Mathematical methods and its applications

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Lecture – 07
Methods for finding particular integral for second-order linear differential equations with constant coefficients II

Hello friends. Welcome to my second lecture on Methods for Finding Particular Integral for Second-order Linear Differential Equations with Constant Coefficients.

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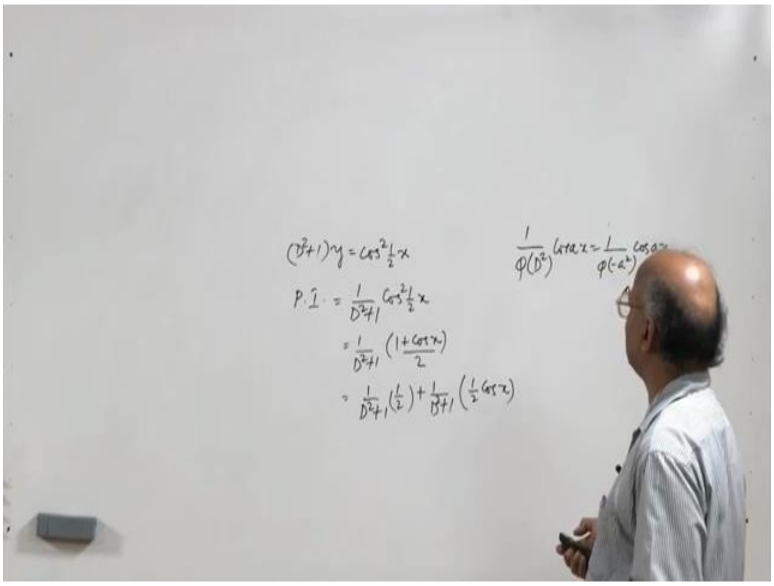

In my last lecture, we were trying to find the particular integral in the case of the differential equation D square plus 1 y equal to cos square 1 by 2 x and we could not find it because we had to write cos square half x in the form of 1 plus cos x by 2 and when we write it in this form, what happens is that we have to find the particular integral for 1 by D square plus 1 cos x, but 1 by D square plus 1 cos x cannot be found from the formula 1 by D square plus a square cos a x equal to 1 over sorry; 1 over f D square equal to write 1 over phi D square equal to cos a x equal to 1 over phi minus a square cos a x, we cannot find the of particular integral in this case because when we replace D square by minus a square that is D square by minus 1 square r minus 1 this D square plus 1 on replacing D square minus a square becomes 0.

So, phi minus a square should not be 0 to apply this formula and therefore, let us now discuss how to find the particular integral in the case of 1 over D square plus c square cos c x.

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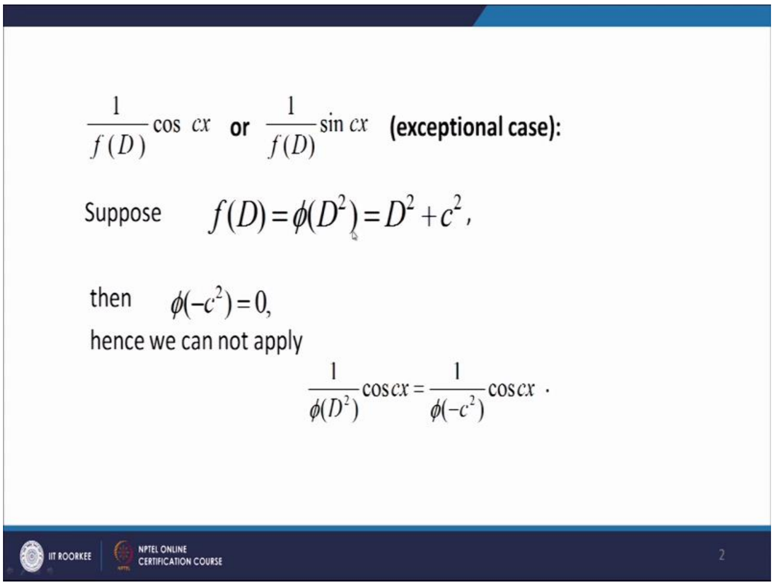

So, let us discuss how to find the particular integral in the case of 1 over f D cos c x or 1 over f D sin c x which are where in the exceptional case we have f D equal to D square plus c square and we see that phi D square when c square D square is replaced by minus c square becomes 0. So, we have not able to apply the formula 1 over phi D square cos c x equal to 1 over phi minus c square cos c x, we will prove that 1 over phi D square cos c x which is 1 over D square plus c square cos c x gives us x sin c x over 2 c.

