Mathematical methods and its applications
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Lecture - 08
Methods for finding Particular Integral for second-order linear differential equations with constant coefficients III

Hello friends. Welcome to my third lecture on this topic Method for Finding Particular Integral for Second-order Linear Differential Equation with Constant Coefficients.

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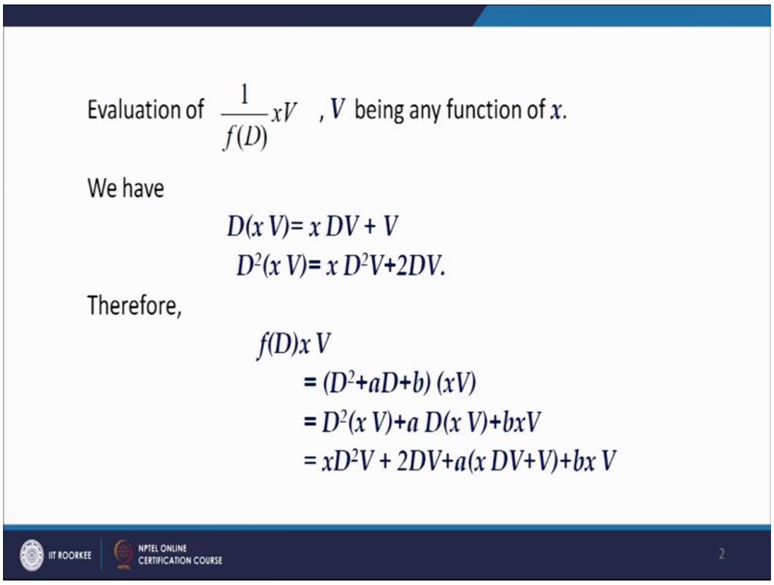

We are trying to find the particular integral for second-order non-homogeneous linear differential equations where r x is of some special forms. So, if r x is of the form x into b where V is any function of x how we shall evaluate the particular integral.

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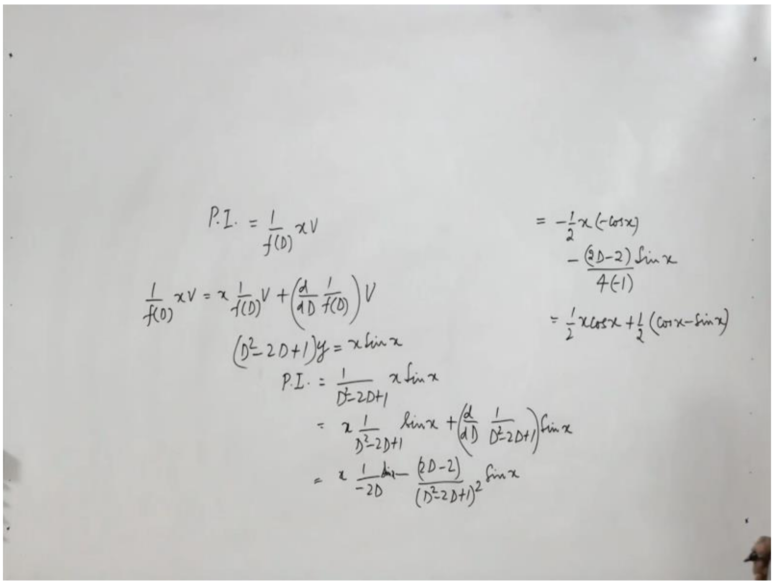

So, particular integral in the case when rx is x into b. So, when you differentiate x into b, what we get is x into dv plus b because b is any function of x then D square when you operate on x into V, you get x D square b plus 2 D b.

So, since D square is since f D is D square plus alpha D plus beta f D here I have taken D square plus a D plus b. So, f D x b f D operating x into b will give you D square plus a D plus b operating on x b. And since, D is a linear operator, so D square operating on x b plus a times D operating on x b plus b x V; now D square xb the value of D square x we can put from here. So, x D square V plus 2 D V and then derivative of x into b can be put it from here; so we have a times x D V plus V plus b x V.

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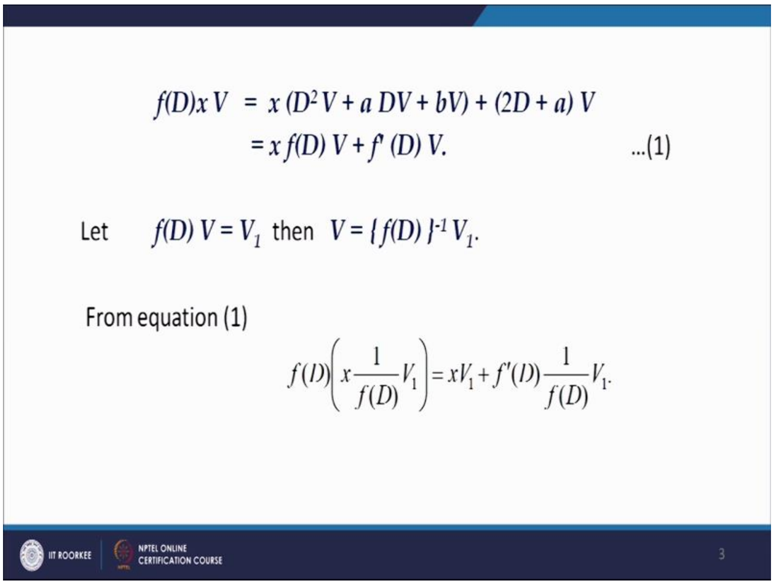

