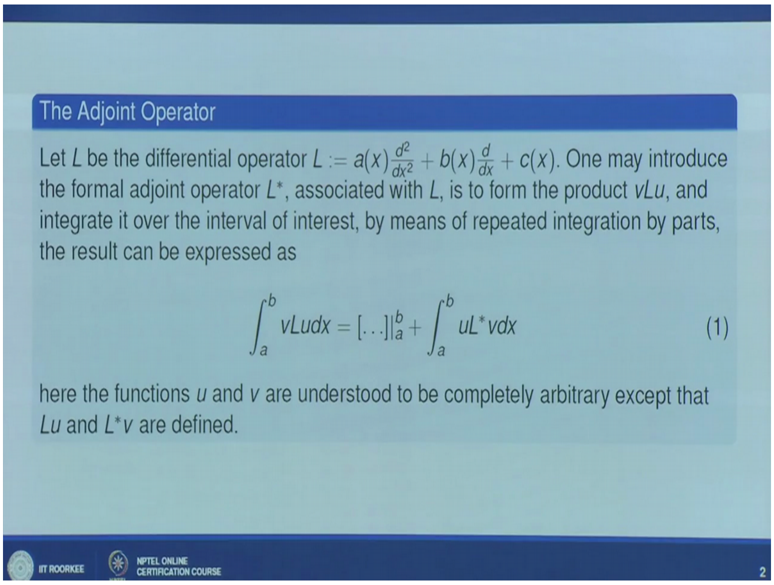
Integral Equations, Calculus of Variations and their Applications
Professor Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 14
Construction of Green's Function-1

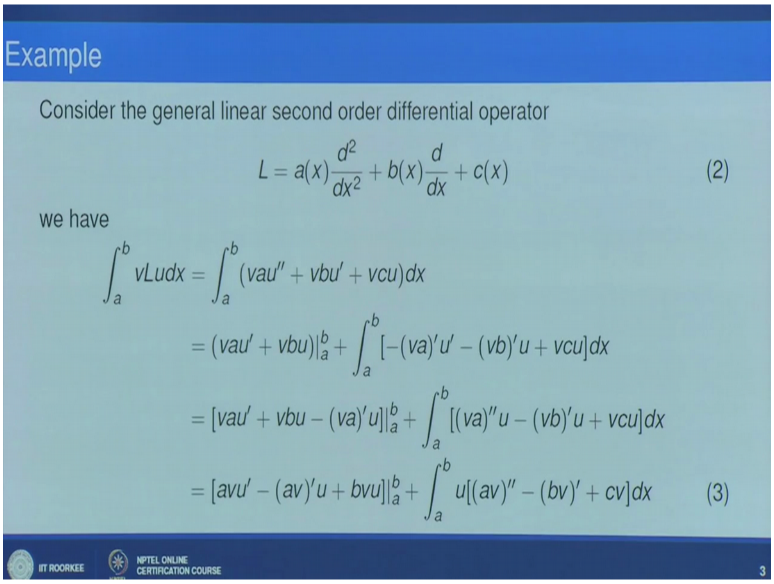
Hello friends in this lecture we are going to discuss construction of Green function and its application in converting linear differential equation into integral equation and in some condition we are able to solve completely the given differential equation. So let us start so before going to start with actual function of greens function we first discuss some basic thing about adjoint operator.

(Refer Slide Time: 0:58)



So for adjoint operator the idea is that let we have L be the differential operator L which is defined as a x d 2 dx square plus b x d by dx plus c x. And here this u and v are considered as say some completely arbitrary only requirement here is that this Lu and L star v exist, okay.

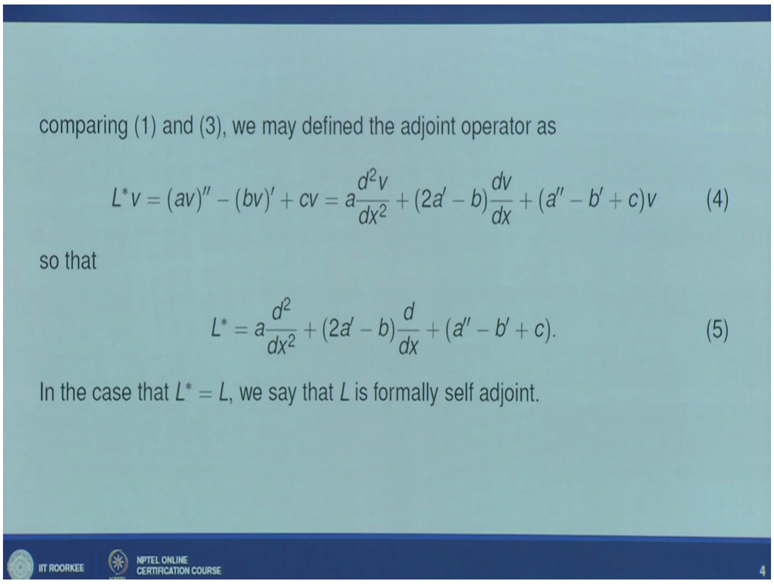
(Refer Slide Time: 1:20)



Now here we are considering this particular example so here if you look at we just multiplying v and integrating with respect to say some interval a to b where a and b are some interval where we want to perform this operation, if you look at here this is done say integration by part we have done once. So here if you look at this vau double dash and integrate by part we will get this vau dash and if we plus a to b minus va dash u dash and if you perform integration by part on this vbu dash you will get this vbu and minus integration of a to b vb dash u.