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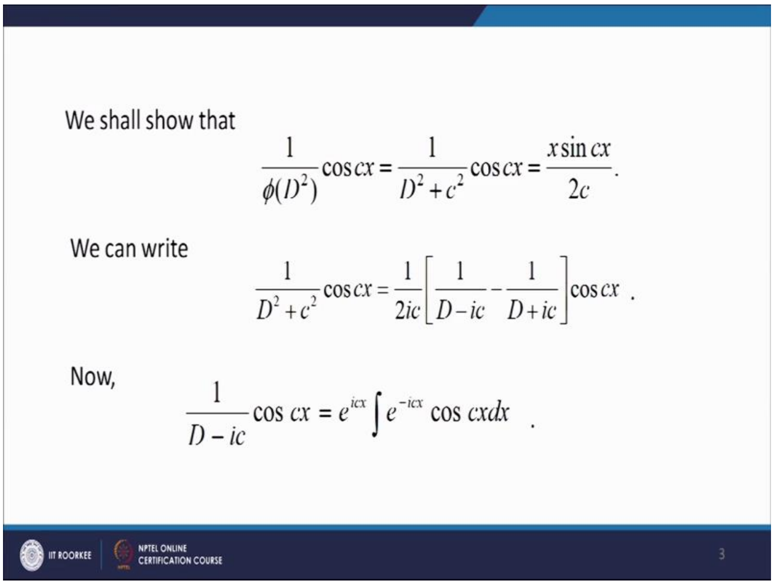

So, let us see how we do this, we can write 1 over D square plus c square as 1 over 2 i c, 1 over D minus i c minus 1 over D plus i c operating on cos c x that is we break 1 over D square plus c square in partial fractions.

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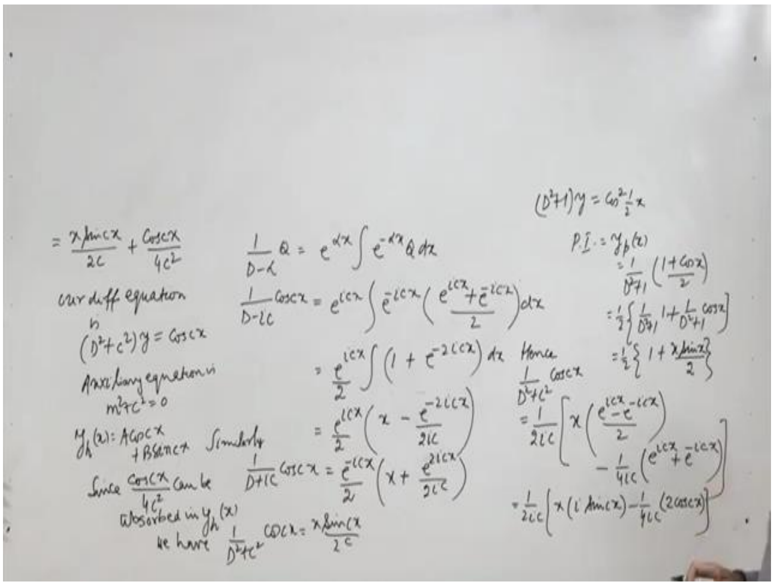

Now let us recall that we had proved that 1 over D minus alpha when operates on q or you can say r x then we had e to the power minus alpha x integral e to the power e to the power alpha x e to the power minus alpha x into q d x.

So, let us apply this formula here and to by using this formula 1 over D minus i c cos c x will be e to the power i c x integral e to the power minus i c x cos c x d x now then what we will have is e to the power minus i c x cos c x we can write as e to the power i c x plus e to the power minus i c x divided by 2 by the Euler's formula we know that for i theta is cos theta plus i sin theta and e to the power minus i theta is cos theta minus i sin theta, so from there cos theta is e to the power i theta plus e to the power minus i theta by 2. Now, this is equal to e to the power i c x integral, this will be 1 plus e to the power minus 2 i c x d x let us integrate it and we get then integral of 1 is x and then e to the power minus 2 i c x divided by 2 i c.

