Integral Equations, Calculus of Variations and their Applications.
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Lecture-11.

Fredholm Integral Equations with Symmetric Kernels: Properties of Eigen Values and Eigen functions.

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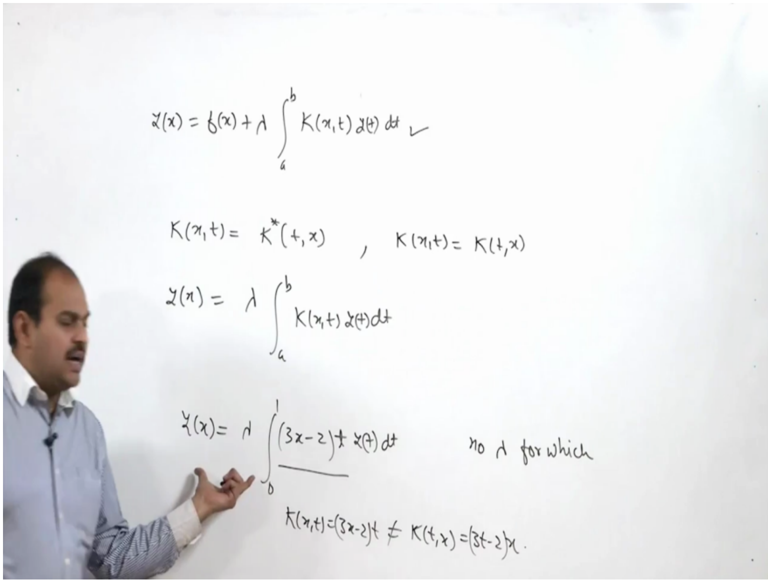

Hello friends, welcome to the lectures of integral equation, calculus of variations and its application. In today's lecture we will discuss the solution of Fredholm integral equation with the symmetric kernel. So what it means here today we will discuss this kind of problem, you can write it yx equal to f of x + lambda a to b K of x of t your y of t, y of t and d of t. This is your Fredholm integral equation of 2nd kind. If you, now here if I assume that K of xt is equal to K star t, x, then we call this kernel as symmetric kernel. You can say that this K star tx is complex complex conjugate of this Kt x.

So if these 2 are equal, we say that I have kernel is symmetric all we can say that kernel is complex symmetric early can say that kernel is a Hermitian kind of time. So kernel is Hermitian kind of kernel. Okay. So here we will discuss some properties of this kernel, solution of this. Here if you remember that if you consider the homogeneous version of this, that is lambda a to b K of xt y of t dt, then we try to find out the solution of this homogeneous Fredholm integral equation of 2nd kind. And the constant lambda for which we have a nontrivial solution, we call that constant lambda as eigenvalue of this kernel K xt or the Eigen values of the Fredholm integral equation even as this.

And the corresponding nontrivial solution, we call as Eigen function corresponding to this lambda. And if you remember, we have seen several case, for example you have this example yx is equal to say lambda 0 to 1, 3x - 2 and t, y of t, d of t. So this is one example which we have discussed in the case when kernel is separable kernel. If you look at this, this is example of separable kernel. So here this is a type of, which does not satisfies this. Here for, particularly here your kernel is real, then for the real kernel, this K xt is equal to Ktx. So symmetric condition is reduced this, so for real kernel, K xt is equal to Kt x, implies that your kernel is symmetric kernel.

So if you look at here, your K xt is 3x - 2 into t. So it is not your symmetric kernel, you can easily see that K xt is basically what, K xt is 3x - 2 t and Kt x is your 3t - 2 , x and clearly they are not equal. So it means that here we have seen that suggests, in this problem your kernel is not symmetric. And if you remember, we have already proved that this has no Eigen values, so no lambda for which this equation has a solution. So it means that there are possibilities that if your kernel, there are kernels available such that we do not have no Eigen values and hence no Eigen function.

But this is not happening when we take kernel of this kind. So it means that symmetric kernel is very very important in the sense that here you can always find out say constant lambda for which we have at least one nontrivial solution. You can always find out at least one Eigen value corresponding to symmetric kernel. So that is why this is very very important topic that we have, we want to discuss it today is the kernel which is symmetric kernel.

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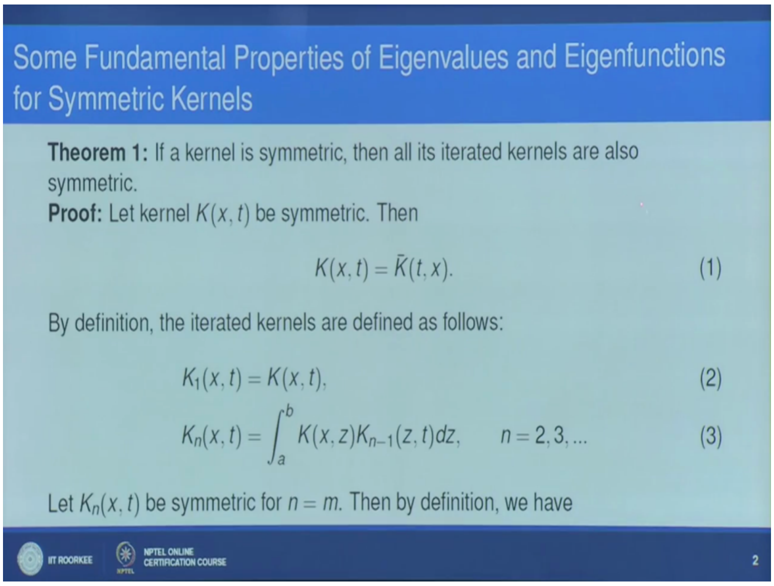

So here if we start, then the 1st result which is very obvious is this, that if your kernel is symmetric, then all its iterated kernels are also symmetric. If you remember, we have already discussed the methods to solve the Fredholm integral equation of 2nd kind. One which is in terms of separable kernels, there we have already discussed and other way to solve this is your method of successive approximation, there we have seen that the concept of iterated kernels. So if we say that, if a kernel is symmetric, then its iterated kernels are also symmetric.

And that is not very difficult to prove, here you can simply say that if it is symmetric, it means that K xt is equal to K bar t x, by definition the iterated kernels are defined as follows. So here if you remember the iterated kernels, we can define it like this. K1 xt we can denote as K xt and Kn xt is given by this a to b kxz, Kn -1 Z t dz where n is from 2, 3… so here we will try to prove this by mathematical induction, so mathematical induction said that for n to 1, your result is trivially true. Your K1 xt we are defining as K xt. And if we assume that the result is true for n equal to m then, so here we are assuming that let Knxt be symmetric for n equal to m, then we want to prove the same for n equal to m +1.