And which can then be a written as x D square V plus a D V plus b V plus 2 D a do 2 D plus a operating on b. Now D square V plus a D V plus b V is nothing the operator f D operating on V. So, we have x into the f D operating on D and then 2 D plus a can be interpreted as the derivative of f D. In short 2 D plus a can be interpreted as f we have differentiated f D with respect to D.

Now let us suppose that when f D operates on the function V, we get another function say V 1. So, that when you apply the inverse operator 1 over f D on this equation f D V equal to V 1, we get V equal to 1 over f D operating on V 1. Now let us put the value of b as 1 over f D operating on V 1 in this equation. So, we get f D operating on x into 1 over f D V 1 equal to equal to x V 1 because f D V is V 1. So, x into V 1 and then f dash V, f dash D operating on V is equal to 1 over f D V 1. Now we will operate on both sides by 1 over f D. So, we operate on this equation by 1 over f D since f D and 1 over f D are inverse to each other. So, we shall get on the left hand side only x 1 over f D V 1.

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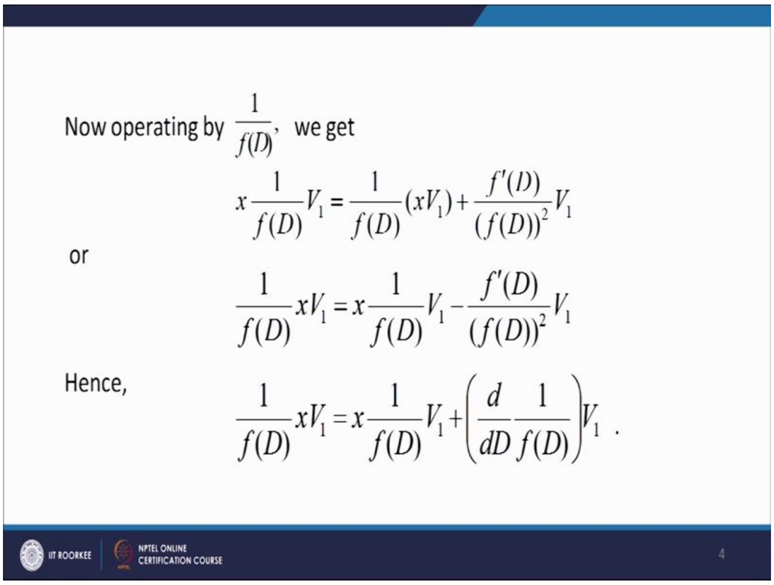

So, we get x 1 over f D V 1 and then the right hand side we get as 1 over f D operating on x V 1 then 1 over f D operating on f dash V and then 1 over f D on V 1 which can be interpreted as f dash d over f D whole square operating on V 1. So, we get this 1 over f D operating on x V 1 then f dash d over f D whole square operating on V 1. Now let us write the value of 1 over f D x V 1 from here. So, 1 over f D x V 1 is x times 1 over f D operating on V 1 minus f dash d over f D whole square operating on V 1. And hence 1 over f D operating on x V 1 is nothing but x 1 over f D operating on V 1, and then this can be interpreted as derivative of 1 over f D with respect to D, like we differentiate 1 over x with respect to x. So, minus f dash d over f D whole square can be interpreted as the derivative of 1 over f D. So, that is operated on V 1.

Now, let us see how we will apply this formula now here. One more thing this formula can since V 1 is an arbitrary function, V for then arbitrary function. So, V 1 is also arbitrary. So, we can write this formula for V. So, 1 over f D operating on V. So, we get the formula following formula V 1 can be replaced by V. So, 1 over f D operating on x into V gives you x 1 over f D operating on V, and then d over d D operating on 1 over f D operating on V. This is the formula now let us see how we shall apply this formula. So, let us look at the equation D square minus 2 D plus 1 y equal to x sin x.