Similarly, we can write the value of 1 over D plus i c cos c x. This is equal to we get this; now let us subtract from the value of 1 over D minus i c cos c x the value of 1 over D plus i c cos c x and then divide by 2 i c hence 1 over D square plus c square cos c x is equal to 1 over 2 i c from 1 over D minus i c cos c x. We subtract the value of 1 over D plus i c cos c x and so this will give you x times cos e to the power i c x minus e to the power minus i c x divided by 2 and when we then we will have minus sin here 2 becomes 4 i c and we get e to the power minus i c x here we get e to the power plus i c x. So, e to the power i c x plus e to the power minus i c x and this is equal to 1 over 2 i c x times. Now e to the power i theta minus e to the power minus i theta by 2 is i sin theta.

So, we get i times sin c x and then here we have minus 1 over 4 i c e to the power i theta plus e to the power minus i theta is 2 cos theta. So, 2 cos c x and thus we have x sin c x by 2 c and we have i this 2 will cancel with that 2 here. So, here cos c x divided by 4 c square, now our differential equation is of the type our differential equation is D square plus a square y equal to cos c x D square plus c square y equal to cos c x.

So, auxiliary equation will be m square plus c square equal to 0. So, complementary function y h x will be equal to a cos c x plus b sin c x and then when we will operate now we will operate 1 over D c square on cos c x to find the particular integral we get x sin c x over 2 c plus cos c x over 4 c square. So, this term cos c x over 4 c square can be observed in the term a cos c x occurring in the complementary function and therefore, when we write the result for 1 over D square plus c square cos c x we just retain x sin c x by 2 c.

So, we say that since of cos c x over 4 c square can be observed in y h x we have 1 over D square plus c square cos c x equal to x sin c x by 2 c and similarly we can prove similarly we can show 1 over D square plus c square sin c x equal to minus x by 2 a cos c x.

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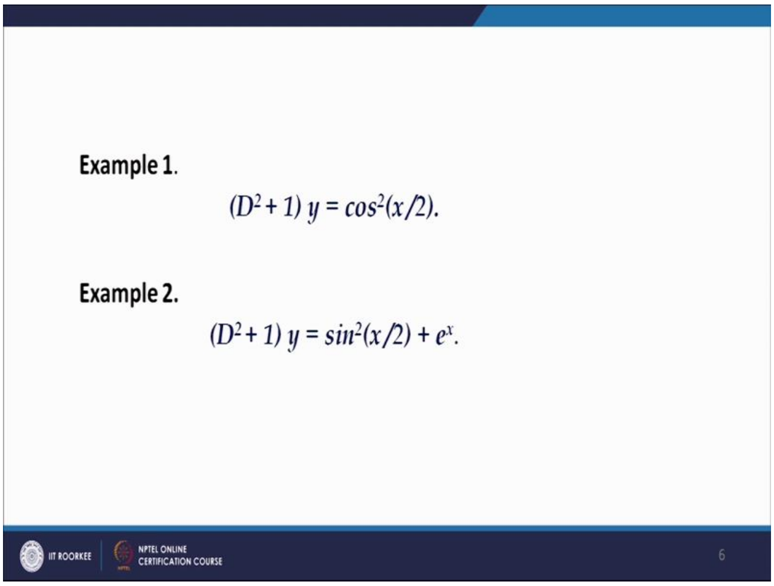

Now, with these formulas we can then find the particular integral easily in the case where phi minus a square becomes 0 for example, one o d let us look at the equation D square plus 1 y equal to cos square 1 by 2 x. So, particular integral y p x is equal to 1 over D square plus 1 operating on cos square x 1 by 2 which we can write as one plus cos x by 2 which is equal to 1 by 2 1 over D square plus 1 operating on one and then we have 1 over D square plus 1 operating on cos x 1 can be regarded as e to the power 0 x. So, this is 1 over 2 d over D is replaced by a 0. So, we have 1 and then 1 over D square plus c square cos c x gives x sin c x over 2 c. So, c is equal to 1 here. So, we have x sin x divided by 2 by using the formula which we have just proved and similarly we can solve example 2.

In the case of example 2, when we find the particular integrals sin square x by 2 is written as 1 minus cos x. So, over 2 and then we apply the formula same formula 1 over D square plus c square cos c x equal to x sin x by 2 c to obtain the particular integral.