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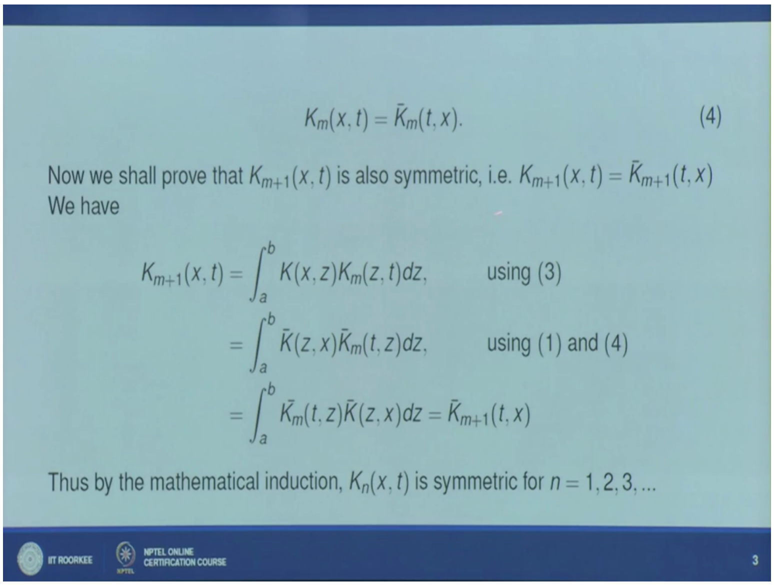

So here if you remove, here we have assumed that for n equal to m, the result is true, means Km xt is equal to Km bar tx. So we want to prove for is Km +1 xt, so this we want to prove, that Km +1 xt is equal to K bar m +1 tx. So for that we simply write of the definition of Km +1 xt which is nothing but a to b, kxz, kmzt dt dz. Now here since Kx that is symmetric, so I can write this as K xz as K bar Zx and kmzt I can write as Km bar t that dz. And if you write it, I can get it like this Km bar tz, K bar Zx dz and this, this is nothing but the bar of complex x conjugate of K bar m +1 t of x.

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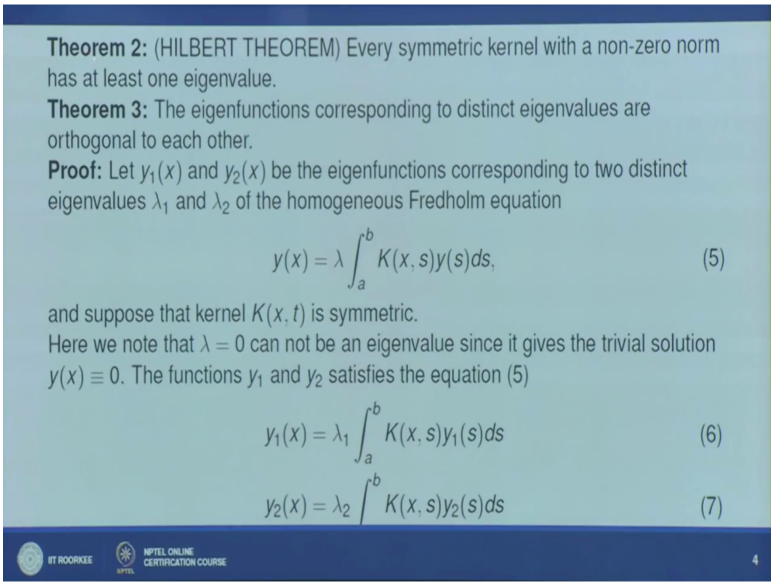

So here if it is true for n equal to1 and n equal to m, then it is true for n equal to m +1. So we have, we can prove our result by mathematical induction that if your kernel is symmetric, then all its iterated kernels are also symmetric, this you are going to utilise later on. Now here the 1st thing which is very very important on the beginning point of the study of eigenvalues and Eigen functions is that every symmetric kernel with a nonzero norm has at least one eigenvalues. This is kind of a license start, licensed to begin. So it means that this is a quite lengthy proof, so I am not going to discuss the proof of this but we are taking this example, this theorem without proof.

And here we assume that whenever we have a homogeneous problem like this, and your K xt is a a kernel which is symmetric and is nonzero, then it always have at least one eigenvalues. And that implies that we have at least one nontrivial solution of this problem. So keeping this thing in mind, so it means that we may, we have at least one eigenvalue and at least one Eigen function corresponding to that. Now we may have, we may have more than Eigen values or we may have finite Eigen values or we may have infinite Eigen values.

For example if you remember we have already discussed the kernel with, which is separable kernel, there we have seen that way may have, we can have only finite many Eigen values. So but if your kernel is not separable, then it may happen that your kernel may have infinite number of eigenvalues and correspondingly we may have infinite number of Eigen functions. So now we want to discuss the properties of Eigen values and Eigen functions and the 1st very very important property is this property is that the Eigen function corresponding to distinct eigenvalues are orthogonal to each other.

So here we are, let us take 2 Eigen values, say here for example, without loss of generality I am assuming lambda 1, lambda 2, you may take as lambda m and lambda n. So I am assuming that lambda 1 and lambda 2 are 2 distinct eigenvalues, so it means that lambda 1 and lambda 2 are not equal and we also consider that corresponding to this we have y1 x and corresponding to lambda 2 we have y2 x as Eigen function. And then we try to show that they are orthogonal to each other. So here what we are assuming that, we are assuming that we have a problem of with the symmetric kernel and we are assuming that suppose more than one Eigen values exist.

So if there are more than 1 eigenvalues exists, so it means more than 1 Eigen functions exist. So here we are assuming that corresponding to lambda 1 by 1 and lambda 2 by 2 are 2 Eigen pairs. And we want to show that they are orthogonal to each other. So if you remember by the definition of Eigen pairs, we can say that yx is equal to lambda a to b Kxs y of s ds and here I am assuming that Kx as symmetric. Now we are assuming that these are Eigen pairs, it means we have these 2 equations, 6 and 7. So y1 x is equal to lambda 1 a to b Kx s y1 s ds and y2 x is equal to lambda 2 a to b Kxs y2 of s ds. Now here we have only these 2 equations and with the help of these 2 equations we want to show that y1 and y2 are orthogonal.