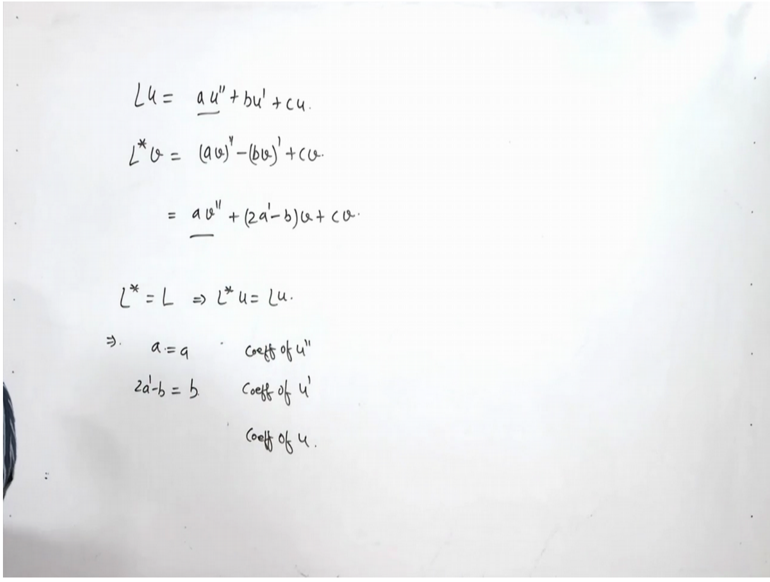
So if we do it all this calculation we will get this, so this is the boundary condition evaluated at the boundary point a to b and this is the integral.

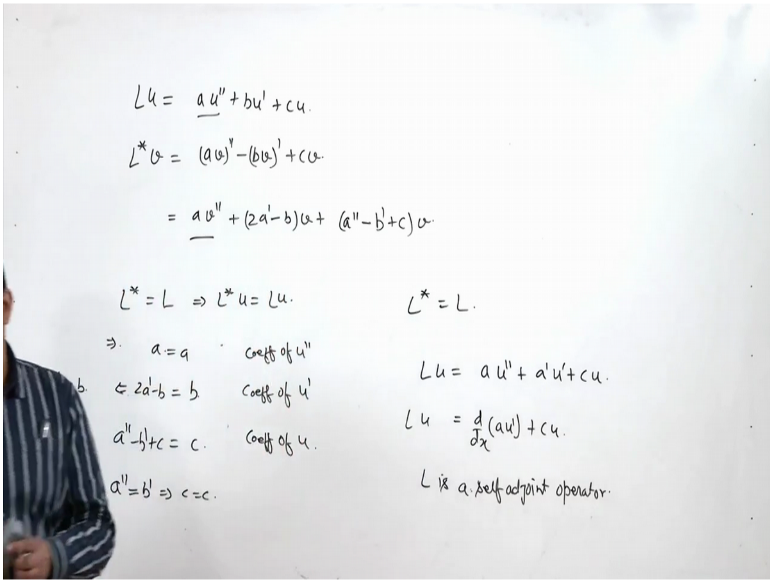
(Refer Slide Time: 2:12)



Here we have seen that corresponding to differential operator L we have defined adjoint operator defined as this L star given as this av double dash minus bv dash plus c of v and if we expand this expression we are going to have this thing equation number 4 that av double dash plus 2a dash minus b v dash plus a double dash minus b dash plus c. Now here in the case when this L star is same as L we call this operator as self adjoint operator.

(Refer Slide Time: 3:00)



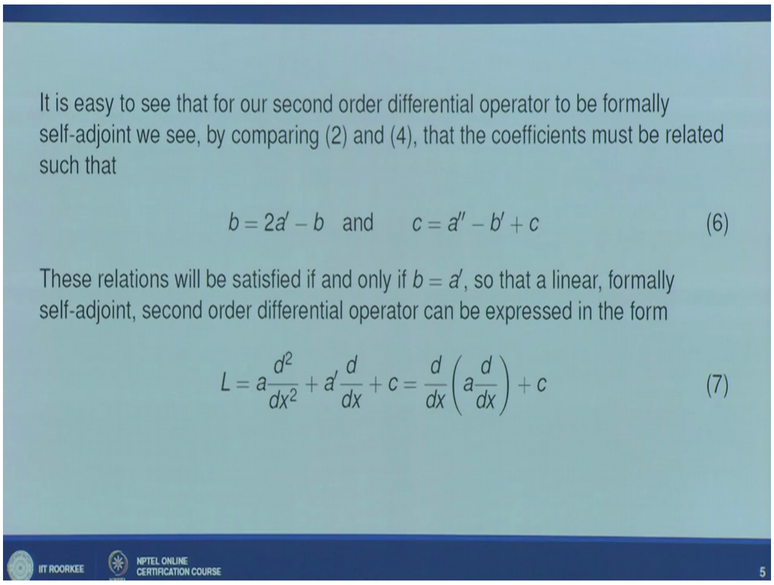


So what it means that let us say that L of u is defined as au double dash plus bu dash plus cu and and calculation we we have this L star v as av double dash minus bv dash plus cv and if you simplify this you will get av double dash plus 2a dash minus b v plus c of v. Now if you say that this L star is same as L this implies that that this this implies that L star operating on u is same as L of u, this implies that your coefficient of u double dash is basically if you compare this this with this you say that a is equal to a, so this is the coefficient of u double dash and if you look at the coefficient of u dash then it is basically here we have 2a dash minus b and here we have say b and if you look at coefficient of u then here we have this is the problem here here it is here it is a double dash minus b dash plus c.

So here if you look at the coefficient of u we have a double dash minus b dash plus c which is given as c here and if you look at if you solve this then here we get what here we get that a dash is equal to b. If you use this condition a dash equal to b then this is nothing but a double dash minus b dash is simply going to be 0, so this imply that c equal to c. So it means that when we assume a dash equal to b then this is trivially true.