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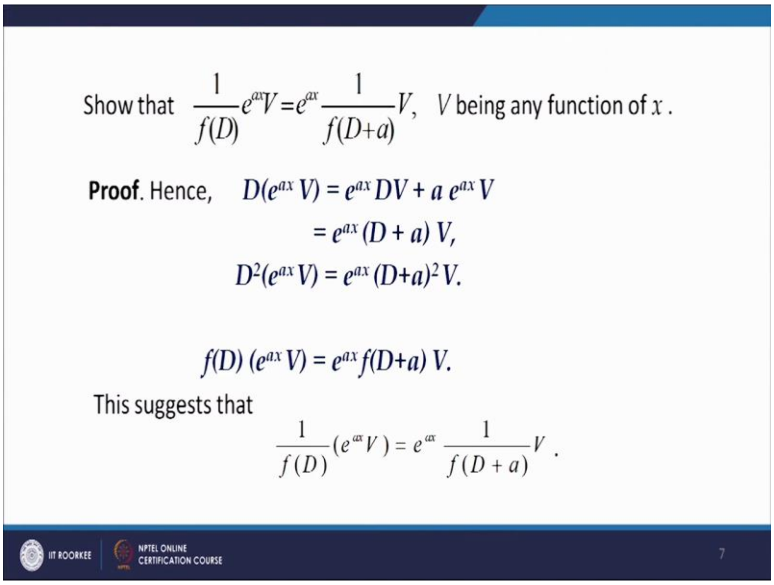

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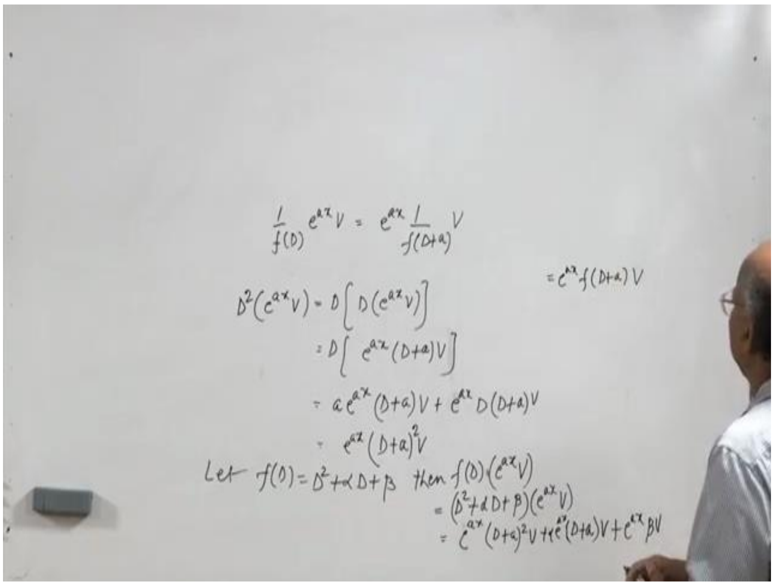

Now, let us look at another formula which is given by 1 over f D e to the power a x into V where V real function of x and a is a is some constant. So, this when 1 over f D operates on e to the power a x into V what we get is e to the power a x into one upon f D plus a operating on V. So, this formula is very useful in the case, earlier we have considered 1 over f D operating on e to the power alpha x this formula gives us 1 over f alpha e to the power alpha x provided f alpha is non 0, but if alpha if f alpha becomes 0 then we cannot apply this formula. So, the case where this f alpha becomes 0 can be handled by the formula which we are now going to prove.

So, let us see how we prove this result when you find the derivative of the function e to the power a x into V you get e to the power a x into derivative of V plus a times e to the power a x derivative into V and this can be expressed as e to the power a x d plus a operating on V and then you again differentiate that is you find the second derivative of e to the power a x into V what you get is if you find D square e to the power a x into V then this is d of and derivate of e to the power a x into V we have found as e to the power a x D plus a operating on V let us operate by D. So, derivate of e to the power a x is a times e to the power a x.

Then D plus a operating on V then e to the power a x D operating on d plus a V. Now this can be expressed as e to the power a x then we have a d V a square d. So, we can write it as d plus a whole square V D, D is D square V and then a D V and then a D V becomes 2 a d V and then we have a square v. So, a to the power a x D plus a whole square we get and thus we if we apply f D operator if we apply suppose f D is equal to D square plus alpha d plus beta.