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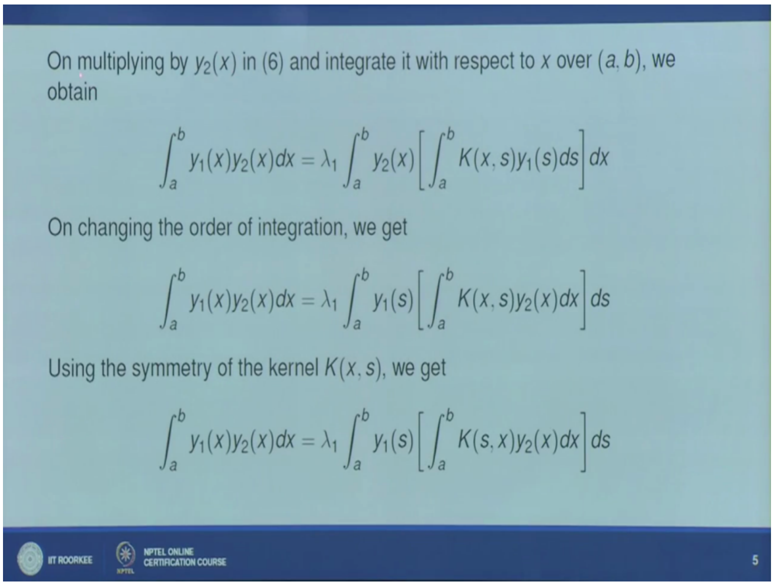

So for that you look at this question number, a question number 6. So what we try to do, we simply multiply by say y2 x and integrate with respect to a to b. So it means that on multiplying by y2 x in equation number 6, that the 1st equation, you can start with the equation number 7, no problem, there is no problem, only thing is that you multiply the other. So if you are taking 6 you multiply by y2, if it is equation number 7 then multiply by y1 and integrate with respect to x from a to b. So if you would do it, we have this, a to b y1 x, y2 x d of x equal to lambda 1, a to b y2 x and this is the right-hand side of equation 6.

Now here is, let me discuss it, so here it is what, here we simply say that it is kind of double integral and one in integral is with respect to s and outer integral is with respect to x here. So here if you look at the limits are finite and it is same as a, b, then we can interchange the order here. So when you interchange the order, then your ds will come out and dx will come in and then y1 s you can take it out and inside you can have a to b K of x, s, y2 xd of x, let me write the same thing here.

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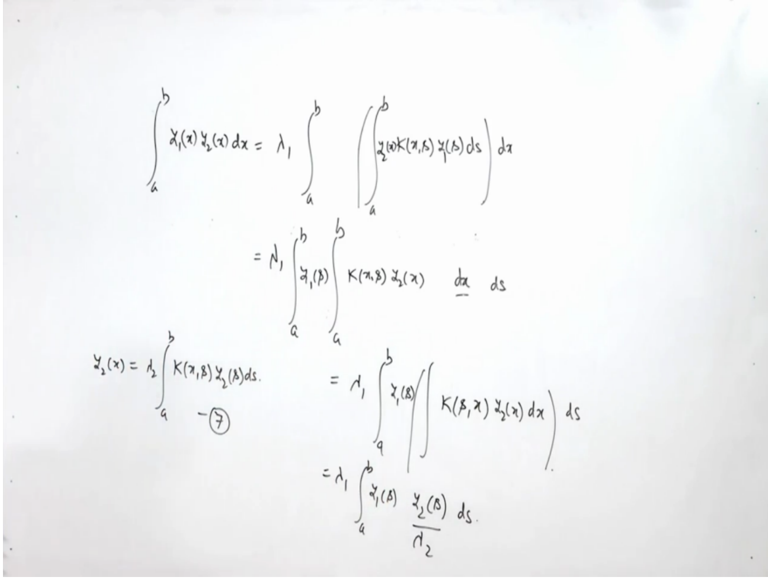

So here we have this thing a to b y1 x, y2 x d of x is equal to lambda 1 is already there, I am just multiplying y2 x and here it is a to b, it is already K of xs, y of s, that is y1 s, d of s and d of x, right. So as we pointed out that we can interchange the order, so we interchange the order, we have a to b and a to b, now I am writing here dx and ds here. So if you look at, the inner integral is with respect to x, then I can take this y1 s out because y1 s is this thing , independent of x.

So I can write this as, 1st of all I am taking this inside, so I can write it here y2 x. So by doing this you can take y1 s out, so y1 s, and here we have x of s, y2 x d of x. So here if you remember, y2 x will satisfy what, y2x is satisfying this property lambda 2 a to b K of xs y2 s d of s, so that we already know, that is equation number 7. Okay. So here is if we use this equation number 7, then this is a small problem here. The problem is this that here your integral is with respect to x and here we have K xs, so there is a problem. So here we can write this as lambda 1 and a to b y 1 s.

Now since we already assumed that we have this as symmetric kernel, then I can write this as Ksx here and y2 x here, dx and d of s. Okay. And once we have this, then I can write my equation number 7, this is simply lambda 1 a to b y1 s and this is nothing but y2 s here, if you… this inner one is s and outer one is x, so by suitably change of say variables here, you can say that this is nothing but your y of s, so y2 of s and this is what, divided by lambda 2. So here lambda 2 is divided, so this is lambda 2 and this is what, d of s.

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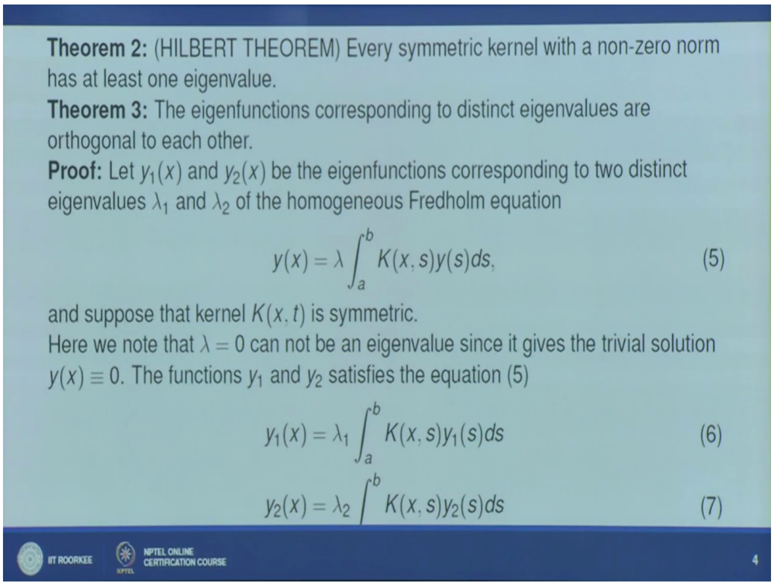