So here let us find particular integral; so 1 over D square minus 2 D plus 1 operating on x sin x. So, this is second-order non-homogeneous linear differential equation where r x is x sin x. So, it is of the form x into V where V is sin x. Now applying this formula 1 over f D x V equal to x times 1 over f D V plus d over d D 1 over f D V, we can write this as x times 1 over D square minus 2 D plus 1 sin x sin x then d over d D of 1 over D square minus 2 D plus 1 sin x. This is equal to D square can be replaced by minus a square, so x times 1 over minus 1 plus 1. So, we that minus 2 D and then we have minus 2 D minus 2 upon D square minus 2 D plus 1 whole square operating on sin x, the derivative of 1 over D square minus 2 D plus 1 with respect to D we have written here.

Now, this is equal to 1 over 2 D operating on sin x. So, we get minus 1 over 2 x then 1 over D sin x this means integral of sin x which is minus cos x. And here we replace D square by minus 1 square that is minus 1. So, minus 1 plus 1 whole square minus 1 plus 1 means 0. So, we get minus 2 D whole square which is 4 D square; so we get minus 2 D minus 2 upon 4 D square D square can be replaced by minus 1 square. So, we get minus 1 sin x and this will then be equal to. So, we can write x 1 by 2 x cos x and minus becomes plus 2 by, so, 1 by 2 and then D minus 1 will operate on sin x will give you cos x minus sin x. So, this is particular integral in this case.

Now, this formula is generally avoided, because of the complicity of this formula. We use alternate method to determine the particular integral in case of the functions where we have x into V.

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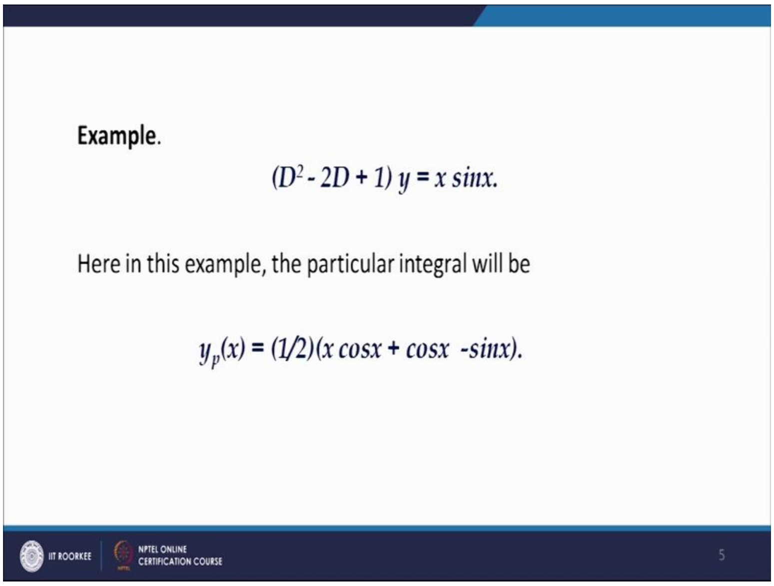

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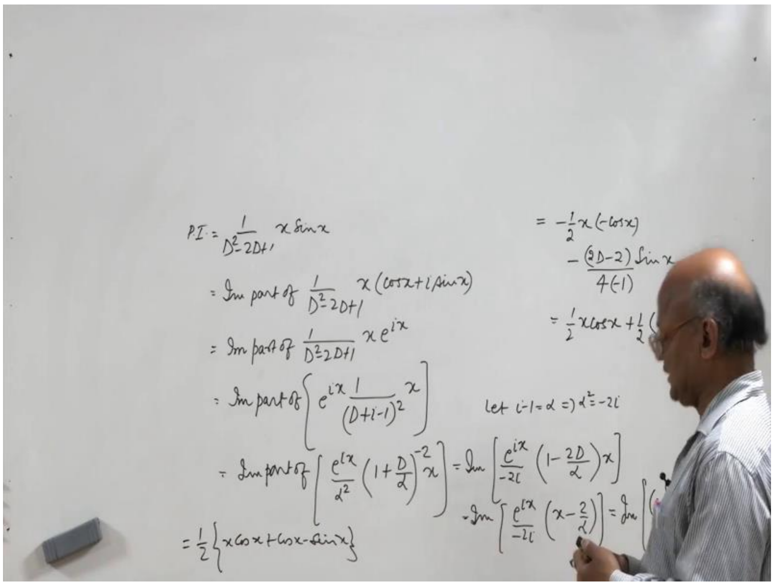

So, let us see alternate method here. So, let us find particular integral by an alternate method. What we do is we can write it as imaginary part of 1 over D square minus 2 D plus 1 operating on x times cos x plus i sin x, x sin x is imaginary part of x times cos x plus i sin x.

Now, this is equal to imaginary part of 1 over D square minus 2 D plus 1 operating on e to the power i x because e to the power i x is cos x plus i sin x. Now let us apply the formula 1 over f D operating on e to the power i x into V. So, that formula would be apply we get this is D minus 1 whole square D is replaced by D plus i. So, we get D plus i minus 1 whole square operating on x let us assume that let us say i minus 1 is equal to some alpha. So, alpha if we assume then this is e to the power i x by alpha square 1 plus D by alpha is to the power minus 1.

