition given at the point x equal to 0 is that your solution is remains bounded (at x equal to) as x tending to 0. So that simply shows that a 2 is equal to 0 and this condition that G x, xi is equal to alpha G x dash x, xi implies that b 1 equal to alpha b 2. So if we use this and c 1, c 2 you can easily write down that your G x, xi is equal to alpha plus ln xi for x less than or equal to xi and equal to alpha plus ln x as xi less than or equal to x, so you can construct our Green function like this.

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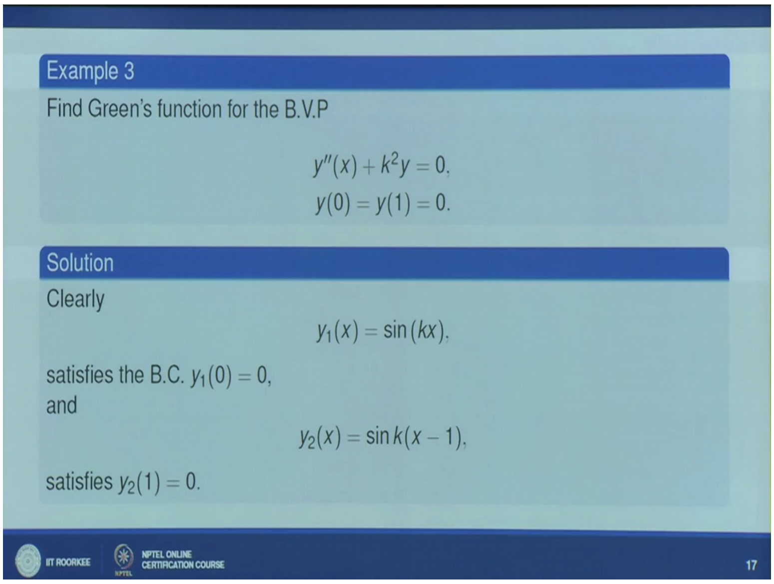

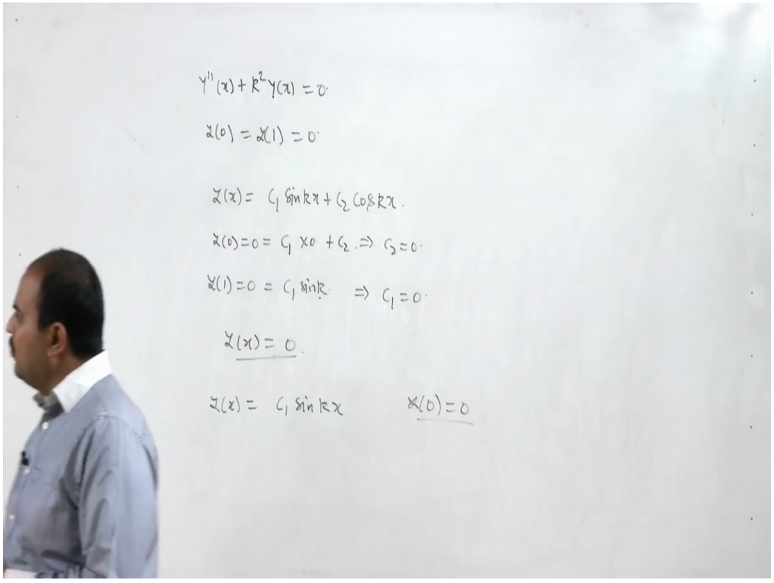

Now let us consider one more problem if you remember the first problem that we have discussed is the fourth order derivative fourth order linear differential equation. Second problem we have considered where your coefficient of highest order derivative is vanishing. Now let us consider (one more condition) one more example so that we have a better understanding of construction of Green function.

So here this is the famous problem y double dash x plus k square y equal to 0, y 0 equal to y 1 equal to 0. So if you look at we can find out the solution and the solution is given by sin of k of x. Now if we use the boundary condition that y of 0 equal to y 1 equal to 0, then we have that the solution given is 0. Let me show it here, so this is your y double dash x plus k square y x equal to 0, here y of 0 equal to y of 1 equal to 0.

So if you solve this the general solution of this is given as y of x equal to c 1 sin of k of x plus c 2 cos of k of x and if you use the condition that y of 0 is equal to 0 this implies that c 1 it is 0 here plus c 2, so this implies that c 2 is equal to 0. Then y of 1 equal to 0 means c 1 sin of k equal to 0. So if k is not your n of pi here k is some given number then you can say that this simply implies that c 1 equal to 0. So it means that your y of x is nothing but a trivial solution.

So now we can proceed for finding Green function for this particular problem. So to start with let us take one solution say we already know that y of x is equal to c 1 sin of k x is the solution which satisfy the condition y of 0 is equal to 0. So let us take y 1 x as this sin of k x which satisfy the boundary condition given at the point x equal to 0. Now take the another solution y 2 x which satisfy the boundary condition given at the end that is x equal to 1. So I can take this sin of k x minus 1, you can take any linear combination of sin k x and cos of k x and ultimately you will get the same thing, okay.