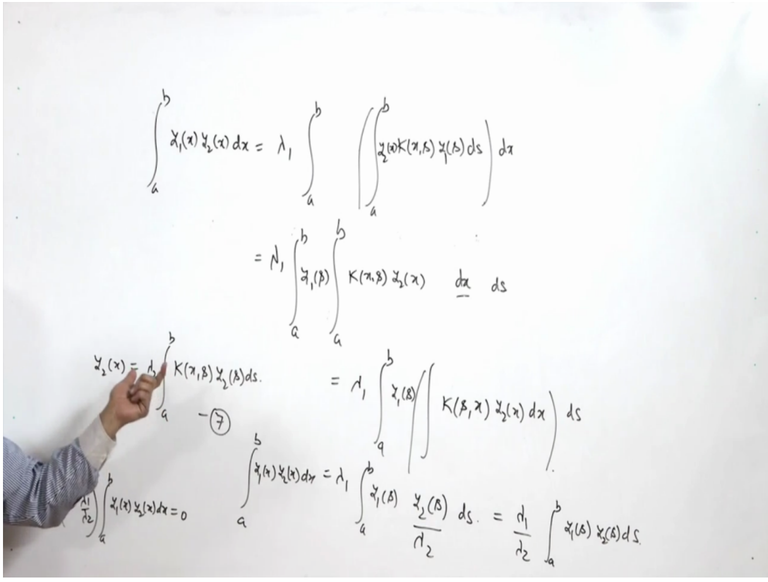

If you look at here, I have assumed only this thing that is K of xs, let a is equal to K x of s, K of sx. So here I am assuming this theorem for real symmetric kernel. If you look at, here we are assuming that this kernel Kxt is real symmetric, that is why I am using here the symmetry of Kx s as Kx as Kxs. Is that okay? So here when we do this, then it is what, it is simply lambda 1 divided by lambda 2 a to b y1 s and y2 s, d of s here. So now you have, which is equal to what, and this is for what, a to b y1 x and y2 x d of x. So here x is kind of dummy variable, so you can write it, and you can take it one side and you can write it here that 1 - lambda 1 divided by lambda 2 and we have a to b y1 x y2 x d of x is equal to 0.

For this step, you please remember that we are assuming that lambda 2 is never 0. And why it is never 0, if you look at here, this lambda 2, if it is 0, then we have only a trivial solution. So this implies that 0 eigenvalue cannot happen. So whenever we have Fredholm integral equation of this kind, then 0 cannot be an eigenvalue. So it means that this division is always possible. So now, here I am assuming, this is nothing but lambda, this lambda 1 is not equal to lambda 2, so this factor is simply nonzero. So it means that we should have a to b y1 x, y2 x dx equal to 0. Is that okay?

So here we can say that this implies that a to b y1 x and y2 x d of x is equal to 0, which says that your y1 and y2 are eigenvalues, Eigen functions which are orthogonal to each other. And this implies that corresponding to distinct eigenvalues we have orthogonal Eigen functions, okay. Now here we may consider that here we have proved only for real case but if it is not real case, then we can also discuss the thing for complex case also but here we are discussing only for real case, is that okay?

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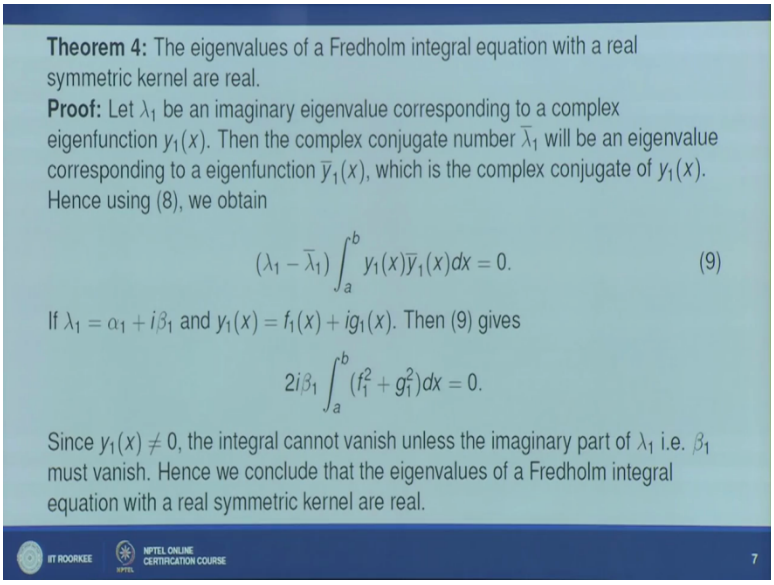

So if you look at the next result, so this is what we have, just to that, since lambda 1 is not equal to lambda 2, we have only this and that implies that Eigen function corresponding to distinct eigenvalues are orthogonal to each other. Now from here only we can say that the eigenvalues of Fredholm integral equation with real symmetric kernels are all real. So it means that whatever eigenvalues we are considering for symmetric kernel, they are all real. So for that let us say, here again I am assuming that lambda 1 is the eigenvalues, you can assume any arbitrary eigenvalue.

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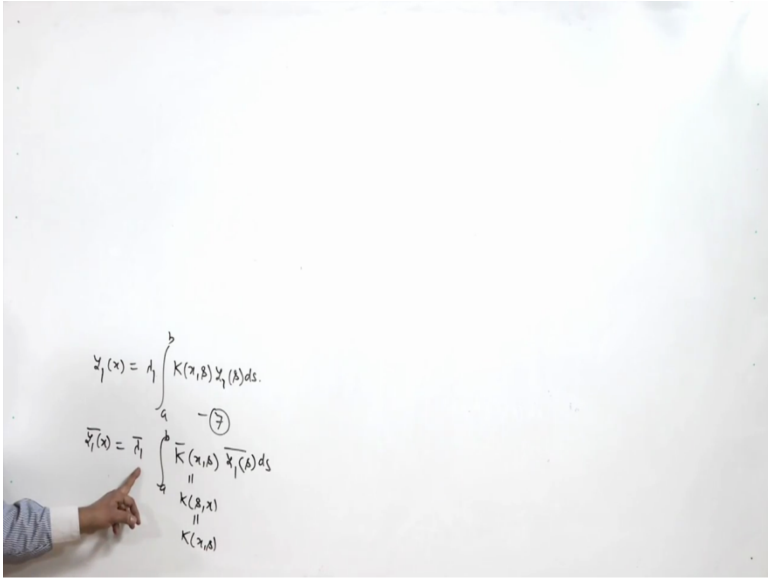

So let us say, I am assuming that lambda 1 be the imaginary eigenvalue corresponding to a complex Eigen function y1 x. So it means that, so here we have y1 x equal to lambda 1 y1 x. And we want to show that this is, when this kernel is real symmetric kernel, then this lambda 1 has to be real. So for that let us take the complex bar, complex conjugate of this. So we simply say y1 bar x equal to lambda 1 bar a to b and K bar x of s and y1 bar s d of s. So here we have assumed that kernel is real symmetric, so in that case it is nothing but K bar, so this is what, K bar, xs is, simply you can write it, it is K of sx and this I can write as, by symmetry city I can write this as K of x of s, is that okay.