Since the power of x is 1. Here when we extend 1 plus D by alpha to the power minus 2 we just retain the power of D as 1. Because the D square when operates on x it will give us 0. So, this is imaginary part of e to the power i x alpha square will be equal to i minus 1 whole square which is i square plus 1 minus 2 i. So, this is minus 2 i.

So, we get e to the power minus i x over minus 2 i and then 1 plus D by alpha minus 2 will give you 1 minus 2 D by alpha. We are not writing higher powers of D operating on x. So, this will be equal to imaginary part of, So, x minus 2 upon alpha now this is equal to imaginary part of cos x plus i sin x divided by minus 2 y into x minus 2 upon i minus

1. So, when you simplify this expression write it as a complex number alpha plus say a plus i V, and then take the imaginary part you will get half of x cos x plus half of cos x minus sin x. So, we get half of x cos x plus cos x minus sin x. So, this is an alternate method of obtaining the particular integral for functions of the type x into V.

Now let us discuss an application. We had earlier discussed an application in mechanics where we had homogeneous linear differential equation of second-order. Now, we are doing considering an application where we will have a non-homogeneous linear differential equation of second-order.

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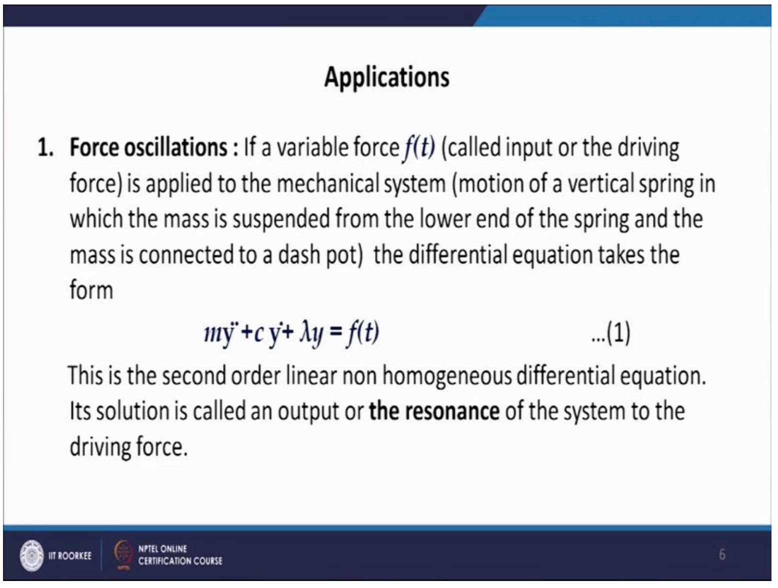

So, let us discuss force oscillations. In the force oscillations we apply a variable force which we call which we can denote by f t, t is the time variable it is called input or the driving force. It is applied to the mechanical system, the mechanical system which we had considered earlier in the case of homogeneous linear differential equation of second-order.

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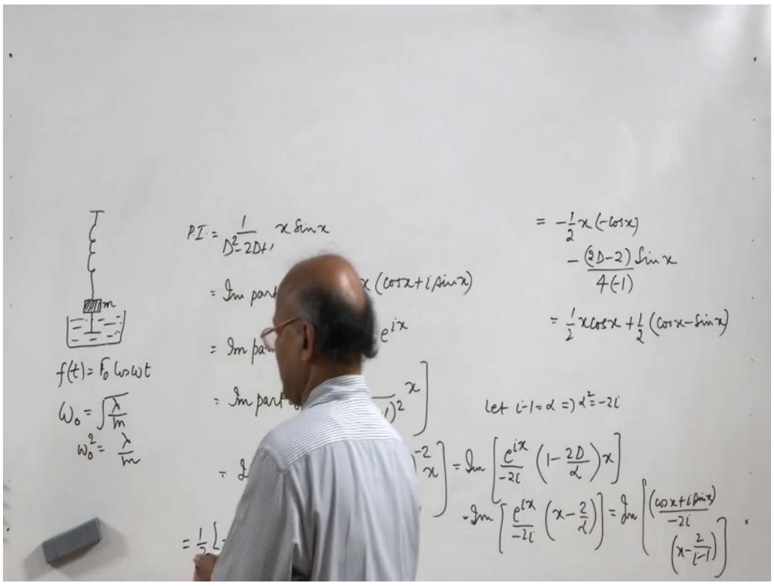

We had considered a particular spring, and with the fix support and then mass m is suspended from the lower end of the spring. Spring and then it is attached to a dead spot, it is attached to a dead spot filled with a viscous liquid.

