Mathematical methods and its applications
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Lecture - 60
Applications of Fourier transforms to BVP – III

Welcome to the lecture series on Mathematical Methods and its Applications. Let us discuss some more application of Fourier transforms. So, we have already discussed some application in last lectures; Fourier series, Fourier integral, Fourier transforms. Now this we have already discussed that how we can find out Fourier derivative of sin or cosine transform of derivatives.

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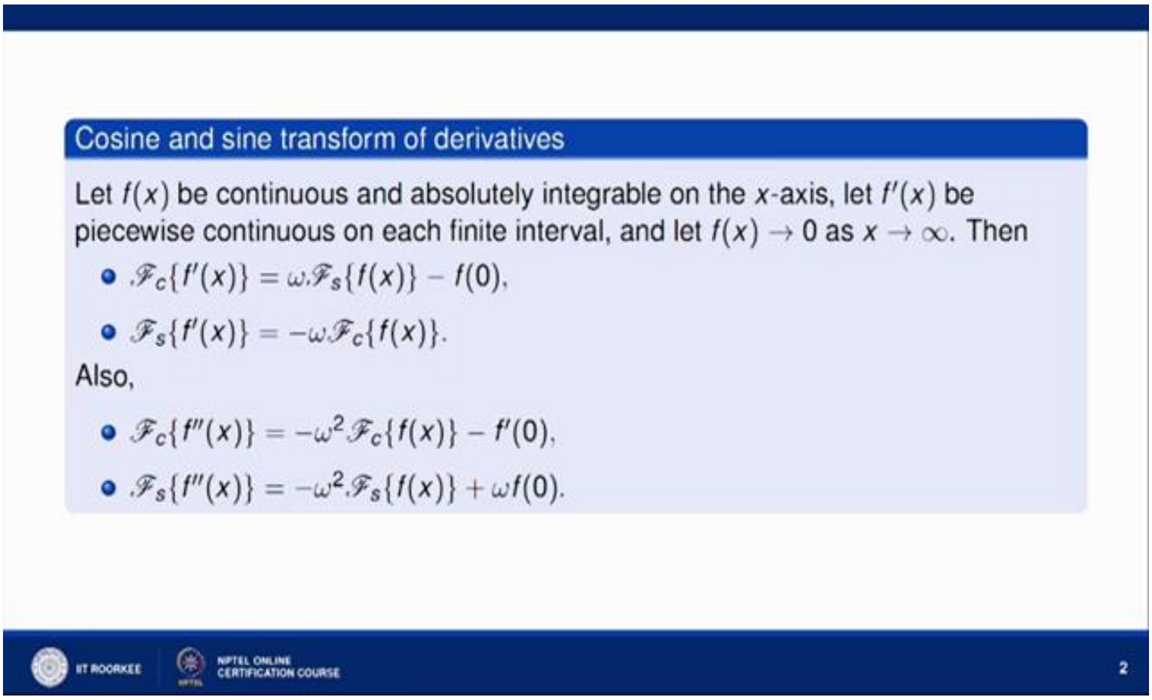

This we have already discussed, that if we want to find out Fourier cosine of f dash x. So, that will be nothing but omega Fourier sin of f x minus f 0. And similarly if you want to find out Fourier sin of f dash x it will nothing but minus omega Fourier cosine of f x.

So, when you replace f by f dash in the first expression and f by f dash in the second expression you will get back to these two expressions. So, what are these two expressions? So, let us this.

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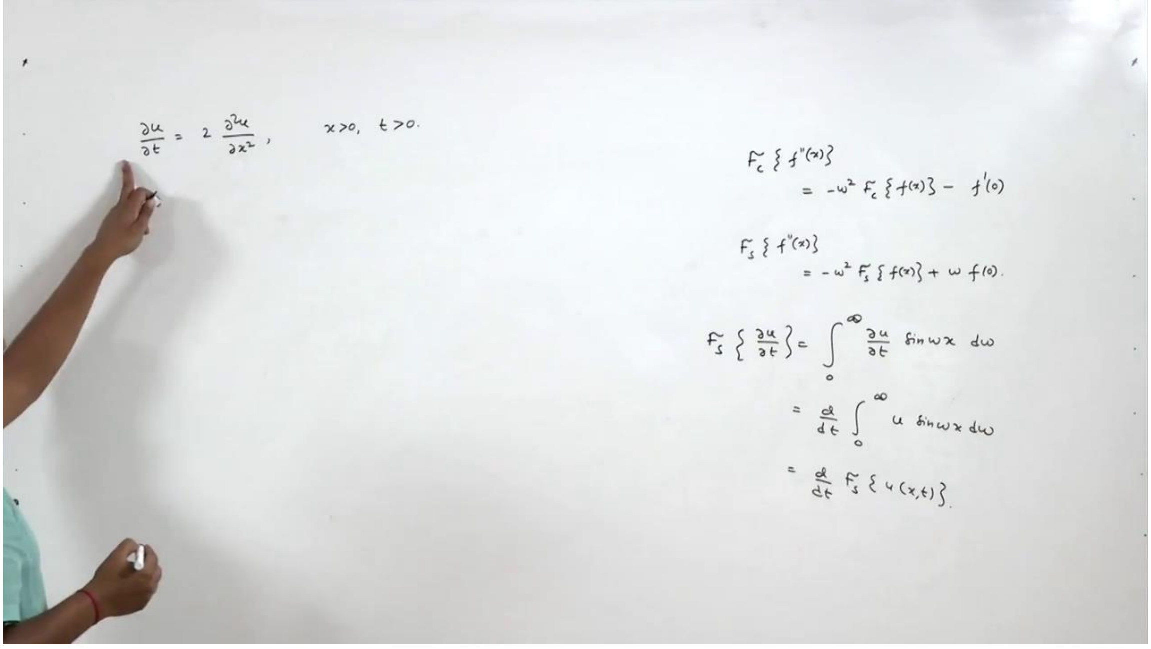