So here we can say that this, if lambda 1 is eigenvalue corresponding to y1, then lambda 1 bar is an eigenvalue corresponding to this y1 bar. Is that okay. So that we have pointed out here that if lambda 1 y1 is an Eigen pair, Eigen pair means lambda 1 is the eigenvalue and y1 is the Eigen function.

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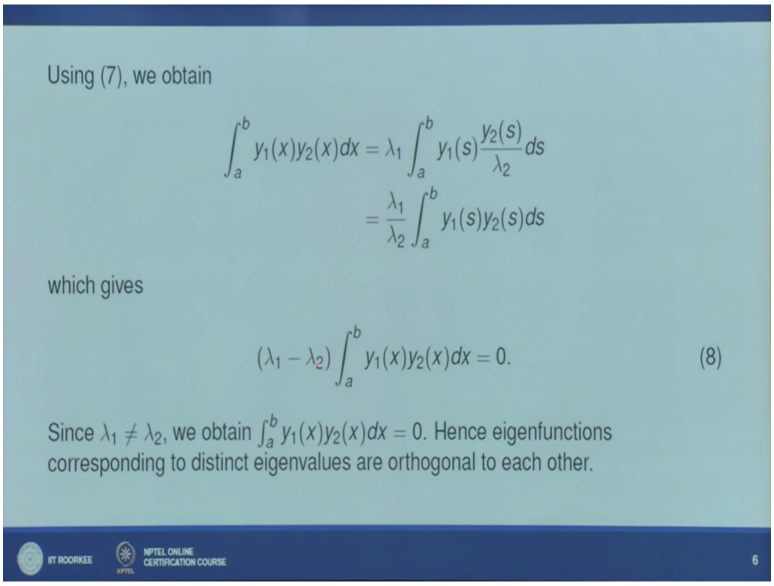

So if lambda 1 y1 is the Eigen pair, so is lambda 1 bar n y1 bar. So using this, if you proceed like if you look at the equation number 8, here I am assuming lambda 2 as lambda 1 bar, so y2 will also be represented by y1 bar. So if you look at here, we have this, lambda 1 - lambda this is I am using equation number 8, this equation. So if you use this equation, then we have lambda 1 - lambda 1 bar, a to b y1 x, y1 bar x dx equal to 0. So here now we assume that suppose lambda 1 is complex. So it means that it has real part and imaginary part, Alpha 1+ I beta 1 and y1 corresponding Eigen function is also say complex Eigen function.

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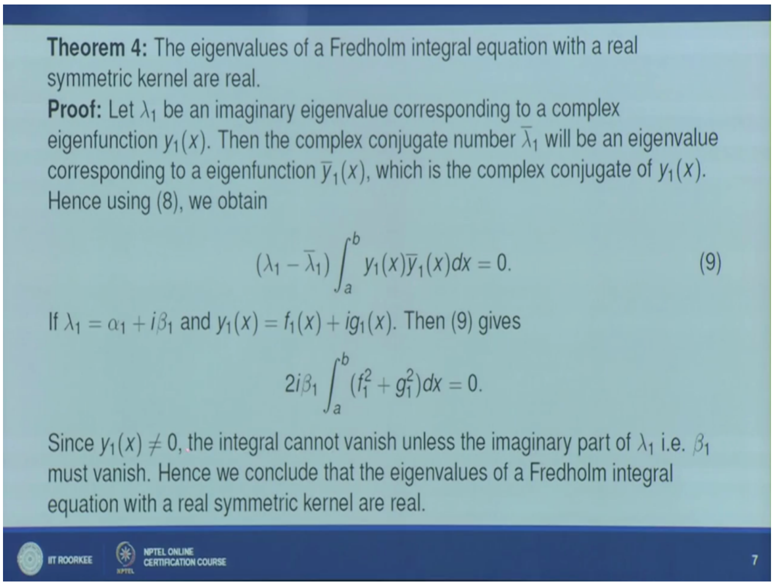

So this also it is written as f1 x + I G1 x. If we use this and put it in equation number 9, then lambda 1 - lambda 1 bar is 2 I beta 1 into a to b f1 square + G1 square d of x equal to 0. Now here, since we know that y1 x cannot be 0, in fact Eigen function means it is nonzero Eigen function. So it means that this is nothing but modulus of y1 x, y1 x square, so this cannot be 0. So the only, this equation implies that beta one has to be 0. So this implies that lambda 1 has no imaginary part. So it means that lambda 1 has to be your real eigenvalue.

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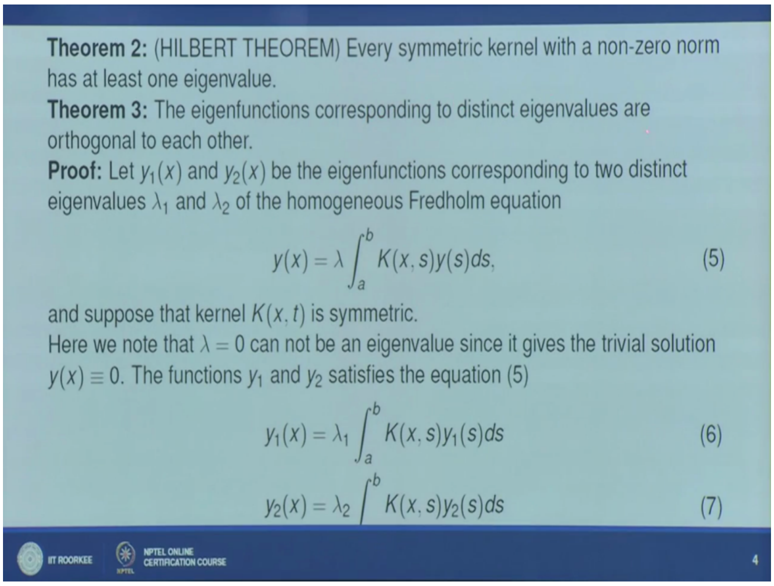

So it means that we can say that the eigenvalues of a Fredholm integral equation with a real symmetric kernel are all real. So if you look at the previous theorem which we have discussed, this implies that if we have Eigen function corresponding to distinct Eigen values are orthogonal to each other. If we do not have symmetric kernels, then this may not be true. Forget about having Eigen function, we may not have even function. So all these results are true only when we have symmetric kernel. So here also we can say that eigenvalues of a Fredholm integral equation with real symmetric kernels are all real.