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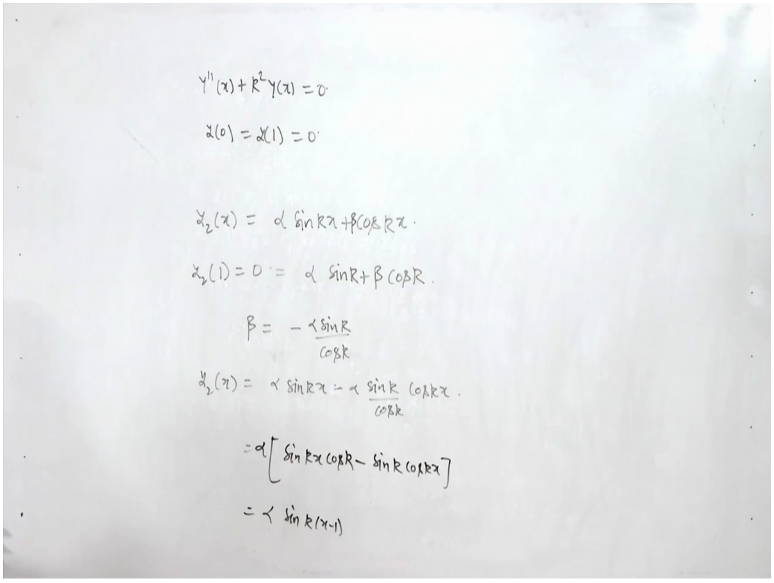

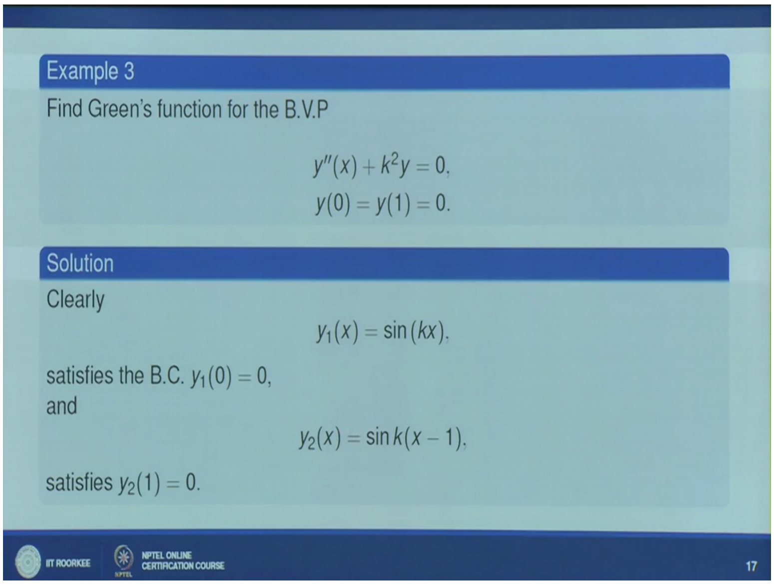

So here you may start with you may choose your y 2 x as some alpha times sin of k of x plus cos of k of x, right. So such that y 2 of 1 is equal to 0, so if you put y 2 1 equal to 0 you will get what, so this is 0 alpha your sin of k plus beta here beta cos of k here. So you can get your alpha and beta you can write it beta is equal to minus alpha sin k upon cos of k, I am assuming that cos of k is nonzero so y 2 of x is equal to I am writing here alpha sin of k x plus beta I am writing here as minus alpha sin of k cos of k and cos of k of x.

If you simplify what you will get if you simplify alpha times here it is what sin of k of x cos of k minus sin of k cos of k x or this you can write it that alpha of sin of k x minus 1. So that is what I am using you may start with this or you can directly write it this y 2 x as sin k x minus 1, it is both way correct you can use anything. So here we assume y 1 x equal to sin k x which satisfy the boundary condition at x equal to 0 and y 2 x equal to sin k x minus 1 as satisfy the condition y 2 1 equal to 0.

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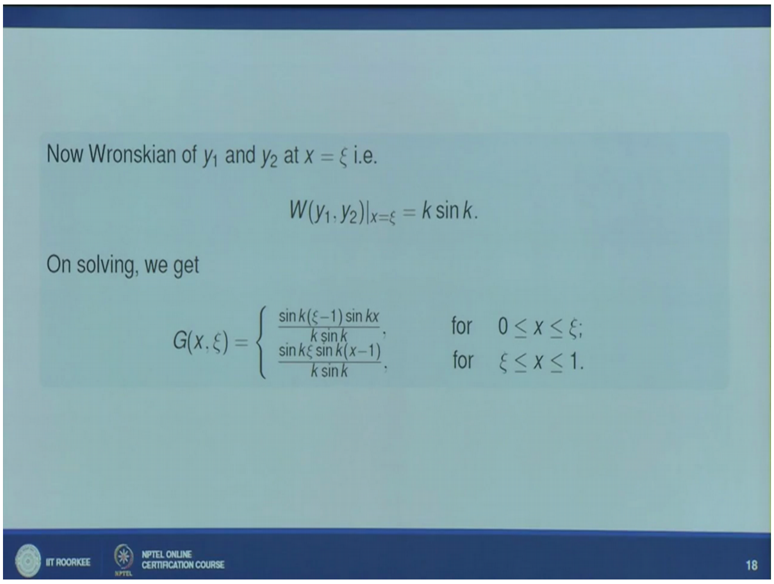

So with the help of this let us find out say wronskian of this, so wronskian of this y 1 and y 2 is coming out to be k sin of k you can easily calculate of this you can easily calculate this and then with the help of this you can write down G x, xi as this, is it okay. So this is all for today's lecture in next class we will discuss the use of Green function for forming of integral equation and solving the boundary value problem, thank you for this.