This we have already derived in the last lectures. So, what is Fourier cosine of f double dash x? Fourier cosine of f double dash will be nothing but x dash we have already derived it is minus omega square Fourier cosine of f x minus f dash 0. And Fourier sin of f double dash x is nothing but omega square Fourier sin of f x; it is minus omega square Fourier sin of this plus omega f 0.

So, these we have already discussed in the last lectures, now what how they are important, and why I am discussing this these things over here let us see. Now in this problem: for the equation u t equal to del u by del t is equal to 2 del square u by del x square, this equation we have to solve, now x is greater than 0 and t is greater than 0. Now x is varying from 0 to infinity; that means, we have either to apply a Fourier cosine or Fourier sin transform, because if this x is not varying from minus infinity to plus infinity; if x is varying if any one of variable is varying from minus infinity to plus infinity then we have to apply Fourier transform.

But this x is varying only in half range from 0 to infinity; that means we have to apply either Fourier cosine or Fourier sin.

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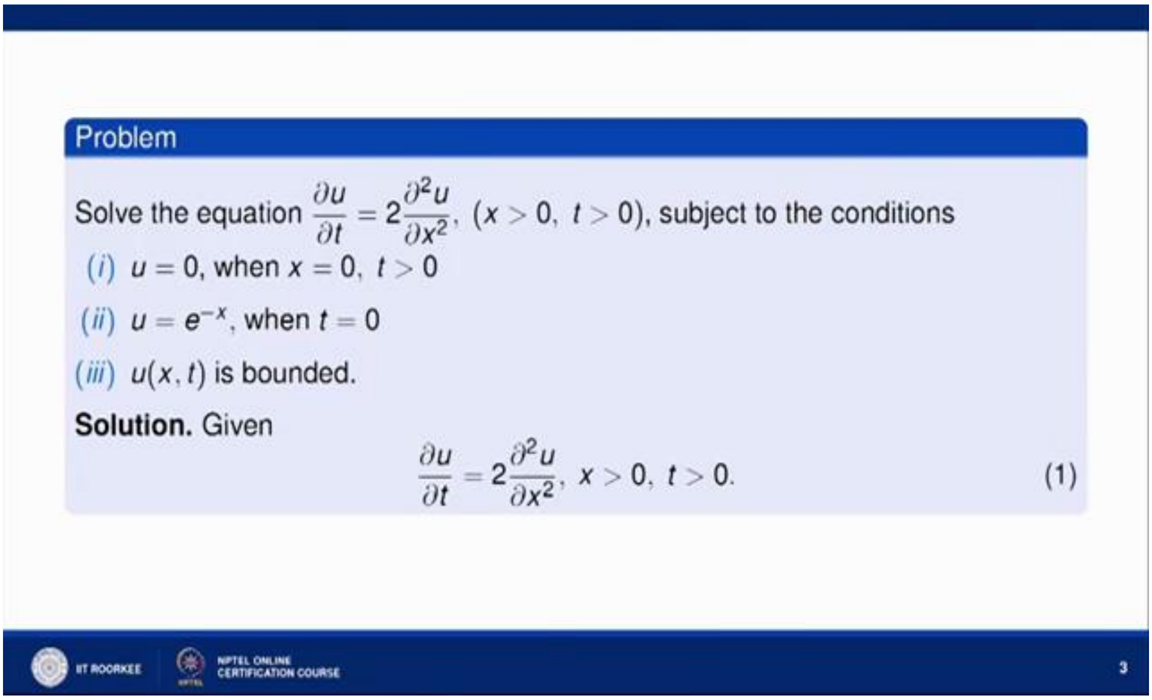

Now, which we have to apply Fourier sin or cosine how we will decide? We will decide by seeing the conditions. If we apply a Fourier cosine in f double dash; here f double dash if this term we are applying Fourier sin or cosine transform respect to x. So, we have a double derivative. When we have a double derivative we apply suppose Fourier cosine. So, in this we need f dash 0. So, here we need del u by del x at t equal to 0; at x equal to 0 sorry. We need the derivative condition which is not here, it is only u equal to 0 when x equal to 0 the first condition. So, if we now observe Fourier sin transform they are we need only f 0, when x is 0. So, in this way we conclude that we have to apply Fourier sin transform in this expression.

Basically, whenever we have a half range; we have to apply either Fourier sin or Fourier cosine. When in the initial condition we have the partial derivative at x equal to 0, so then we have to apply Fourier cosine otherwise we have to apply Fourier sin transform. So, in this particular problem we have to apply Fourier sin transform. So, take Fourier sin transform both the sides, because we have the condition when x equal to 0 u equal to 0. If we have derivative condition at x equal to 0 then we apply Fourier cosine transform.