If we do not have real symmetric matrix, we may have Eigen values which are not real, it may happen that we have a complex eigenvalues. Okay. So now let us next result, which says that the multiplicity of any nonzero Eigen value is finite for every symmetric kernel for which this quantity is finite.

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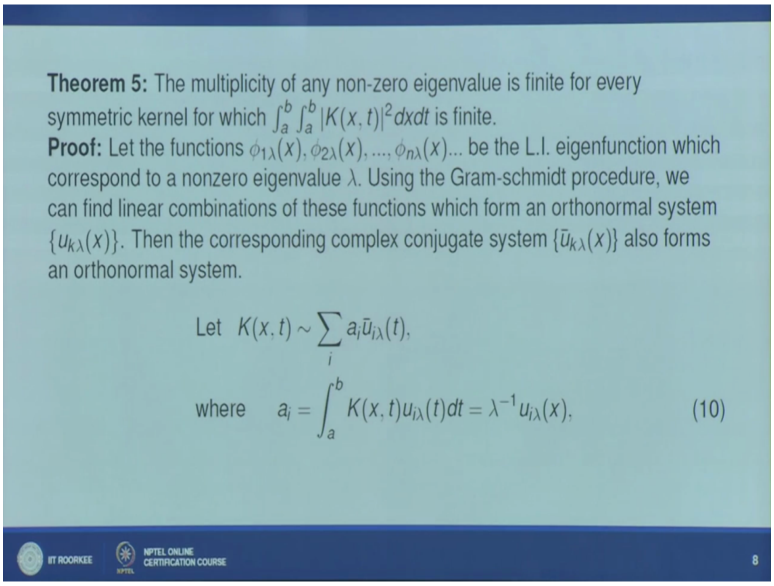

Or, if you look at, what it means, what it means by multiplicity? What we have seen is that we have eigenvalue and we have an Eigen function. Now it may happen that corresponding to one eigenvalue, we may have more than one Eigen functions. So then what we can say that we consider the linearly independent Eigen function corresponding to a particular eigenvalue. So the number of linearly independent Eigen function corresponding to one eigenvalue, we call this as multiplicity of this eigenvalue.

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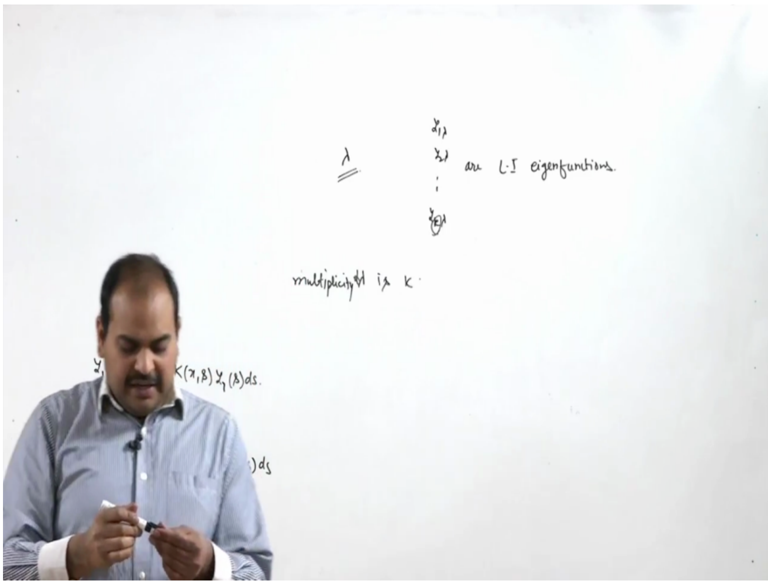

Let me show it here, so suppose we have say lambda and corresponding to this we have say y1 lambda, y2 lambda and so on, say yk lambda, all these are L I Eigen functions. So in this case we simply say that multiplicity of lambda is your the number K. So here we say that multiplicity of lambda is, so multiplicity of lambda is K, is that okay. So that this number of linearly independent Eigen functions corresponding to this lambda. So this theorem 5 says that if we have nonzero eigenvalues, which is always true here, then for every symmetric kernel for which this quantity is finite, if you look at, this quantity is finite for every kernel which is L2 kernel.

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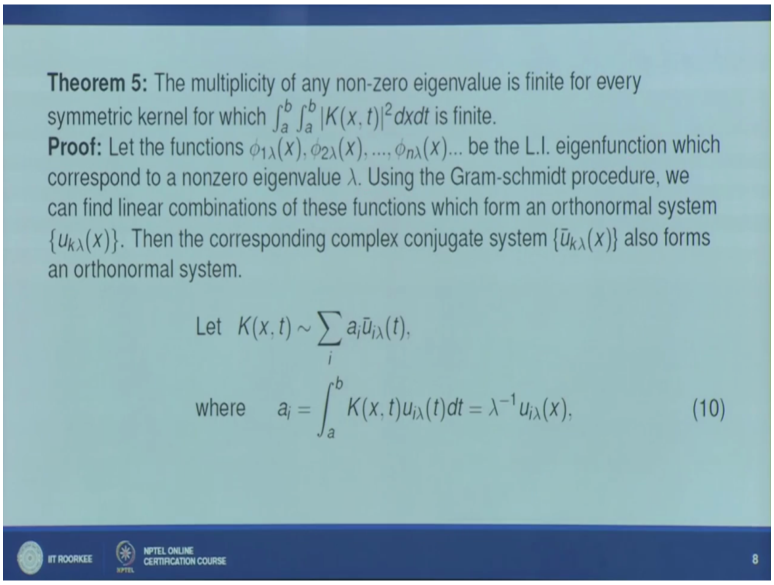

So for L2 kernel, we want to show that the multiplicity has to be finite, it cannot be say infinite. So corresponding to one eigenvalue we cannot have infinite Eigen functions, infinite linearly independent Eigen functions. So for that let us try to prove this. So here let us assume that we have say phi one, Phi 1 lambda x, Phi 2 lambda x and these be the L I Eigen functions with corresponding which corresponds a nonzero eigenvalue lambda. So here it is, we have Eigen functions, not Eigen function. So corresponding to a nonzero eigenvalues lambda. So now I hope you remember the Gram schmidt procedure.