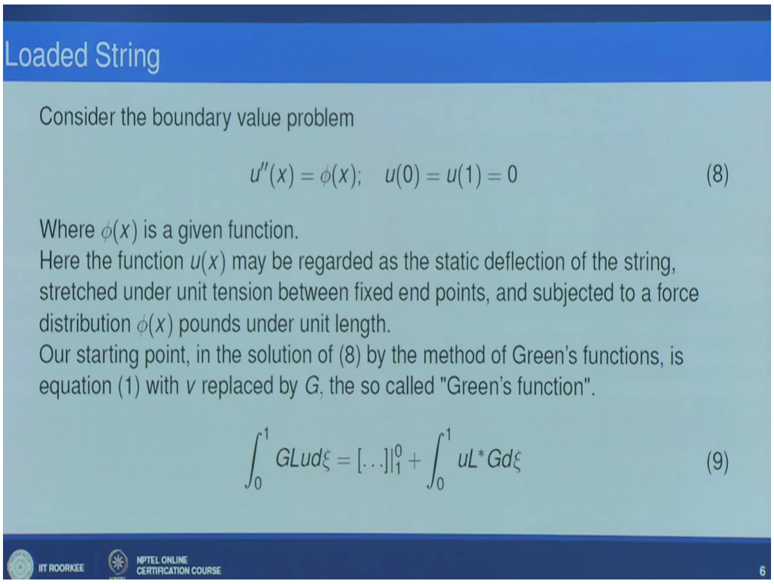
So it means that in the case of when when we have L star is equal to L or we can say that we have a self adjoint operator then we can write our Lu as this that au double dash plus now b is your a dash, so it is a dash u plus c of u (dash). So here I can rewrite this as d of dx of au dash plus c of u. So this is the this form of operator L is known as self adjoint form. So we say that here L is the self adjoint form we do not say that it is the we say that L is a self adjoint operator defined as this.

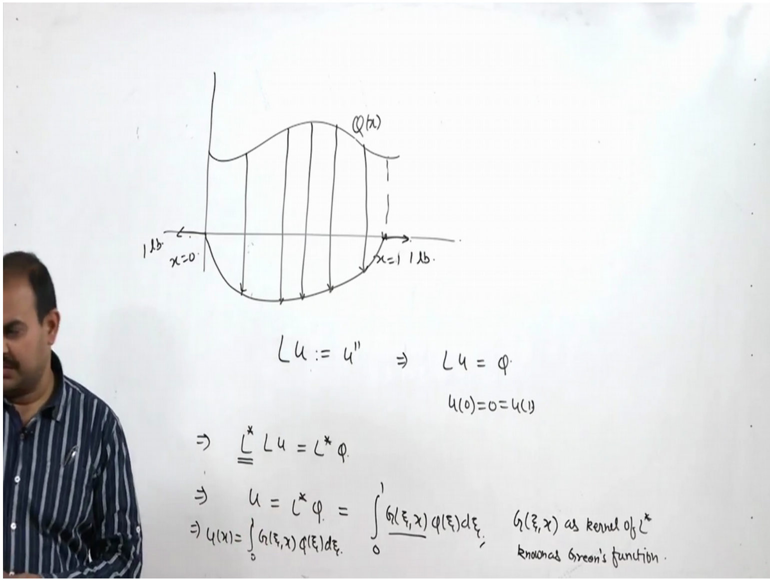
(Refer Slide Time: 6:10)



Now this we have seen that when b equal to a dash then we call this L operator as self adjoint operator and in this particular case your L can be written as d by dx of a d dx plus c.

(Refer Slide Time: 6:28)





Now let us come to construction of Green function, so we initiate we take an example of one boundary value problem given as u double dash x plus phi of x with the boundary condition that u 0 equal to 0 and u 1 equal to 0, where phi x is some given function. Now here we may consider u x as say static deflection of the string and this string is stretched under unit tension between fixed end points and here we are putting this phi x as a force distribution under unit length.

Here we can write like this, so here we have x equal to 0 and here we say that x equal to 1 and between this to fixed end we have string attached here and we have a force say phi of x, right force distribution and it is acting on this string and here we are have a this is unit length.

So now we try to find out the solution of this particular problem u double dash x equal to phi x u 0 equal to u 1 equal to 0.

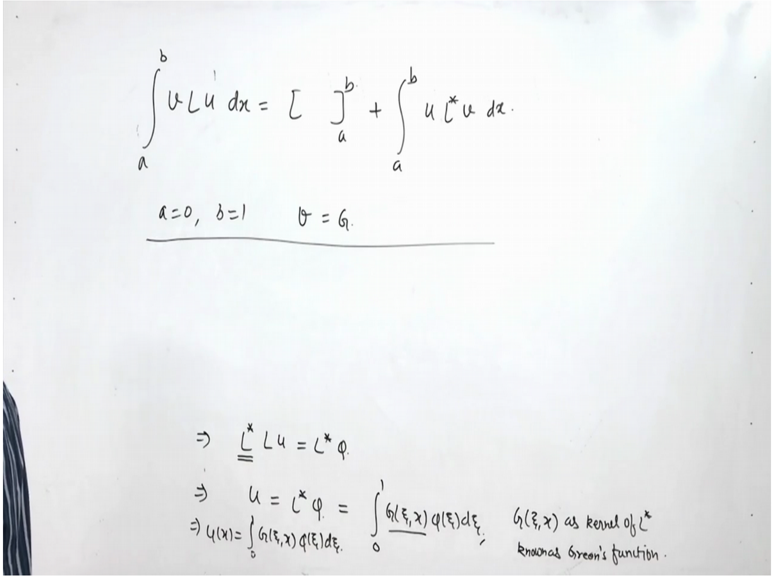
So we may try to find out the solution using variation of parameter method but here we try to solve it in in a different contest. So here we try to solve this using Greens function method. So if you look at the idea idea is something like this. Here we have this L of u which may defined as u double dash, is it okay. And when we define Lu as u double dash, then this problem is given as L of u is equal to phi with a condition that u of 0 equal to 0 equal to u of 1 and we try to find out the solution of this nonhomogeneous linear order in a differential equation.