Now take Fourier sin transform both the side, what will be the Fourier sin transform of del u by del t? It will be nothing but 0 to infinity del u by del t of sin omega x into d omega. So, that will be nothing but d by d t, because this term is free from t so 0 to infinity u into sin omega x into d omega which is nothing but d by d t of Fourier sin transform of u x t.

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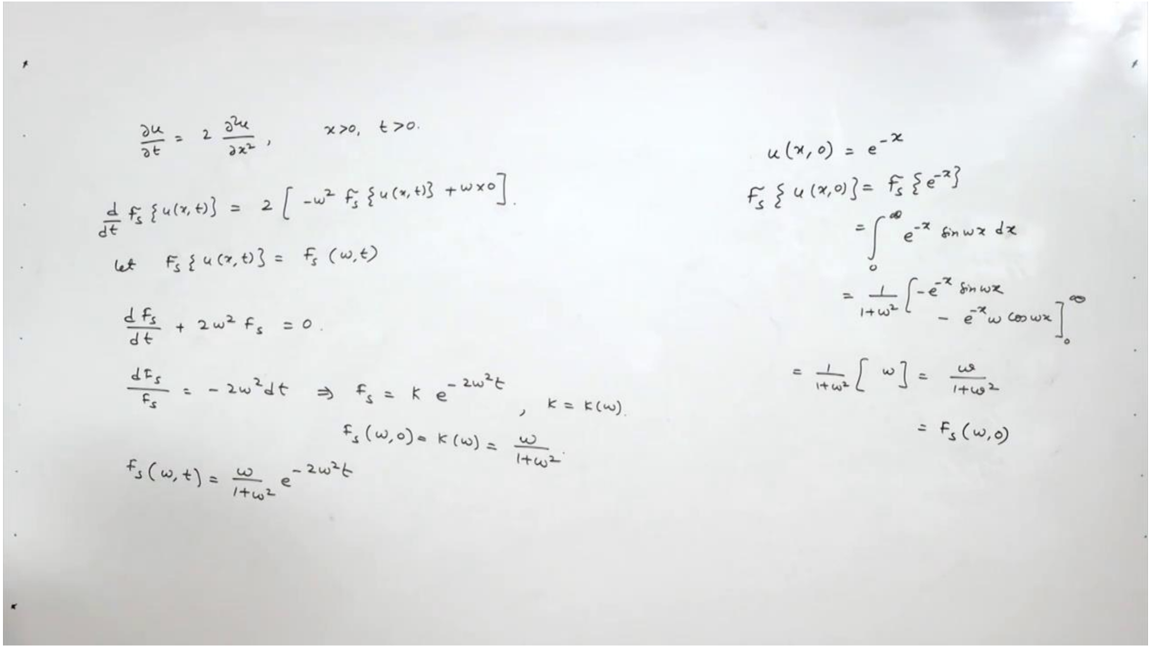

So, now when you take Fourier sin transform both the side it will be nothing but d by d t of Fourier sin transform of u x t which is equal to 2 times minus omega square; apply this result, so Fourier transform of sin transform of u x t plus omega into 0 because u is 0 when x is 0.

Now, let Fourier sin transform of u x t as suppose you take it 4 f of s it is omega t, because we are applying Fourier transform with respect to x. So, that is it is d upon f x upon d t which is minus minus plus 2 omega square f s which is equals to 0. So that implies; d f s upon f s is equals to minus 2 omega square d t, when w e integrate both sides f x is nothing but you obtain as k times e k power minus 2 omega square d.

Again, this k will be nothing but function of omega. So, now we have and the next condition where when t equal to 0 u equal to e k power minus x where this condition. Now apply this condition over here. What we obtain? So, u x 0 is e k power minus x. So, apply Fourier sin transform both the sides. So, Fourier sin transform of this expression will be nothing but Fourier sin transform of e k power minus x, which is nothing but 0 to infinity e k power minus x sin omega x into d omega into d x sorry it will be d x. So, it will be equal to 1 upon 1 plus omega square and it is minus e k power minus x to integrate by parts and minus e k power minus x omega cos omega x from 0 to infinity, so this is nothing but 1 upon 1 plus omega square.

So, when x is tend to infinity both will tends to 0 when x is tends to 0 it will 0 it will be nothing but minus minus plus it is omega. So, that will be nothing but omega upon 1 plus omega square. Now when you take t equal to 0 here, so Fourier sin transform of u x 0 will be nothing but f s omega 0; this is nothing but basically f x omega 0. When you take f x omega 0 here; f x, when you substitute t equal to 0 here basically it will be k omega. And k omega is nothing but from here will be nothing but omega upon 1 plus omega square.

Therefore, what will be f s omega t? It will be omega upon 1 plus omega square e k power minus 2 omega square t. Now take inverse Fourier sin transform both the side to find out the solution. So, this implies u x t will be nothing but 2 upon pi integral 0 to infinity omega upon 1 plus omega square e k power minus 2 omega t and it is sin omega x d omega. So, this will be the final solution of this problem.

So, here is a here is a solution, the same solution I have discussed here. So, this is a solution 2 upon pi f 0 to infinity omega upon 1 plus omega square e k power minus 2 omega square d sin omega x into d omega.