Let f D V equal to D square plus alpha d plus beta then f D operating on e to the power a x into b will give you D square plus alpha d plus beta operating on this and. So, what you will get actually D square plus alpha d plus beta operating on e to the power a x into V. So, this will give you e to the power a x e to the power a x D plus a whole square operating on V then alpha times d plus a e to the power a x e to the power a x D plus a V and then e to the power a x beta b which is equal to e to the power a x f D plus a operating on v. So, this is what we get.

And so thus ultimately we get f D operating on e to the power a x into V, it is equal to e to the power a x f D plus a into V. So, this suggests us this formula 1 over f D operating on e to the power a x into V equal to e to the power a x 1 over f D plus a into V. Now let us see how we get this.

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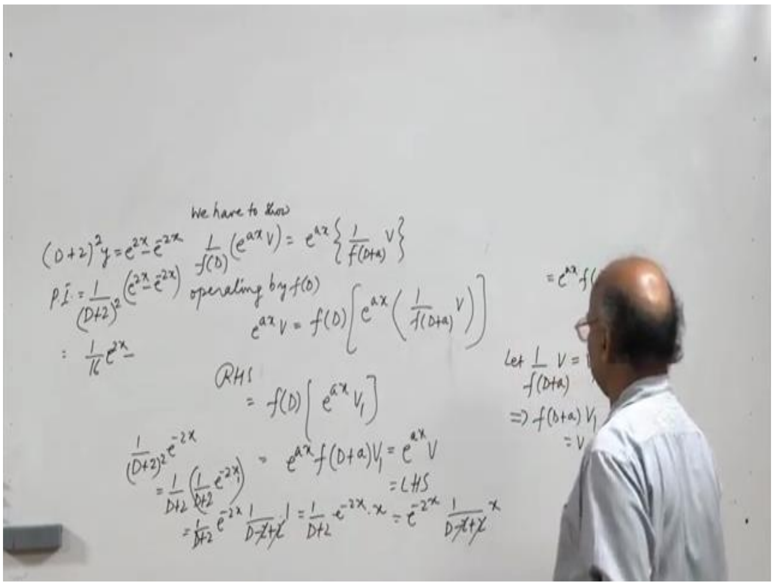

So, we have to prove 1 over f D e to the power a x into V equal to e to the power a x 1 over f D plus a operating on V now what we do is let us operate by f D on both sides. So, operating in by f D; f D in 1 over f D are inverse operators. So, we get e to the power a x into V equal to f D operating on e to the power a x 1 over f D plus a V now let us look at the right hand side.

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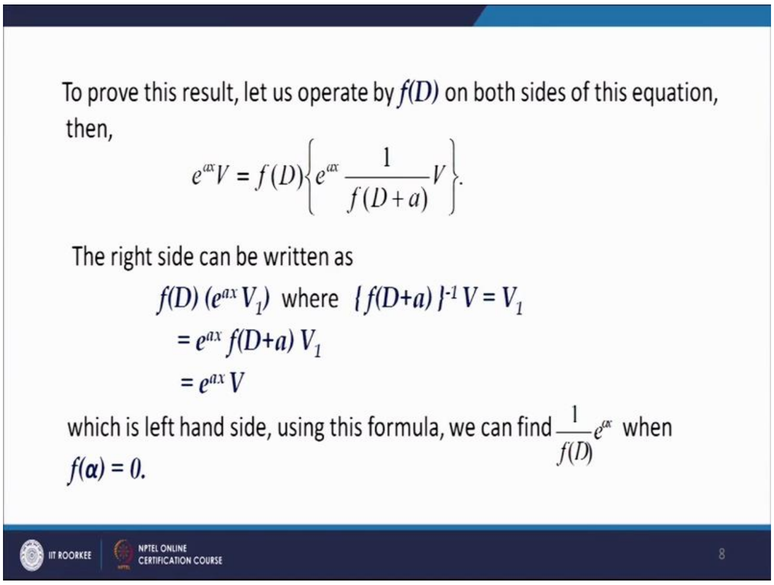

So, let us assume that 1 over f D plus a operating on V is equal to V 1, let us assume that 1 over f D plus a operating on V is equal to V one. So, we shall get f D right hand side;

right hand side here will be equal to f D operating on e to the power a x into V 1 and when f D operates on e to the power a x into V what we had got earlier that if this is e to the power a x f D plus a V 1 we have proved this earlier.