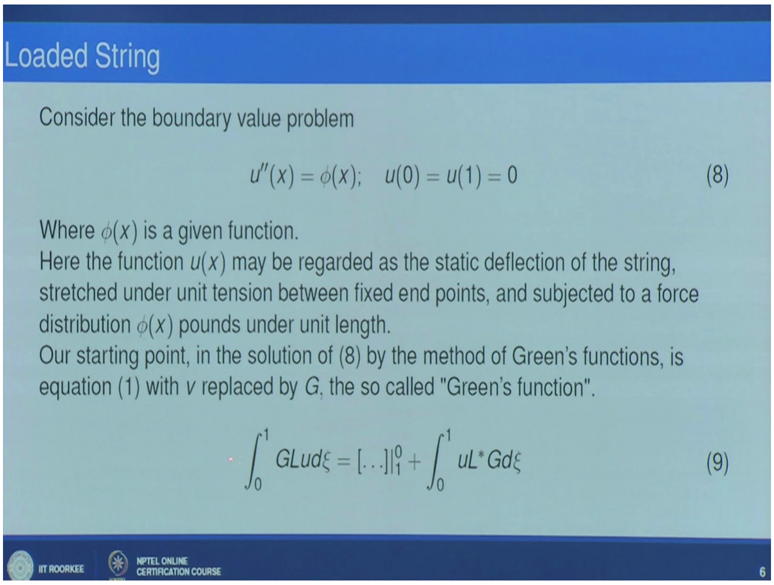
So what we try to do is we try to operate another operator say L star I am just denoting it by L star which we operate on this so we will get what so we can write it like this. So L star operating at Lu is same as writing L star phi. Now what we try to do is we try to define we need to find out L star in a way such that L star L is going to be identity or I can say that this L star is going to be right inverse to this L so we write it like this. So here we simply say that u is going to be L star phi, right.

Now since this L is defined as a differential operator then we can consider we may consider that L star is going to be an integral operator. So when we operate L star phi L star on this phi we may write it like 0 to 1 some G xi, x operating on phi of xi d of xi, right. So here we say that this L star is L star phi is given as this integral thing. Now here I can say that this G xi, x is a kernel of this L star and we call this G xi, x as as kernel of L star and commonly known as Green function Greens function.

So it means that if we can find out this Green function then we may find out our solution u x given as this. So here we can write u x as 0 to 1 G xi, x phi of xi d of xi. So our aim is to find out this green function. So how to do it? So for that use this relation which is known as Greens formula.

(Refer Slide Time: 11:24)

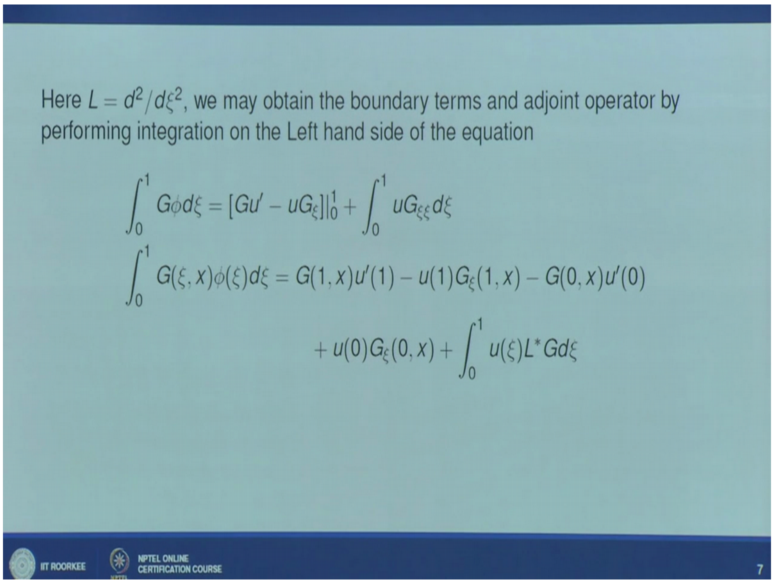


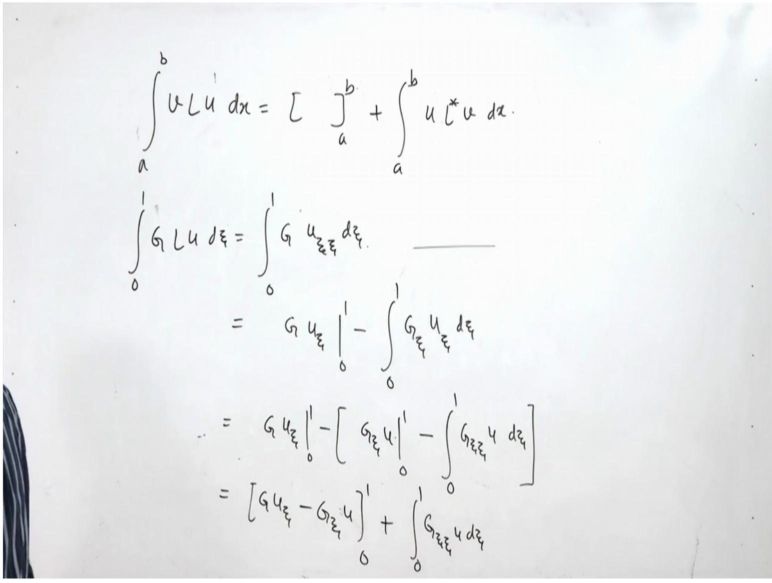


So if you look at you remember this the process of finding adjoint operator what we have done here we have v of Lu dx and we have written it some boundary terms between a to b if the limit is a to b plus here we have a to b and we have u L star v d of x. Now here in this particular case your a is 0 and b is equal to 1. So what we try to do is we try to look at the same formula but now we simply say that v is replaced by this Greens function G.

So here we have seen that while defining the adjoint operator we have this relation that a to b v Lu dx equal to boundary terms plus a to b u L star v dx. For this particular problem we have a equal to 0 and b equal to 1, what we try to do is we just replace our v by Green function G and try to solve this. So here we have 0 to 1 GLud xi equal to boundary term between 0 to 1 and plus 0 to 1 uL star Gd xi.