So, the differential equation now when we apply the variable force f t, will be of the form m y double dot plus c y dot plus lambda y equal to 0 in this absence of this variable force f t we had obtained the second-order differential equation homogeneous m y double dot plus c y dot c plus lambda y equal to 0. So, now, this is second-order linear non-homogeneous differential equation. It is solution will be called an output are the resonance of the system to the driving force.

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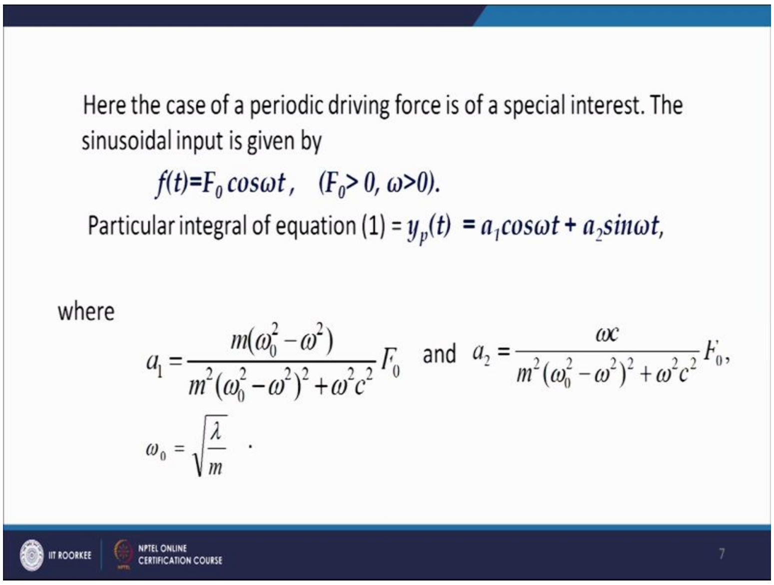

Now, in the case of this non-homogeneous differential equation, we generally consider periodic driving force, but it has a special interest, we have interest specially interest in So, let us considered f t to be equal to F naught cos omega t where F naught is positive omega is positive, then the particular integral. So, when you take f t equal to f t equal to F naught cos omega t, then particular integral will come out to be a 1 cos omega t.

We can check it very easily the particular integral will come out to be a 1 cos omega t plus a 2 sin omega t where a 1 is equal to this and a t equal to this. And omega naught is the root lambda y m the natural frequency. Now provided the denominator is non 0, the denominator here this denominator is this denominator is non 0, we get this y p t equal to a 1 cos omega t plus a 2 sin omega t.

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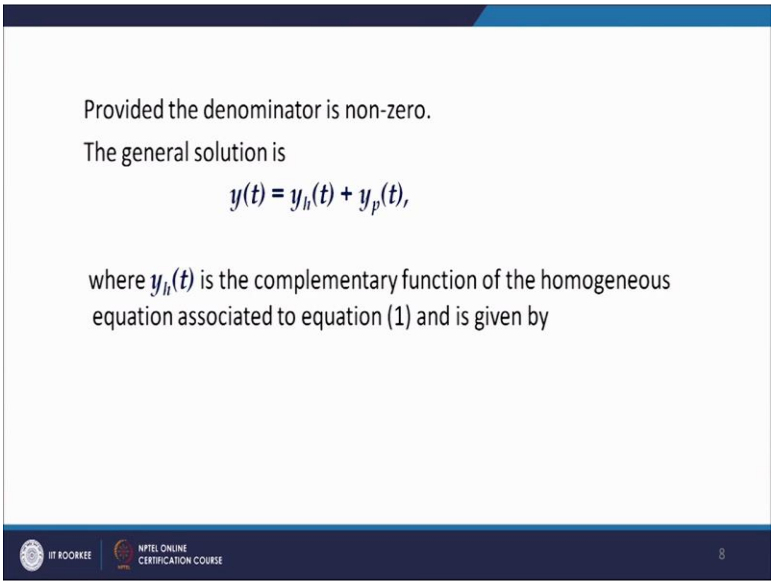

Now, the general solution as we know the general solution will when be the sum of the complementary function, which is the general solution of the homogeneous linear differential equation and the particular integral y p t. So, this is the complementary function of the homogeneous equation associated to equation 1.