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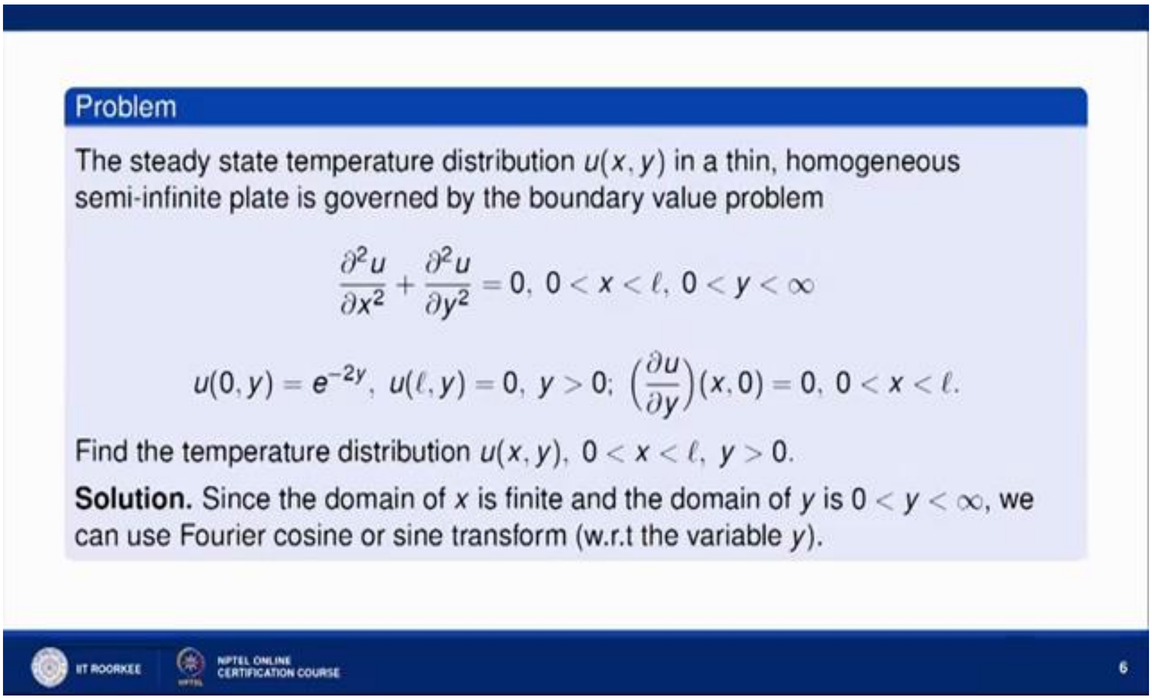

Now, the next problem: it is again in half range. Now here the half range is in y, y is varying from 0 to infinity. That means, we have to apply Fourier sin or Fourier cosine transform with respect to y. First we have to decide that for which variable we have to apply Fourier sin or cosine transform and that depends on the range. Here x is in 0 to l and y is varying from 0 to infinity. So, we have to apply Fourier sin or cosine transform with respect to y, number 1. Number 2: we have to apply Fourier sin or Fourier cosine that will decide by the conditions.

Now, here we will apply Fourier sin or cosine transform with respect to y and the partial derivative with respect to y when y is 0 is 0, it is given to us. That means we will apply a Fourier cosine transform here. Because, when you see this expression in Fourier cosine transform the second derivative here we have the derivative at t equal to at x equal to 0 that means we have to apply Fourier cosine transform when we have the derivative term in the initial conditions; in the conditions.

So, here del u by del y at y equal to 0 is given; that means, we have to apply Fourier cosine transform. Now, in this problem apply Fourier cosine transform both the sides. So, what is the problem basically?

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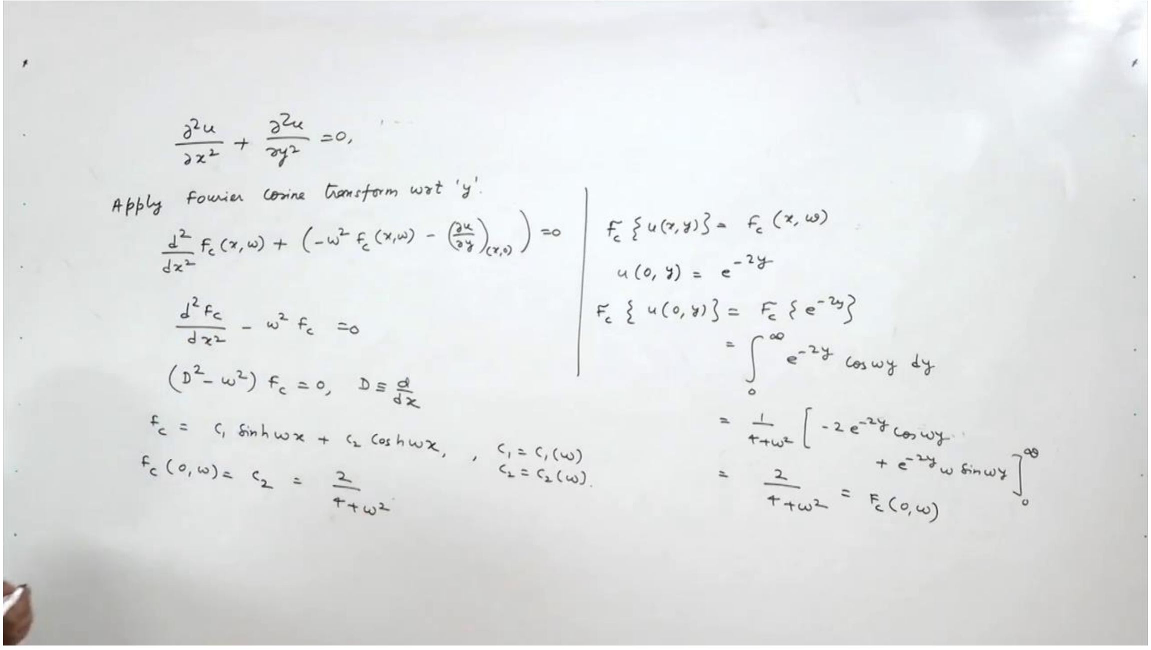