(Refer Slide Time: 12:46)



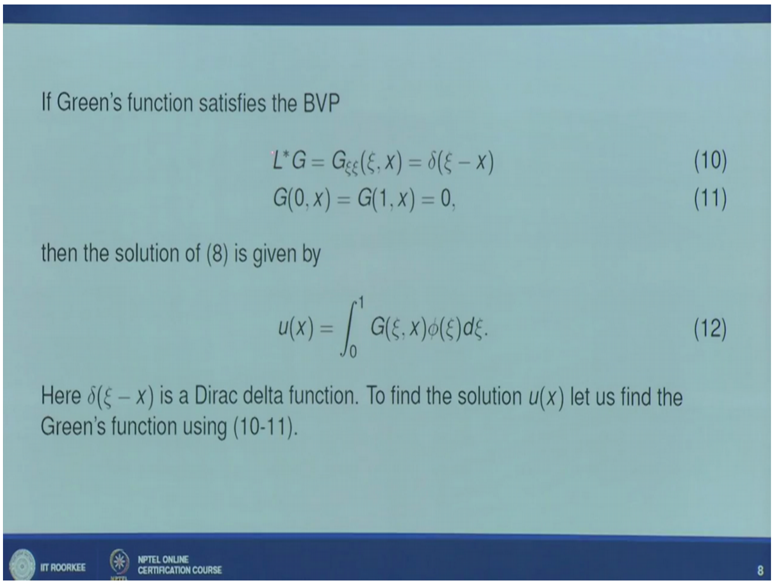


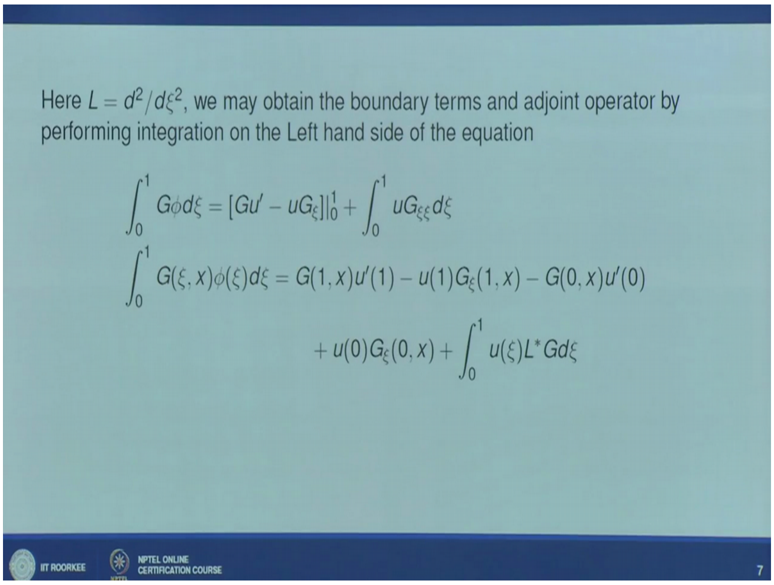
For this particular problem we have L as d 2 d xi square and if you simplify this what you will get you will get here. So here we have 0 to 1 v which is replaced by G, so G L u dx is equal to I am writing here 0 to 1 G and L is u xi xi d xi I am denoting here that variable is xi here, so this I can write it as first integration of second so u xi 0 to 1 minus 0 to 1 G xi u xi d xi. If you perform one more time integration by part you will get G u xi 0 to 1 minus here if you do G xi u between 0 to 1 minus 0 to 1 G xi xi u d xi. If you simplify this you will get G u xi minus G xi u evaluated between 0 to 1 plus 0 to 1 G xi xi u d xi.

So it is written like this so here if we write down the boundary points here then we have G 0 to 1 G xi, x phi xi d xi equal to G 1, x u star 1 minus u 1 G xi 1, x minus G 0, x u dash 0 plus u 0 G xi 0, x plus 0 to 1 u xi L star G d xi. So here if you remember we have u double dash u double dash is given by phi, so I am writing here phi in place of u double dash xi. So if you look at here than if we look at the boundary condition boundary condition is that u 1 is equal to 0 and u 0 is going to be 0.

So it means that these two term is going to drop out the only thing left here is this and this and if you look at this term I want that this should give the solution of the problem. So it means that use 0 to 1 u xi L star G d xi should somehow give the function u of x here.

(Refer Slide Time: 15:54)