Now, this implies that f D plus a when you operate on both sides you get f D plus a V 1 equal to v. So, this will be equal to e to the power a x into V and this is what we had to prove this is the left hand side. So, left right hand side left hand side are same and therefore the formula 1 over f D e to the power a x into V equal to e to the power a x 1 over f D plus a operating on V is true.

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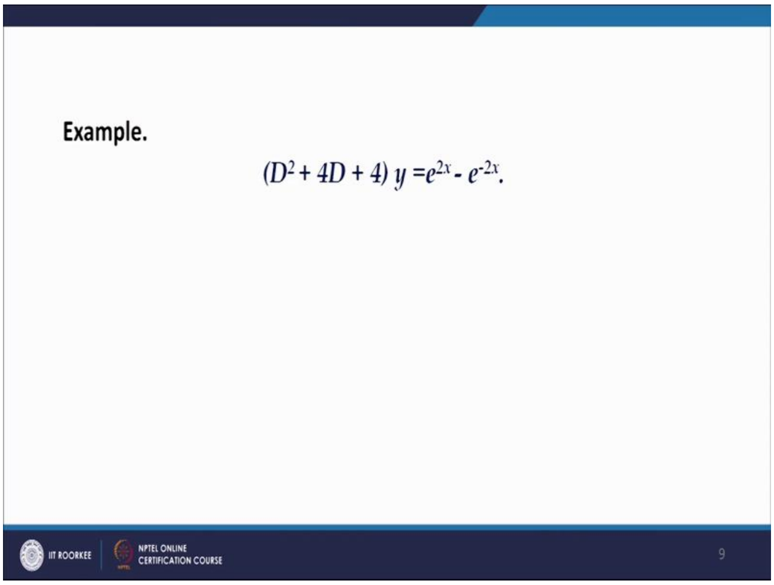

Now, let us apply this formula to solve this equation D square plus 4 D plus 4 which we can write as D plus 2 whole square y equal to e to the power 2 x minus e to the power minus 2 x i will just find particular integral here complementary function is easy to find. So, this is 1 over D plus 2 whole square e to the power 2 x minus e to the power minus 2 x now this is equal to when 1 over D plus 2 operator a square operates on e to the power 2 x we get 2 plus 2 4 square. So, we get 1 by 16 into e to the power 2 x minus.

Now when 1 over D plus 2 whole square operates on e to the power minus 2 x on replacing D y minus to D plus 2 become 0. So, we will apply this formula. So, we will let us see how we get this 1 over D plus 2 whole square V operates separately on e to the power let us operate on e to the power minus 2 x. So, this is 1 over D plus 2 1 over D plus 2 operating on e to the power minus 2 x, we can we can assume here 1. So, 1 over D plus 2 operating on e to the power minus 2 x into 1. So, V, we are taking as 1. So, this is equal to 1 over D plus 2 e to the power minus 2 x D will be replaced by D minus 2 and this cancels we get 1 over D plus 2 e to the power minus 2 x 1 over D is D means derivative 1 over D means inverse of the derivative that is integral of 1 is x.

Now, again we will apply the formula which we have just proved 2 e to the power minus 2 x into x let us take V equal to 2 x. So, then we will have e to the power minus 2 x d will be replaced by D minus 2. Now 1 over D is again integral of we have to make integral of x. So, that is x square by 2 o x square by 2 e to the power minus 2 x and thus the particular integral is. So, this is how we can make use of this formula in the case where under placing d by a in the expression one by f D e to the power a x f a becomes 0 now let us find another result.