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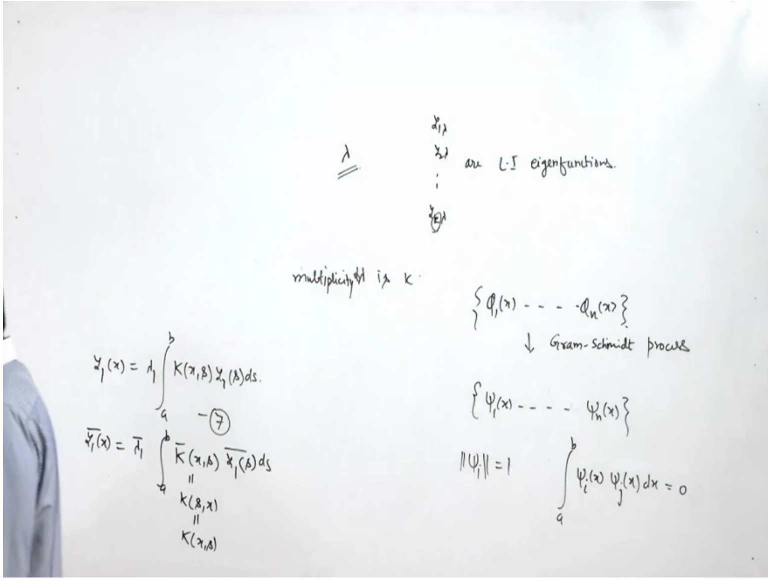

So what is Gram schmidt procedure, Gram schmidt procedure is to, we given any n number of Eigen functions, n number of functions, you can always say generate a new set having equal number of functions but with the new property that they are orthonormal to each other. Orthonormal to each other means, 1st of all I look at this, say phi 1x to Phi nx. So suppose in the beginning we have this set, then you can always construct a new set with equal number of say functions but with the property that norm of psi 1 psi I is basically 1 and that in the integral like, right now I am assuming the interval is between a to b, that psi I x psi Jx dx is simply 0.

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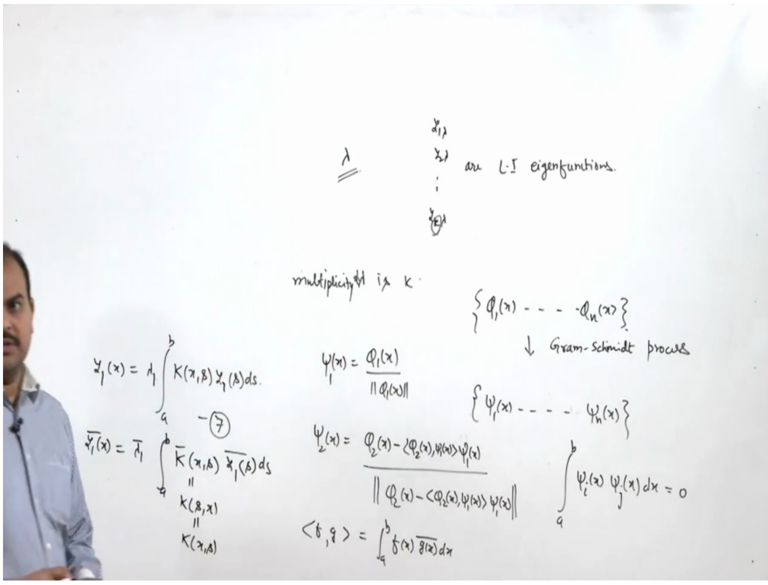

Now if we are assuming real, then this is this, if it is complex then we have considering this. So here we are studying only say real case. So just assume that they there are orthogonal to each other means this, that psi I x, psi Jx dx equal to 0. So this procedure is then known as Gram schmidt procedure. And I think this we have done, if you want we can discuss. I think you can recall it like this that your psi one is nothing but you can take it Phi 1x divided by norm of phi 1x. And psi 2, so psi 1x is this, psi 2 x is basically what, psi 2 x, psi 2 x, you can take it psi 2 x as phi 2 x - your Phi 2x, psi 1x, inner product of this.

And here you take it your, this is psi 1x, okay, divided by the norm of this. Okay, so what will be, whatever, so here we are defining this inner product as say f and G as this, a to b fx G of x d of x. Here I am defining this as a real inner product, otherwise in complex case it is like this. Okay. So this we can, so basically what we are doing, we are taking out the part which is in the direction of psi 1. So this we can do it for any n number of results.

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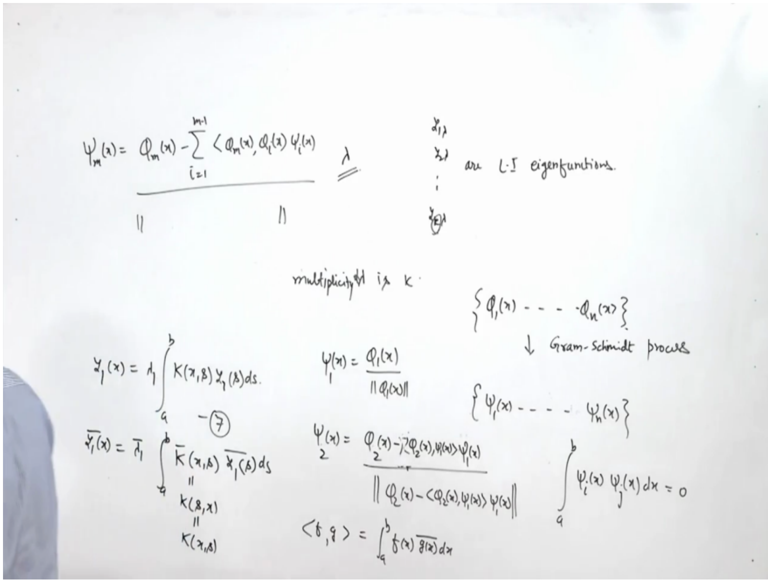

Psi mx is equal to your Phi mx - summation I equal to 1 to say m -1 and we are just taking out say Phi mx and Phi I x and your psi I x and divided by the norm of this, whatever be the norm of this. So you can write it psi mx as new set which are having the property that their norm is 1 and they are orthogonal to each other. So we call it orthonormal set and this procedure is known as Gram schmidt or orthonormalisation process. So here we are assuming that we have function Phi 1 lambda x, Phi 2 lambda x, Phi n lambda x be the L I functions which corresponds the nonzero eigenvalue lambda.