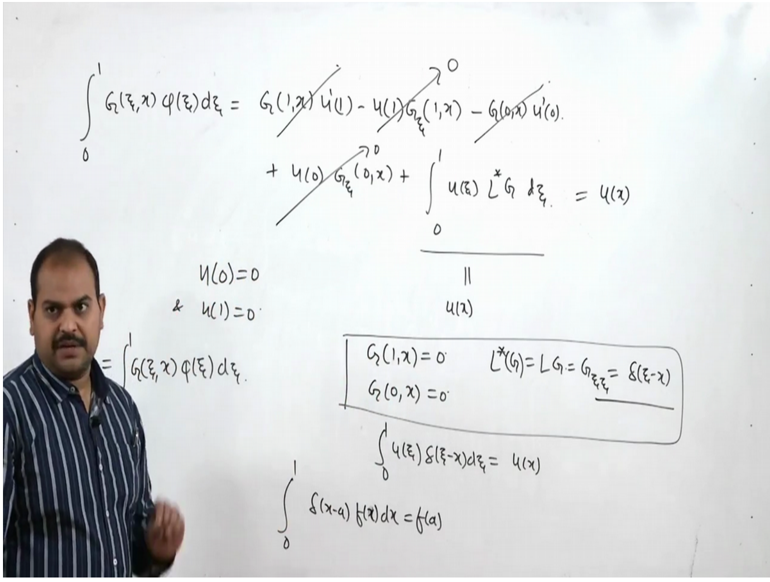


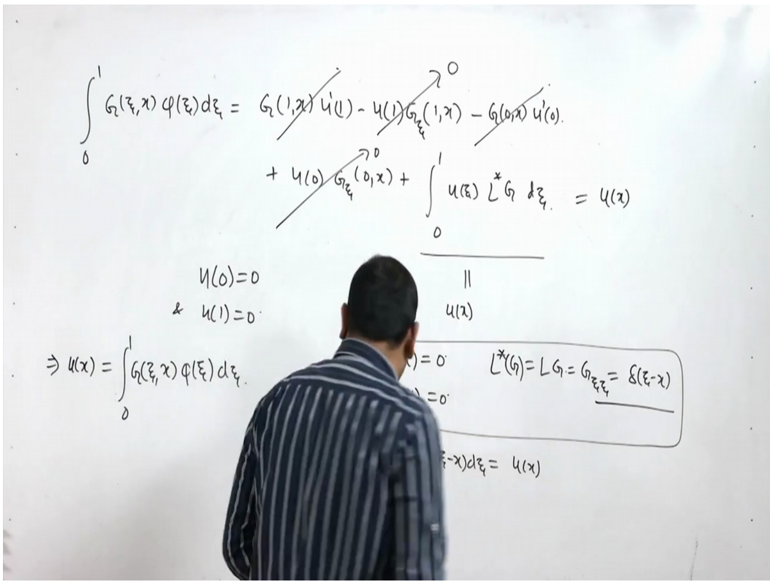


So to show that this will give you u of x here we assume that our Green function L star G is given by G xi xi xi, x here because here we are dealing with self adjoint operator so L star is nothing but same as L which is double derivative with respect to xi here, so G xi xi equal to delta xi minus x. And the boundary condition just to make these two term vanish that G 1, x and G 0, x we are assuming that it is 0.

So this is an additional condition we are putting here that G Green function should satisfy this second order ordinary differential equation plus G 0, x equal to G 1, x given as 0. If we use this then this term last term this 0 to 1 u xi now this is what delta xi minus x.

(Refer Slide Time: 16:50)





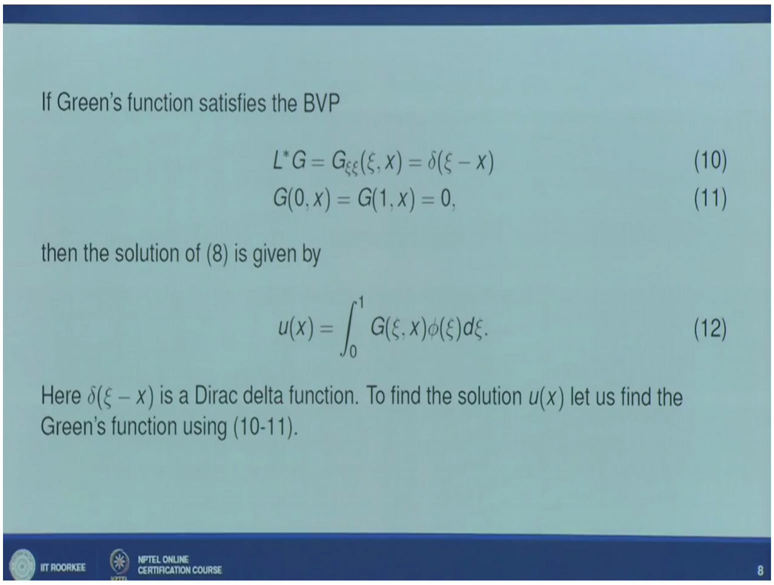
So here delta is your Dirac delta function 0 to 1 G xi, x phi xi d xi equal to G 1, x u of 1 u dash 1 minus u 1 G xi 1, x minus G 0, x u dash 0 plus u of 0 G xi, 0 x plus 0 to 1 u xi L star which is same as L here G d xi. Now since we already have u 0 equal to 0 and u 1 equal to 0. So if we use this this condition is simply 0 and this is again going to be 0, right.

So now if we want that this will give us the solution here what we want is that we choose our G in a way such that this will give you u of x and and G 1, x should be 0 so that no boundary term will be there and G 0, x is equal to 0. So here what we want is this this we can achieve by saying that if L star G which is same as L G here so it is G xi xi here and we want that this should be given as delta xi minus x.

So if we assume this then 0 to 1 u xi delta xi minus x d xi is going to be u of x, so here this is the filtering property we are assuming here that 0 to 1 delta x minus a (f of a) d of f of x d of x is going to be f of a. So here we are using the filtering property of Dirac delta function. So it means that if we assume our Green function which satisfy this equation that G xi xi equal to delta xi minus x and satisfying these two boundary condition that G 1, x equal to 0 and G 0, x equal to 0 then this will also vanish and this is coming out to be u of x, so it is coming out to be u of x.

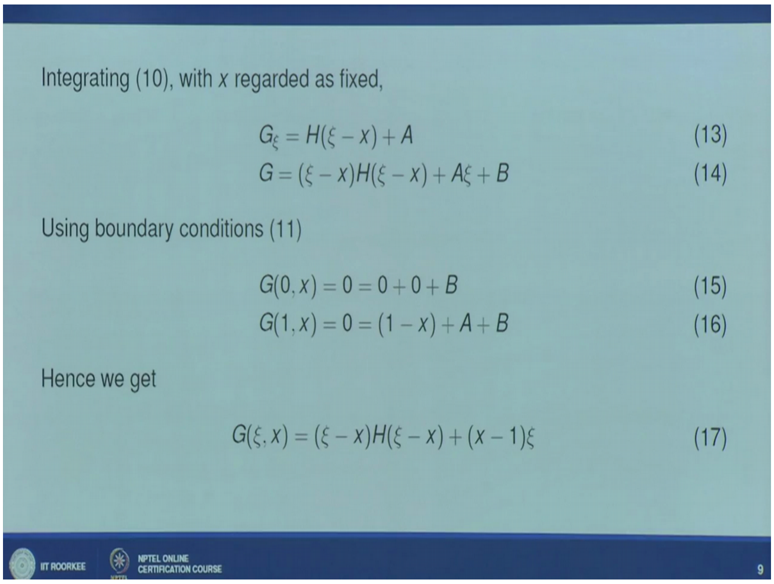
And we can rewrite this as u of x is equal to 0 to 1 G xi, x phi of xi d xi. So this is our solution provided we should know what is G of xi, x. So let us try to find out G xi, x with the this boundary value problem here and try to find out what is G xi xi, okay.