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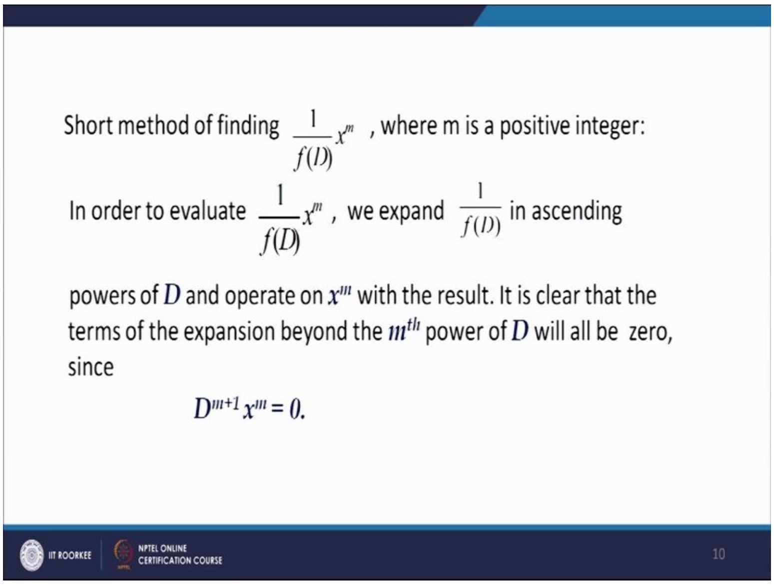

So, now we discuss how to find the particular integral in case r x is of we form x to the power m where m is a positive integer.

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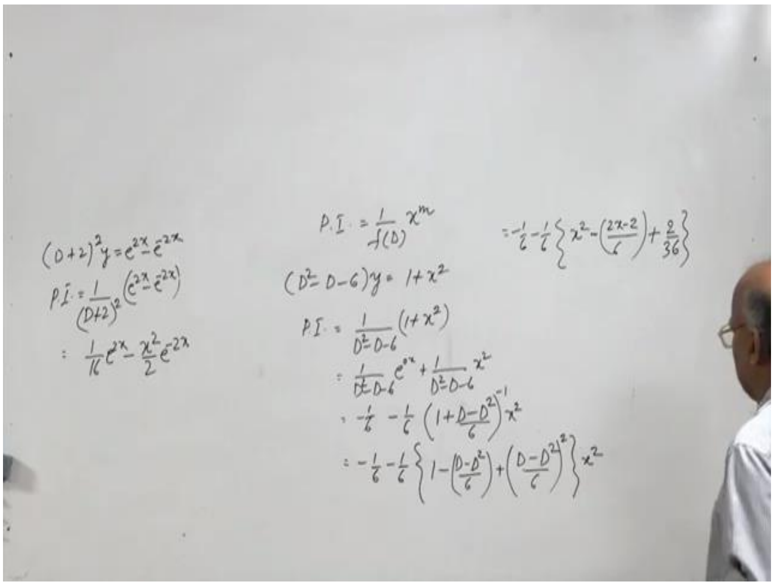

So, when we have to find, but particular integral of this type where m is positive integer what we do is in order to evaluate 1 over f D x to the power m we extend f D in ascending powers of D and then operate on x to the power m. Since the derivative m plus 1 of the derivative of x to the power m is 0, all derivatives more than 1 plus are also 0 while writing the expression in the ascending powers of d we need to write the derivate of the powers of V only up to m rest of the terms will be 0. So, we can evaluate particular integral by writing in the ascending powers of D.

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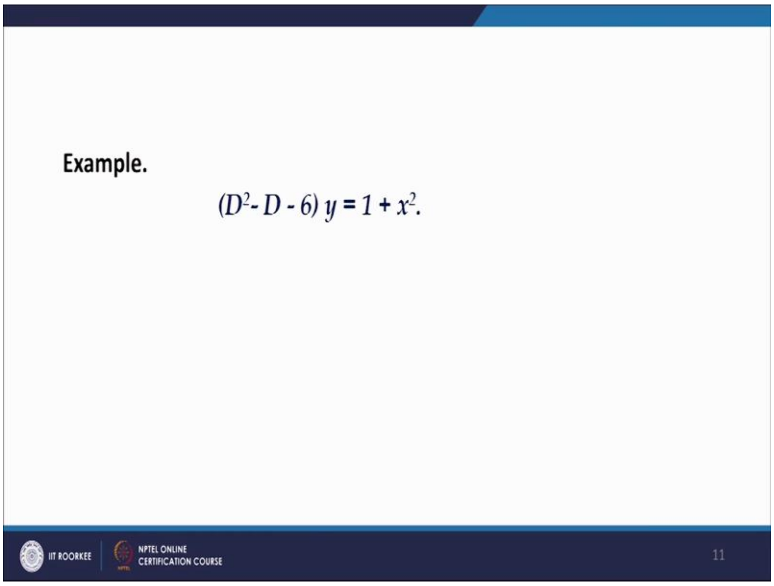

Let us see; we how apply this formula suppose we have the equation D square minus D minus 6 y equal to 1 plus x square.