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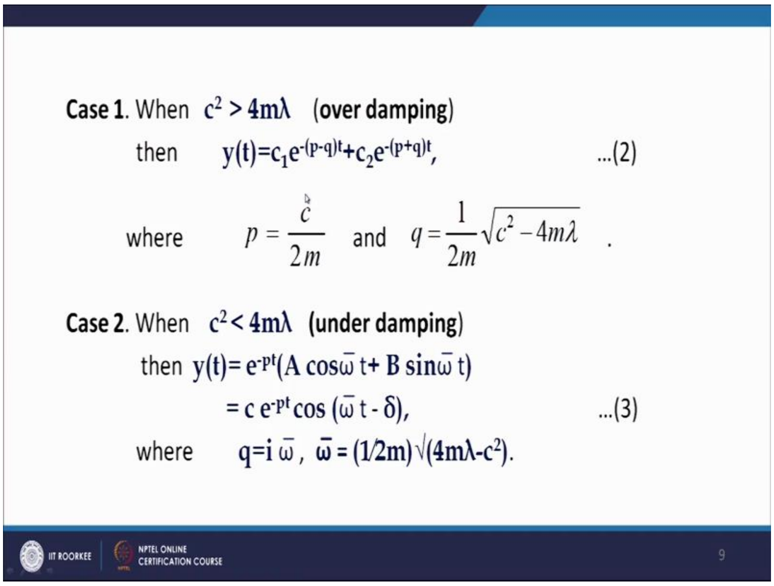

Now, let then in the case of free oscillations, we had 3 cases when c square is greater than 4 m lambda. We had over damping y t came out to be this where p we had taken as c by 2 m q equal to this and then we had the second case c square less than 4 m lambda which was the case of under damping because c is the damping constant. So, we had the solution y t equal to c times e to the power minus p t cos omega over t minus lambda. And the q was equal to I omega bar omega bar is equal to this.

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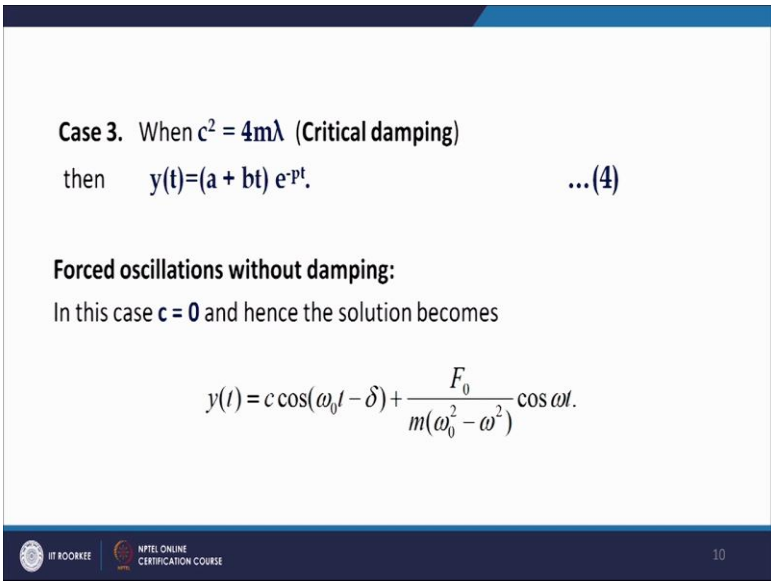

Now, in the third case the critical damping when c square becomes 4 m lambda, we had y t equal to a plus b t e to the power minus p t. So, these are the solutions of the associated homogeneous equation. Now forced oscillations without damping first we have considered the case of forced oscillation without damping; that means, the damping constants c is equal to 0, and then the solution will take the form y t equal to c cos omega t minus 1, we can see it from here y t equal to y h t plus y p t, y h t plus y p t we are found y p t as a 1 cos omega t plus a 2 sin omega t. So, in that you put c equal to 0 you get the value of y p t.

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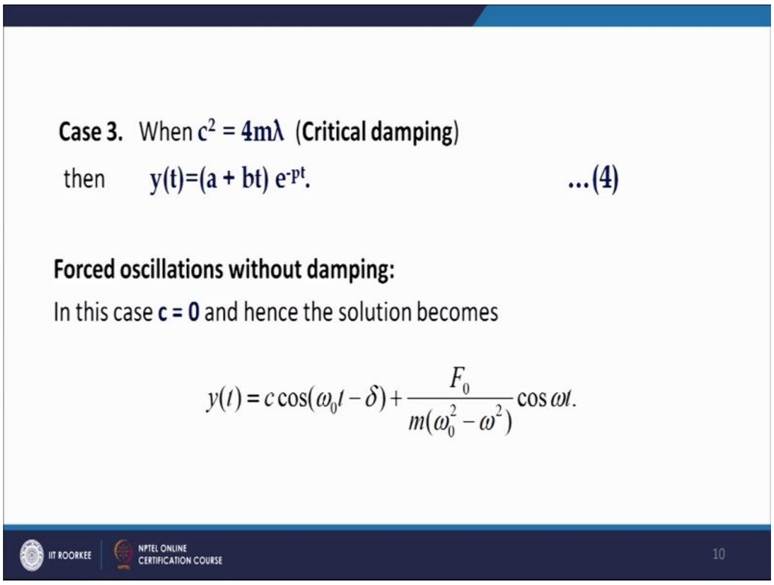

And y p t comes out to be F naught over m omega naught square minus omega x square cos omega t. And when c is 0 c square if you look at this c square is 0 then c square will be 0. So, 0 will less than 4 m lambda and therefore, y h t will be equal to c e to the power minus p t cos omega t bar t minus delta. So, we will get this as c times because p is equal to 0 here. This p is equal to 0 because c 0. So, we get c times cos omega bar t minus delta.