Now, the problems here is del square u upon del x square plus del square u upon del y square equal to 0. So, you will apply Fourier cosine transform that is clear and with respect to variable y that you have to apply. Apply Fourier cosine transform with respect to y.

So, what we will obtain? So, x will be 3, so it is d square f upon d x square. Now let us suppose Fourier cosine transform of u x y will be nothing but suppose it is f c x omega, because we are apply with respect to y so x we will treat as it is. So, it will be f c x omega plus. Now when we apply Fourier cosine transform here we will apply that result that is nothing but minus omega square Fourier sin cosine transform of x omega plus and minus it is del u by del y when y equal to at x comma 0 equal to 0. This condition we have applied for Fourier cosine transform.

So, what we obtain finally? Sorry, so basically, now this term is 0. So, what we obtained? D square f c upon d x square minus omega square f c will be 0 because this is 0, so this gives d square minus omega square f c is equal to 0 where d is nothing but d by d x. So, f c will be nothing but c 1 e k power or you can break in terms of hyperbolic functions. So, it is sin hyperbolic omega x plus c 2 sin hyperbolic omega y. And again c 1 and 2 are the arbitrary functions and f c is a function of x and omega, so c 1 and c 2 are the nothing but functions of omega.

So, it is c 1 sin hyperbolic omega, sorry; it is cos hyperbolic omega y. Now use the conditions; conditions are u 0 y equal to e k power minus 2 y is given to you, you apply a Fourier cosine transform both the sides. So, it is Fourier cosine transform of e k power minus 2 y which is given as 0 to infinity e k power minus 2 y and it is cos omega y into d y. So, when you integrate it what we will obtain? It is 1 upon 4 plus omega square and again you integrate. So, it is minus 2 minus 2 y it is cos omega y minus minus plus e k power minus 2 y omega sin omega y from 0 to infinity.

At infinity both will tend to 0 and at 0 this is 0 and this will be 1 and minus minus plus, so it is 2 upon 4 plus omega square. So, this we will obtain here as a Fourier cosine transform of this term. And again when you substitute x equal to 0 both the sides this will be nothing but is equal to f c of 0 omega. So, when you take f c of 0 omega here, so sin hyperbolic 0 is 0, and cos hyperbolic 0 is 1 that is nothing but c 2; it is it is also actually x, we are putting x equal to 0. When you put x equal to 0 it is 0 and it is 1, so it is c 2. And c 2 will be nothing but 2 upon 4 plus omega square.

So, in this way c 2 can be obtained. Now for c 1 we apply another condition.

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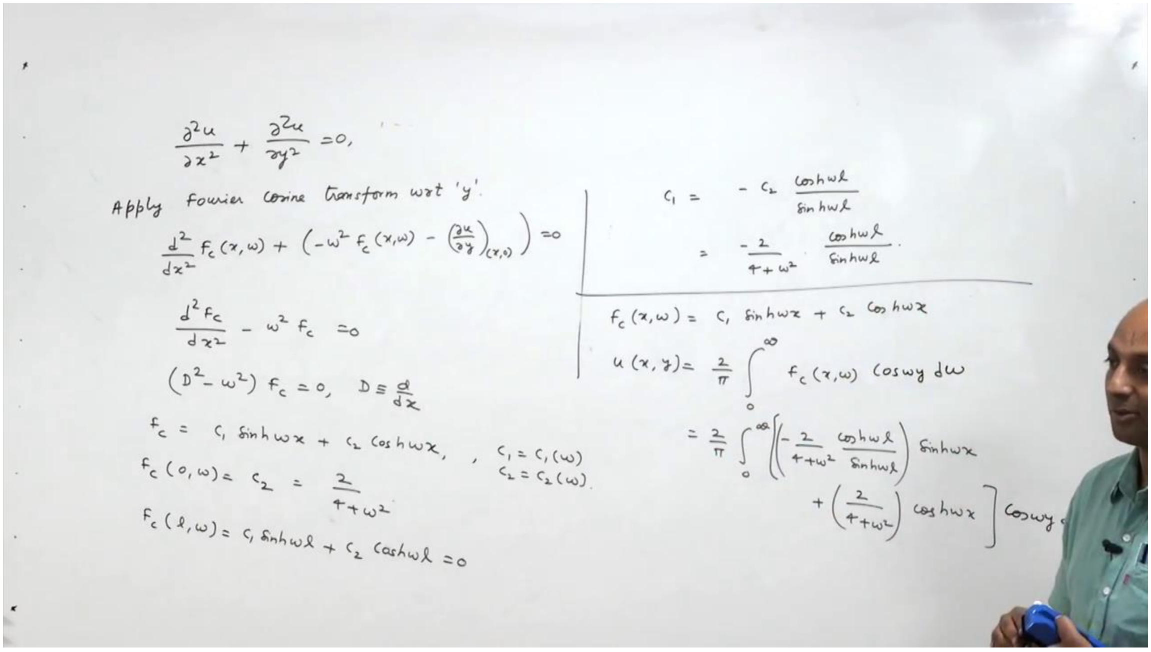