So, this is let us say; let us find particular integral here. So, 1 over D square minus D minus 6 operating on 1 plus x square 1 can be taken it again by writing e to the power 0 x. So, 1 over D square; D square minus d minus 6 e to the power 0 x and then 1 over D square minus D minus 6 x square. So, this is give by replacing d by 0 we get minus 1 over 6 and this is and 1 over D square minus D minus 6 when operates on x to the power 2, we have to expand it in ascending powers of D. So, that we can do like this 1 over minus 1 over 6 1 plus d minus D square divided by 6 raise to the power minus 1.

Now, we will expand it like the expand 1 plus x to the power minus 1. So, this 1 will be 1 plus x to the power minus 1 is 1 minus x plus x square minus x cube. Now x cube I am not writing because x cube in the x cube when you do, D minus D square to the; by 6 to the power cube will minimum power of D will be three, but we have x square here third derivative x square is 0. So, we need not write those terms we can stop here and then.

So, this operates on this operates on x square and so what we will get is minus 1 by 6 minus 1 by 6 and then x square and then D minus D square when operates on x square D or in square gives 2 x D square on x square gives 2. So, 2 x minus 2 divided by 6 and then we have D minus D square 1. We find; we get D square then we get D cube then we get D to the power 4. So, D square 1 operates on x square gives you 2. So, 2 by 36, all other terms will contribute 0 because when they operate on x square we what we get is 0. So, this is the particular integral in this case. So, when 1 over f D operates on x to the power m where m is an integer we have to extend 1 over f D in ascending powers of D.

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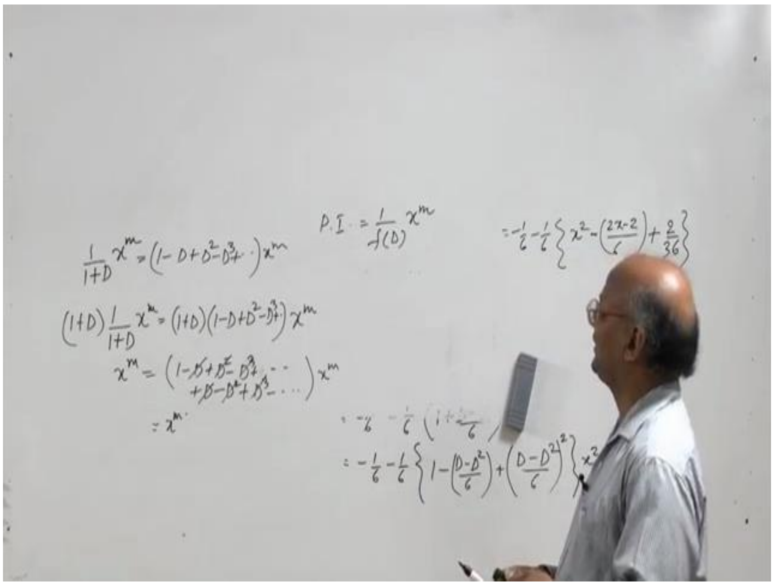

Now, I will tell you how this expression is justified suppose we have 1 over 1 plus D operates on x to the power m and we are writing it as 1 minus D plus D square minus D cube and so on, we are expanding 1 over 1 plus D as 1 minus D plus D square minus D cube and so on like we expand 1 plus x to the power m minus 1. So, how this expansion is valid? Now what you do is to prove this, you operate by 1 plus D on both sides, so operating by 1 plus D.

We will have this, now 1 plus D and 1 over 1 plus D are inverse of operators. So, we will have x to the power m and in the right side, let us operate 1 plus D on 1 minus D plus D square minus D cube, what you get is when you operate by 1 you get 1 minus D plus D square minus D cube and so on and when you operate by D, you get plus D minus D square plus D cube minus and so on x to the power m. So, these terms they go on cancelling and what we get is x to the power m. So, both sides are equal, and therefore 1 over 1 plus D, if we write as 1 minus D plus D square minus D cube like we expand binomial expansion, this just extension is valid justified.

So, then we have to find 1 over f D operating on x to the power m we have to expand 1 over f D in ascending powers of D.

Thank you.