So, we get y t equal to c cos omega naught t minus delta and then F naught over m omega naught square minus omega square cos omega t in the case c equal to 0. Now here this is a super position of 2 harmonic oscillations. This is harmonic oscillation and this is also harmonic oscillation. So, we get a supper position of 2 harmonic oscillations with frequencies omega naught over 2 pi here the frequency is omega naught over 2 pi, here the frequency is omega over 2 pi which comes from the driving force.

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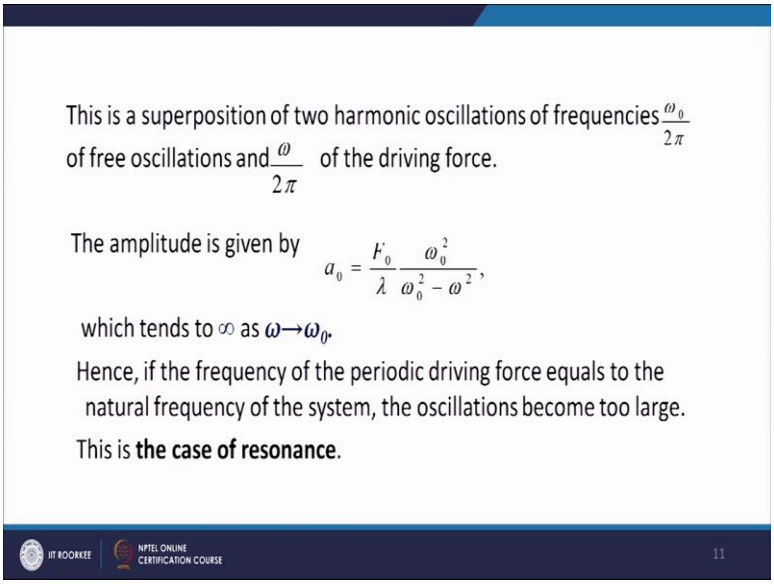

The amplitude is given by a naught; the amplitude is given by this one. This amplitude F naught over m times omega naught square minus omega square. Since lambda y m equal to omega naught square, we have taken omega naught equal to square root lambda y m or omega naught square equal to lambda y m. So, from here if you substitute the value of m in the expression this one in the expression F naught over m times omega naught square minus omega square when you substitute the value of m, from the relation this one, omega naught square equal to lambda y m we get F naught y lambda omega naught square y omega naught.

Now this we can see from here as omega tends to omega naught, this a naught tends to infinity. So, amplitude becomes very large. Hence if the frequency of the periodic driving force omega equals the natural frequency omega naught, the oscillation becomes too large. So, this case is the case of resonance. The differential equation in the case of resonance then become if you look at the differential equation, these are my differential equation. We are taking c equal to 0.

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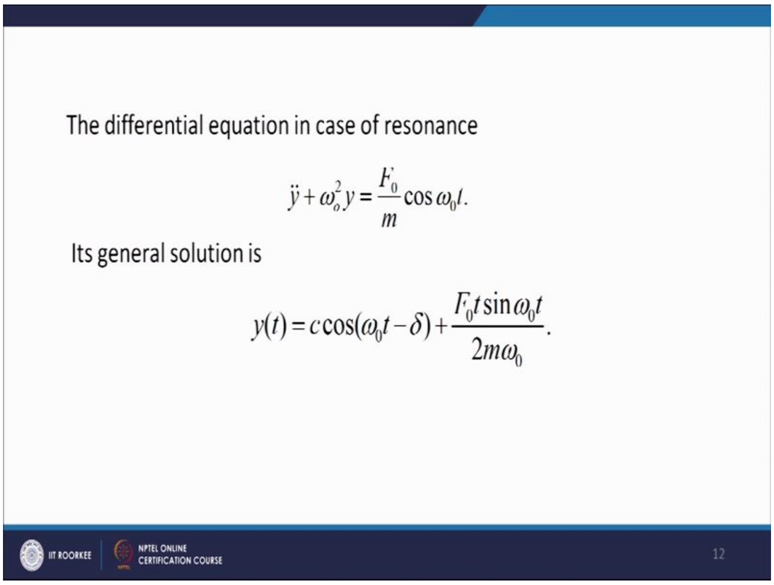

So, m y double dot plus lambda y equal to f t. This equation will then turn into this equation, y using the relation omega naught square equal to lambda y m. We can write it as y double dot omega naught square y F naught m over cos omega naught t. And it is general solution is then y t equal to c cos omega naught t minus delta F naught t sin omega naught t over 2 m omega naught.