Another condition is u at l y is 0. Now again take Fourier cosine transform both the sides. So, Fourier cosine transform of u l y will be nothing but Fourier cosine transform of 0 which is nothing but 0. And this is also equal to; where you substitute x equal to l both the sides this is nothing but f c l omega. So, when you find f c l omega from here, f c l omega so that will be nothing but c 1 sin hyperbolic omega l plus c 2 sin c 2 cos hyperbolic omega l which is equal to 0 from here.

Now, we know c 2; c 1 can be find out. So, what will be c 1? C 1 will be nothing but minus c 2 cos hyperbolic w l upon sin hyperbolic w l. And c 2 is nothing but this term, so it is minus 2 upon 4 plus omega square into cos hyperbolic omega l upon sin hyperbolic omega l. So, in this way once we obtain c 1 and c 2 which is here, then the only thing is you take inverse Fourier cosine transform both the sides to find out the final solution; that is the only thing remain now.

So, what is f c omega t? F c omega t here is nothing but f c x omega sorry, because f c is a function of x in omega. So, x and omega is nothing but c 1 sin hyperbolic omega x plus c 2 cos hyperbolic omega x, here c 1 and c 2 are given by these two expressions so the case are inverse cosine transform both the sides. So, u x y will be nothing but 2 upon pi integral 0 to infinity and that will be f c x omega into e k power eta omega y into d omega. So, that will be 2 upon pi integral 0 to infinity and f c is nothing but c 1 times, c 1 is this term. So, this is minus 2 upon 4 plus omega square cos hyperbolic omega l upon sin hyperbolic omega l c 1 at this term into sin hyperbolic omega x and plus c 2; c 2 is 2 upon 4 plus omega square into cos hyperbolic omega x and whole multiplied by e k power eta omega y in to d y d omega.

So, we can simplify this expression and that will give the final solution u x y which is a solution of this partial differential equation. So, hence we can solve such type of problems. So, whenever the half range is given to us you first make sure that respect to which variable you will have to apply Fourier transform x or y and that will be decided by the conditions given to us. Now you apply either Fourier sin or Fourier cosine transforms and for finding the arbitrary constants c 1 and c 2 you have the conditions given the problem. And final take initial sin or cosine transforms to find out the final solution of the given partial differential equation.

So, this is what I have done here also. This is when you simplify this; when you simplify this expression so we will get this form of answer, this one can easily solve. So, this is the final answer which we can obtain after solving this. Ok sorry, one thing I have missed here, it is not e k power eta y it is basically, because we are taking inverse cosine transform it will be nothing but cos omega y and it is d y, d omega. Similarly it will be cos omega y into d y into d omega, because we are taking inverse Fourier cosine transform; it will be cosine term will be here. So, that we have missed, this way we will get the final answer.

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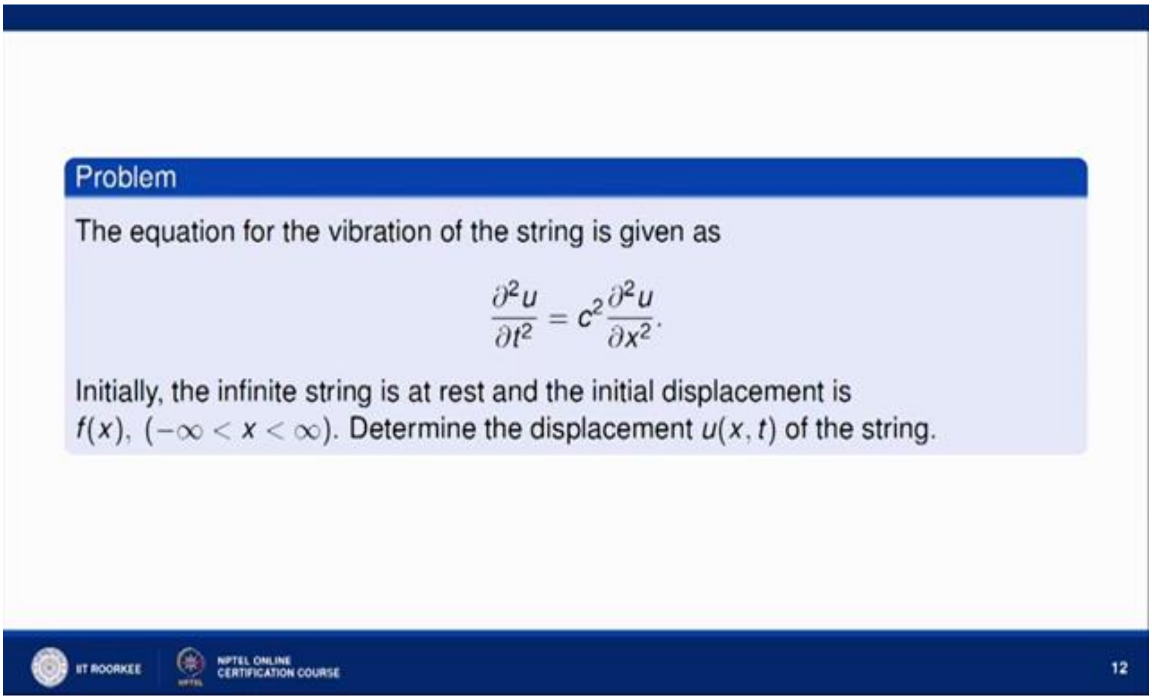