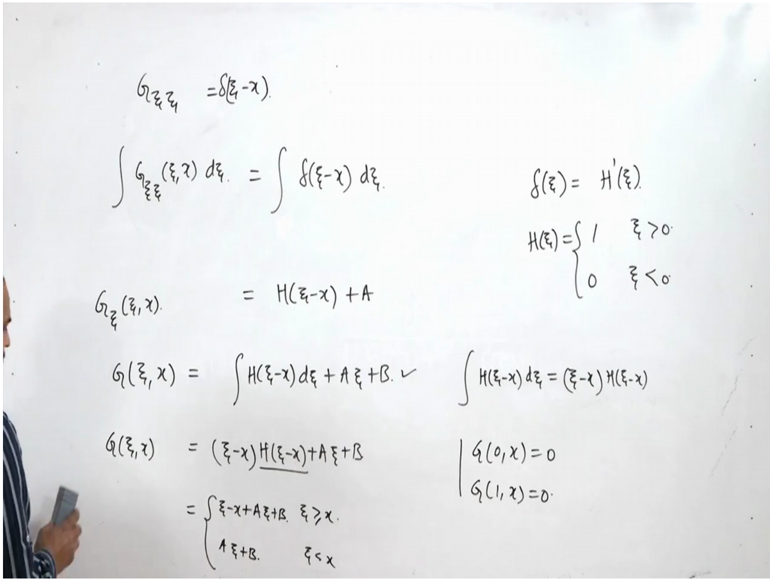
(Refer Slide Time: 20:26)

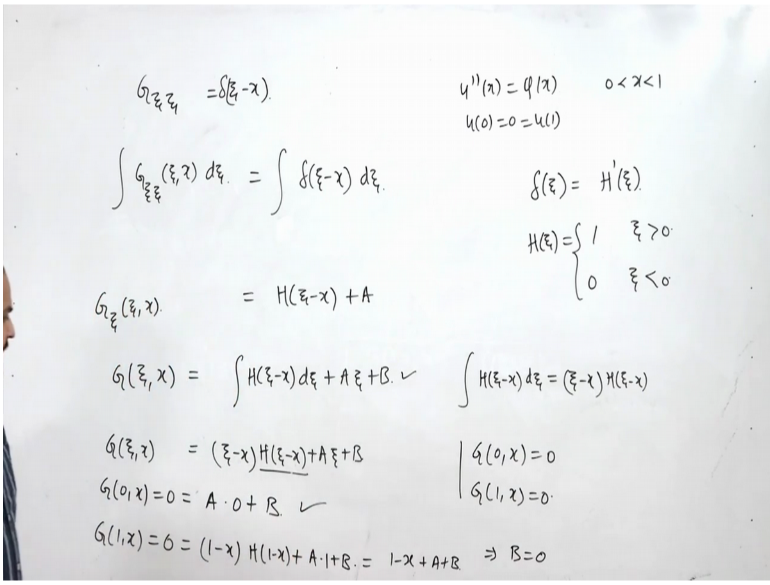


So for that here just look at here here we these these are the Greens function satisfies these boundary conditions L star G equal to G xi xi equal to delta xi minus x with the boundary condition and solution is given this, okay.

(Refer Slide Time: 20:40)







Now if we look at this boundary condition your G xi xi is equal to delta xi minus x, so we integrate with respect to xi keeping x as fixed variable. So we are integrating this G xi xi equal to (xi) delta xi minus x. So here we integrate with respect to xi keeping x as fixed, so G xi xi xi, x d xi between 0 to 1 is equal to 0 to 1 delta xi minus x and d xi and if we recall here that this Dirac delta function delta xi can be considered as derivative of Heaviside function.

So what is Heaviside function here? Here Heaviside function is defined as 1 when xi is positive and 0 when xi is less than 0. So we can this is available in any of the standard text book of differential equation that delta xi is defined as H dash xi, where H xi is a Heaviside function given defined as this. There is a small hint for this particular problem, so I can write this delta xi as limit say epsilon tending to 0 I can write it here H minus epsilon H minus epsilon minus H epsilon sorry H epsilon minus H minus epsilon divided by 2 epsilon 2 epsilon, right so here xi minus epsilon and xi plus epsilon minus H xi minus epsilon by 2 epsilon. So you can take this delta xi as derivative of this.

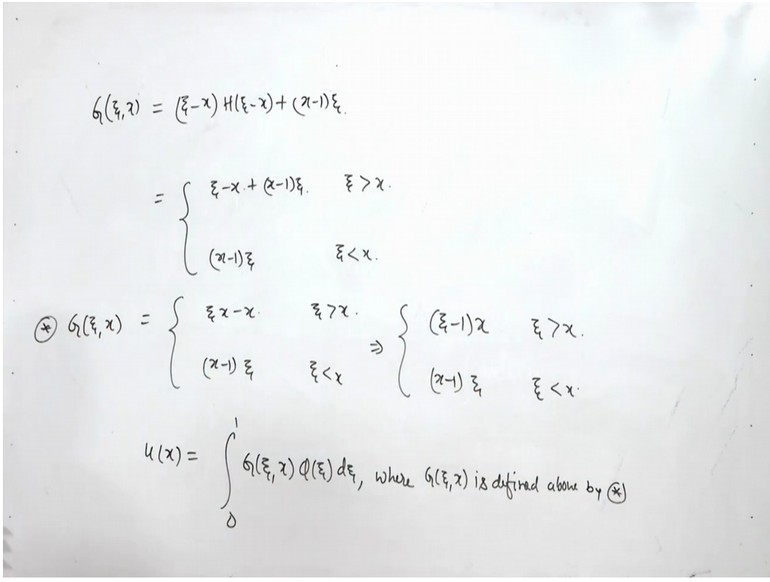
So using this concept we can write this as H of xi minus x plus some constant A. Now so this is nothing but G xi xi, x is given by this I am not putting the limit here 0 and 1 we simply integrating in a definite as definite integration, so that is why I am putting this as integration constant here. So now we want to we want to find out G xi only, so integrate one more time so we have G xi, x given as integration of H xi minus x d xi plus A xi plus B. So this you can get this as xi minus x H xi minus x plus A of xi plus B so I can write it G of xi x is this.