So using this procedure we can find linear combination of these functions which form an orthonormal systems UK lambda. So here we already, considering that we are worrying about only say corresponding orthonormal Eigen functions. So here, let us say that if this is orthonormal, then the corresponding conjugate system UK lambda bar is also an orthonormal sister. So let us consider that approximation of K xt by these orthonormal conjugate systems, complex conjugate systems.

So let us say that K xt is approximated by the summation a I U I lambda t bar where a I is the fourier coefficient, fourier coefficient corresponding to this kernel K xt and it is given by this, AI equal to a to b, K xt U I lambda t dt. Now if you look at, since U I lambda t is the Eigen function corresponding to K xt with the eigenvalue lambda, so this I can write it, what, this I can write it U I lambda divided by lambda. So A I if the things but lambda inverse UI lambda x. So in this case when K xt is approximated by this linear combination where A I is the fourier coefficient given by this, then since U I lambda is Eigen functions, we can write AI as lambda as U I lambda x.

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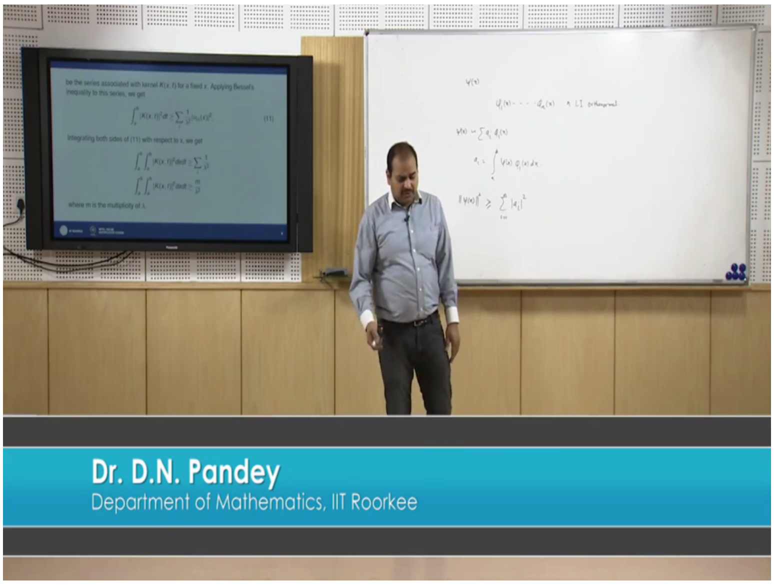

So using this I can write it that, now we can use bessel's inequality and we can write it like this. Now what is bessel's inequality, if you recall, let me write it here. So Bessel's inequality means, you take any function psi of x and let say that Phi 1x to say Phi nx are simply L I orthonormal functions. Right, so you can say that, you can always approximate this psi of x by summation AI your Phi I of x varies AI I am assuming as the limit, I am assuming a to b here psi of x, your Phi I x d of x, so this I am writing here.

Then you can always say that psi of x square is greater than or equal to, this is what summation of I square, I is equal to, in this particular case it is equal to 1 to n, so it is 1 to n. So this is the inequality which is known as Bessel's inequality. This is, this we can get it in any function analysis book or any, for example here we are using the book linear integral equation theory and examples by RP Kawal. So you can get the proof of this, this is known as your Bessel's inequality, so we keep on using this, okay. So now here I am assuming psi of x as K of xt. So here we are writing K xt as this, A I U I bar lambda t, here A I is this fourier coefficient, so we can apply our Bessel's inequality on this equation and we can say that a to b modulus of K xt square dt is greater than this quantity.

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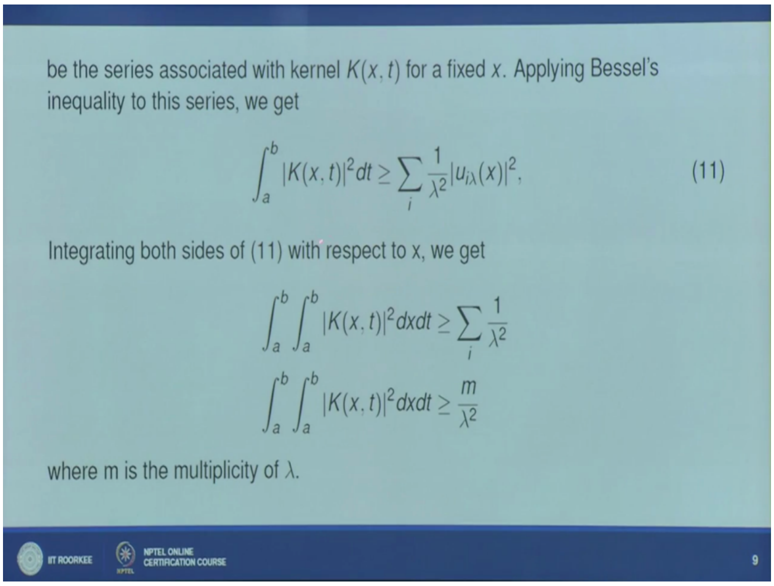

Here you are AI is a sickly, AI is what, AI is lambda inverse U I lambda x. So using this I can write 1 upon lambda A square U I lambda x. So this is by Bessel's inequality. So now what we try to do, we just integrate with respect to x, so we have a to b , this a to b Kxt square ds dt equal to this. Now here if we integrate this a to b modulus of U I lambda x whole square dx, this is nothing but 1 because we have already assumed that it is normalised. Then we have only this. Now if I, I is basically representing what, I is representing the multiplicity of this.

So if multiplicity is say m, then we have this that this quantity is greater than or equal to m by lambda a square. Now we already assumed that K is L 2 kernel or for which this is finite, then this m cannot be infinite. So it means that the multiplicity m cannot be infinite. Okay. So that is the result with me want to prove, that is, that the multiplicity of any nonzero eigenvalue is finite for every symmetric kernel for which a to b, for which this quantity is finite or you can say for which this kernel is L2 kernel. So in next lecture we will discuss some more properties of this and then some more properties of eigenvalue and Eigen function and then we will try to have a solution method to solve Fredholm integral equation of symmetric kernel, we want to see that. Thank you for listening to us, thank you.