We know how to determine the solution of the differential equation particular integral here will be, 1 over D square plus omega naught square operating on cos omega naught t and we know the formula for that. So, we can make use of that formula 1 over D square plus c square operating on cos c x, we have cos omega naught cos c x that we have found as x sin c x divided by 2 c. So, make use of this formula to get the particular integral this one, F naught t sin omega naught t omega naught.

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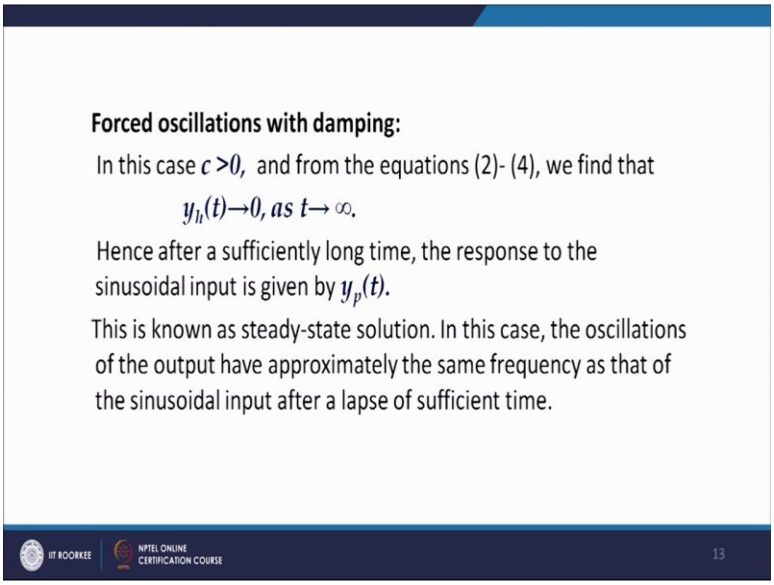

Now, force oscillation with damping. So, here in this case c will be positive and from the equations 2 to 4 let us look at the equations 2 to 4 here.

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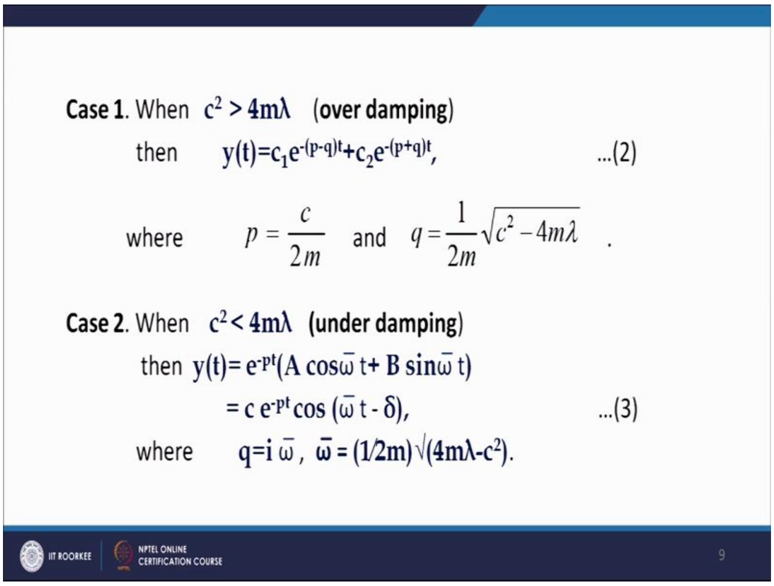

So, in this case when damping is present, then we have there arise 3 cases corresponding to the discriminant c square greater than 4 m lambda c square less than 4 m lambda then c square is equal to 4 m lambda. In all the 3 cases you can see that y t tends to 0, because when q is less than p, q is 1 by 2 m under root c square minus 4 m lambda. So, q is less than p. So, this p minus q is positive. So, when t tends to infinity this goes to 0 this also goes to 0 and here, when t goes infinity e power minus p t goes to 0 and cos omega bar t minus delta is bounded. So, this is tending to 0.

And here also when t tends to infinity because of the exponential function y t goes to 0. So, now, the y t there is nothing but the complementary function because y t is the general solution of the associated homogeneous linear differential equation. So, y t is y h t. So, y h t goes to 0 h t goes to infinity. Hence after a sufficient sufficiently long time the response to the sinusoidal input is given by y p t, because y h t term tends to 0. So, y t is nothing but y p t this is known as the study state solution in this case the oscillations of the output have approximately the same frequency as that of the sinusoidal input after a lapse of sufficient time.

So, with this I will come close this lecture.

Thanks.