So here I am using this this integration that H of xi minus x d xi is given as xi minus x H of xi minus x. So this you can proof as an simple exercise of Heaviside unit step function. So once we have G xi, x given in this particular form then using the definition of H xi minus x I can write this as say this will activate when this argument of H is going to be positive, so it means that when xi is greater than x we have xi minus x plus A xi plus B and when we have xi less than x we have A xi plus B, right.

So here we have to how to find out this A and B for that we have condition given to us that G of 0, x is 0 G of 0, x is 0 and G 1, x is equal to 0, so G 1, x equal to 0. So if we operate if we use this two boundary condition then we can find out A and B how we can find out let us see, let me write it here then put G 0, x equal to 0 then here when we put xi as then this is going to be a negative then this will be as 0 and you can simply write it here as 0 times (sorry) this I can write it here A into 0 plus B.

Similarly G 1, x it is 0 here then we can write 1 minus x and this H 1 minus x here plus A into 1 plus B. Now look at here this H 1 minus x whether it is positive or not. So if you remember your problem your problem is basically this u double dash x equal to phi of x and condition was u 0 equal to 0 equal to u 1 here, so here your x is between 0 to 1. So it means that this 1 minus x is going to be positive. So if 1 minus x is going to be positive then this is going to be 1, so I can write it here this 1 minus x plus A plus B here. So and that is going to be 0, so this I can write it here as B equal to 0 this implies that B equal to 0 which is clear from this and you can write here A as this x minus 1.

(Refer Slide Time: 27:42)

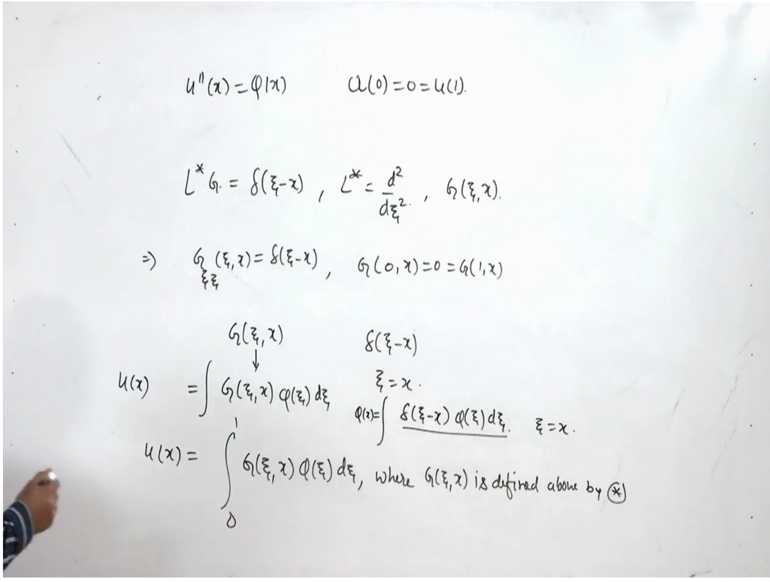


So I can write it here that G xi, x is defined as xi minus x H xi minus x plus x minus 1 xi. Now again using the formula definition of xi minus x I can write this as xi minus x plus x minus 1 xi when xi is greater than x and x minus 1 xi when xi is less than x or you can simplify this further and it is what this is simply xi x minus x, so xi x minus x and when xi is greater than x and x minus 1 xi when xi is less than x.

So this I can further write in a different manner, so I can write xi minus 1 x when xi is greater than x and x minus 1 xi when xi is less than x. So here our Green function is coming out to be this so if we remember our solution is going to be written as u of x is equal to in integral format 0 to 1 G xi, x phi xi d xi. So here where G xi, x is defined above we can say it is defined by this formula. So we can write it here G xi, x is equal to this, so we can call this as star, so defined above by star.

So it means that our solution of the problem is given by this so we have seen that u x is given by 0 to 1 G xi, x phi xi d xi where G xi, x is defined above by this. So here so it means that I can write down the solution here once we know what is phi of xi here then we can write down this solution in this particular integral equation Fredholm integral equation.

(Refer Slide Time: 30:26)



Now here we have a very interesting interpretation of G xi, x if you remember here in the beginning we have started all this for solving this particular problem u double dash x equal to phi of x and (phi) u of 0 equal to 0 equal to u of I and here we simply said that this u x is nothing but static deflection from the equilibrium position under the force phi of x. But when we calculate it our G xi, x then it is what it is basically L star G and we have delta xi minus x, here L star is given as d 2 by d xi square and G is nothing but G xi, x.

So it means that it is given as G xi xi xi, comma x equal to delta xi minus x and with the condition that G 0, x is equal to 0 equal to G 1, x. So here if I look at the same interpretation for this particular problem then I can consider as that G xi, x is basically deflection at a point xi here when a unit force delta xi minus x is applied at the point xi equal to x. So it means I can just repeating it one more time we consider that this G xi, x is denoting the deflection in string when your unit when unit force is applied at xi equal to x.

So I can write that when we apply G xi, x phi xi d xi then we can say this is the deflection in string for the force delta xi minus x phi xi d xi, right. So it means that I can consider that G xi, x phi xi d xi is the deflection in string when you apply delta xi minus x phi xi d xi force at point xi equal to x. So this is the force we are acting at xi equal to x.

Now this xi is a variable point right now we are considering only the fixed point xi equal to x. Now if we vary for every point between 0 to 1 we can say that this is the deflection this will represent the deflection if this is the force acting on the string. So this will represent your u of x and the force it is given as if you apply your filtering property then this is nothing but your phi of x, is it okay. So it means that u x is going to be a deflection in your string if phi x is the force applying on the string per unit length. So that is the that maybe we can consider as a explanation for this G xi, x.

So here in todays lecture in this lecture we have started the concept of Greens function and see that how we can solve a ordinary differential equation with the help of integral equation and if your function phi xi is known to us then we can find out the complete solution, thanks for this lecture, thanks for listening us and in next lecture we will meet.