Now, the last problem the equation for the vibration of the string is given as this and initially the string is at rest and the initial displacement is f x. Now here x is ranging from minus infinity to plus infinity; that means we have to apply Fourier transform. So, initially string is at rest; these are the conditions given to us so simply apply Fourier transform both the sides to solve this problem.

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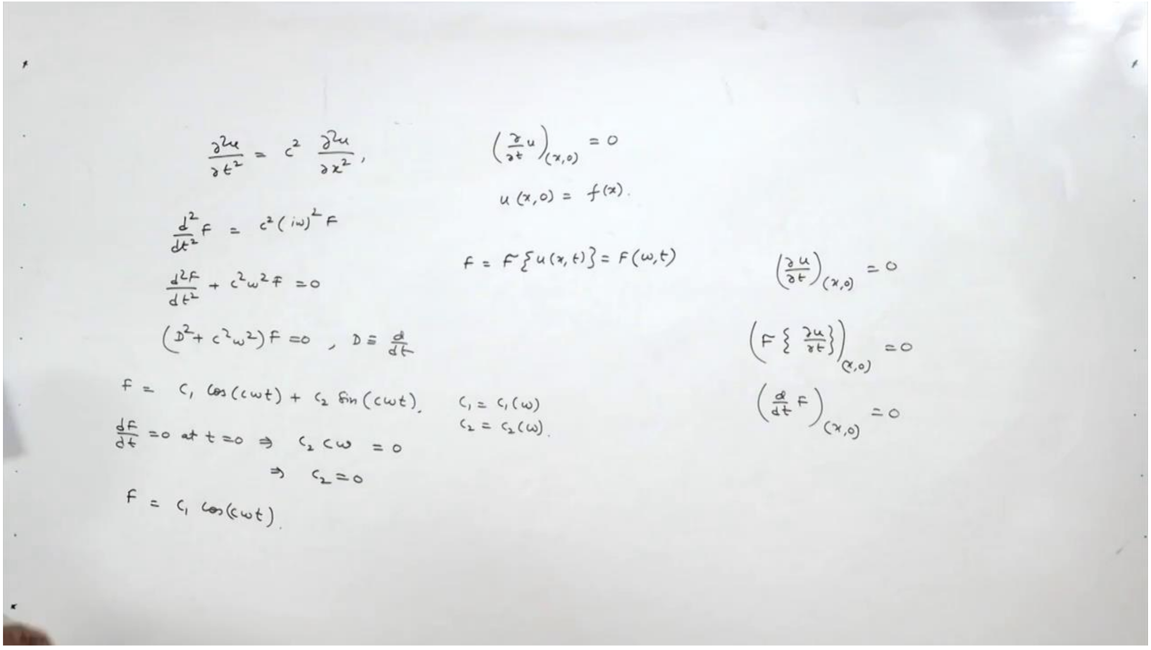

So, what is the problem del square u upon del t square is equals to c square del square u upon del x square. And initially the infinity string is at rest; that means, del by del t of u at x comma 0 is 0. And initial displacement is f x; that means x comma 0 is f x. So, these are the conditions given to you. And x is varying from minus infinity to plus infinity; you have to solve this equation.

So, again we will apply Fourier transform technique. You will take Fourier transform both the sides respect to x, because x is varying from minus infinity to plus infinity. That means, it is d square upon d t square of f which is equals to c square eta omega square f; here f is nothing but Fourier transform of u x t which is a function of basically omega and t. When you simply d square f upon d t square and it is minus 1, so it is plus c square omega square f equal to 0, so it is d square plus c square omega square into f equal to 0. So, where d is nothing but d by d t. When you simply this; so f we can obtain as c 1 e k power, it is cos; so it is cos c omega t plus c 2 sin c omega t. Again c 1 and c 2 are the functions of omega.

Now, we will apply these conditions. So, we will take the first condition is del u upon d t at x comma 0 is 0. When you take Fourier transform both the sides of this expression you will obtain Fourier transform of del u by del t at x comma 0 is 0. That means, d by d t of capital F at x comma 0 is 0. A capital F is nothing but Fourier transform of u x t; it is

Fourier transform of u x t. That means, when you take d by x by d t that is 0 at t equal to 0.

So, when you apply this condition over here. So the derivative of this will be sin, minus sin; when t equal to 0 it will be 0 and it will be cos when t equal to 0 that will be 1. So, we will left with c 2 c omega which is equal to 0; c 2 c omega which is equal to 0. Of course, this implies c 2 equal to 0, because c 1 c and omega are constant; so c cannot be 0 and omega cannot be 0, c 2 is the only thing which can be 0. So, f is nothing but c 1 cos c omega t.

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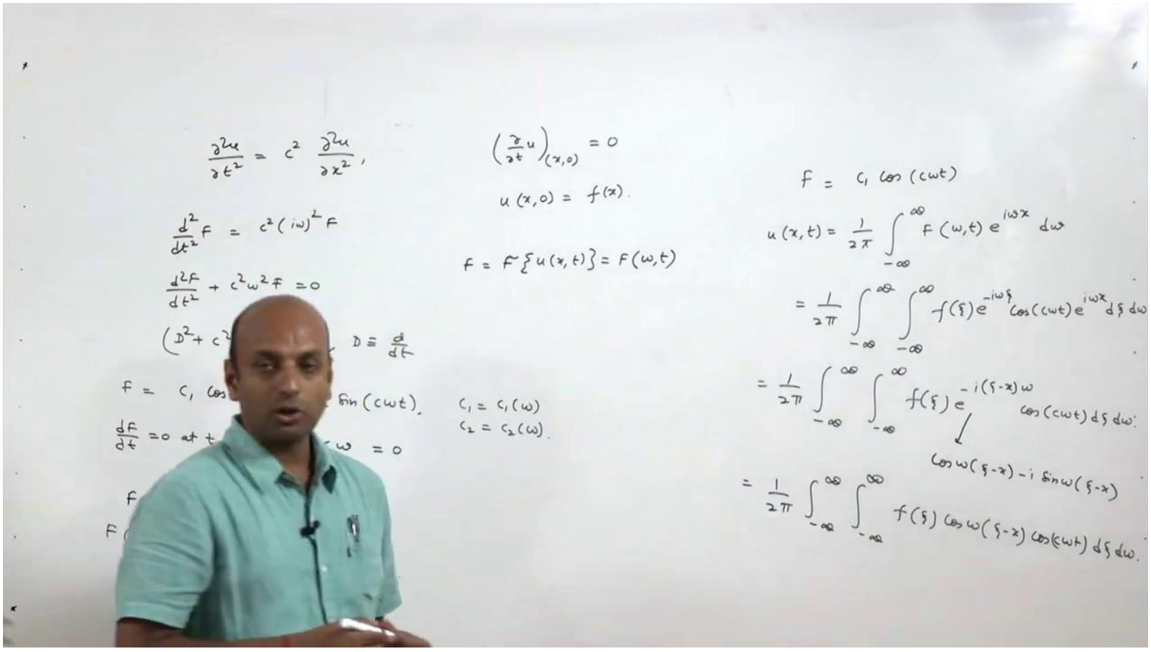

Now we have one more condition to find out the value of c 1; that is u x 0 equal to f x. Now u x 0 is f x, so take Fourier transform both the sides. So, Fourier transform of u x 0 will be nothing but Fourier transform of f x and that will be nothing but minus infinity to plus infinity f x into e k power minus eta omega x into d omega into d x sorry; so that will be nothing but f of, because you are putting t equal to 0 so that will be o f omega 0. So, when you take f omega 0 that will be c 1 and that c 1 will be nothing but this expression. So, that will be nothing but minus infinity to plus infinity f x e k power minus eta omega x into d x so.

Now we have the value of c 1 and c 2 in our hand, so we will simply take inverse Fourier transform to find out the final solution. So, what is f? F is nothing but c 1 cos c omega t where c is given by this expression take inverse Fourier transforms both the sides, so it is u x t will be nothing but 1 upon 2 pi integral minus infinity to plus infinity f omega t e k power eta omega x into d omega; that will give the final answer.

It is nothing but 1 upon 2 pi integral minus infinity to plus infinity. And f omega t is c 1 into cos c omega t and c 1 is given by this expression. So, we can substitute c 1 also; it is f psi e k power minus eta omega psi and cos c omega t e k power eta omega x d psi d omega. So, that will be here. Now when you further simplify it, so it is 1 upon 2 pi integral minus infinity to plus infinity integral minus infinity to plus infinity f psi e k power minus eta psi minus x into omega cos c omega t into d psi into d omega.

Again when you write e k power minus eta k eta as cos theta minus eta sin theta then cos; theta cos theta will contain omega and it is also even in omega even into even remain even. However, when you have a sin term now, this can be written as cos omega psi minus x minus eta sin omega psi minus x. And this into this is a even function in omega and this into this is an odd function in omega. So, this will be 0 in terms of omega, so we will left with only one term that is 1 upon pi integral minus infinity to plus infinity minus infinity to plus infinity as psi cos omega psi minus x cos c omega t d psi into d omega. So, that will be the final solution of the given differential equation.

In this way we can solve several problems, several partial differential equation using Fourier series, Fourier integrals, or Fourier transforms. I hope in this course you have learnt lot of things about mathematical matters, lot of techniques to solve ordinary differential equation, partial differential equation, Laplace transforms its applications, Fourier series, Fourier transform, and Fourier integrals their applications that all we have seen. So, I hope you might have enjoyed the course. I wish you all the best for your bright career ahead.

Thank you very much.