Mathematical methods and its applications
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Lecture – 06
Methods for finding particular integral for second-order linear differential equations with constant coefficients I

Hello friends. Welcome to my lecture on Methods for Finding Particular Integral for Second-order Linear Differential Equations with Constant Coefficients. On this topic there will be 3 lectures; this is my first lecture on this topic.

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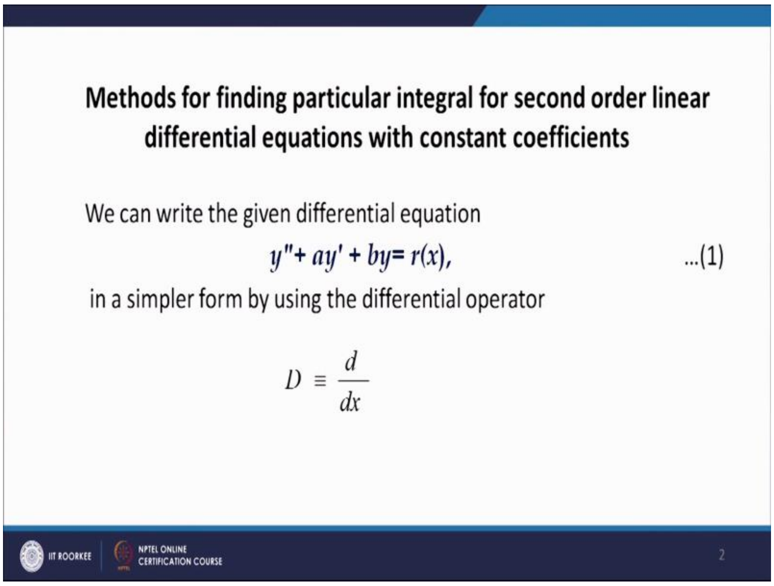

Let us consider the second-order differential equation y double dash plus ay dash plus by equal to r x here a and b are constants and r x is any function of x. So, it is a second-order linear differential equation with constant coefficients. Now we have already discussed how to find the general solution of the associated homogenous linear differential equation that is y double dash plus ay dash plus by equal to 0. Now we are going to consider; we are going to find a particular solution of this non-homogenous equation which is known as the particular integral and we have already seen that once the general solution of the associated homogenous linear differential equation is known and a particular solution of the non-homogenous equation is known, then their sum gives us the general solution of the non-homogenous linear differential equation of second-order.

So, this equation of second-order can be written in a simple form by using the differential operator D; D is the differential operator d over dx.

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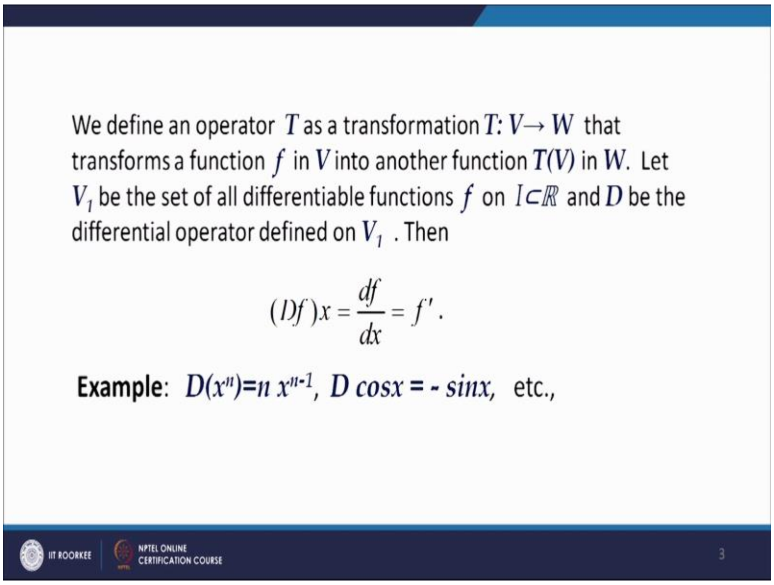

So, this what do we mean by D? What do we mean by an operator? An operator T is a transformation from a space of functions V into the other space of function W. These are vector spaces that transforms a function f in V to another function T V in W. Let us say V 1 is the set of all differentiable functions f on I. I is any subset I is any interval subset of R and D be the differential operator defined on V 1 then D f at x; D f at x means the derivative of f with respect to x, V also denoted by f dash.

For example, if you take f x equal to x to the power N then D of x N that is the derivative of x to the power N with respect to x is N times x to power N minus 1, similarly derivative of cos x with respect to x is minus sin x, etcetera.

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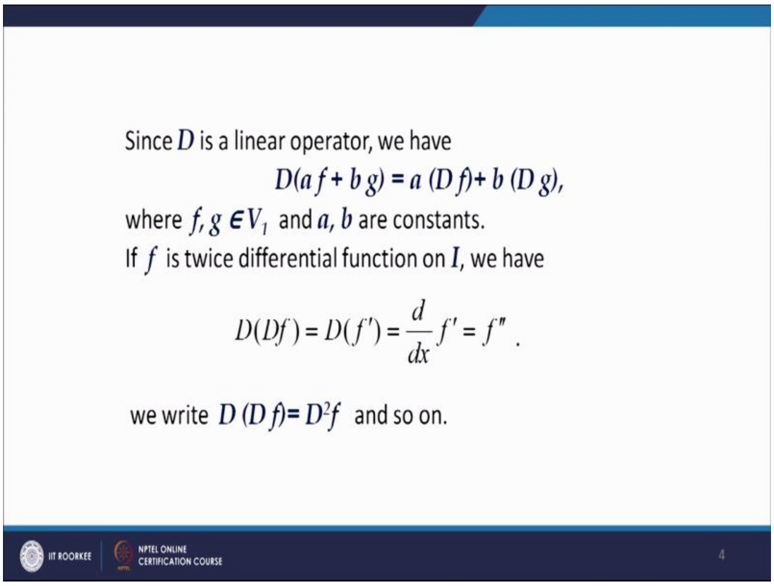

Since D is a linear operator, we can see that for example, if you use you can see here D is an operator from b into W, whenever you take any 2 function f and g in b since b is a vector space and T is a linear operator T, D a f plus b g becomes a D f plus b D g where f g belongs to b 1 and a b are constants. If I is a twice differentiable function on I, we write D D f as D of f dash or we can write it as d f dash over dx which is same as f double dash, yes f double dash x, the twice f double dash as D square f over dx square, we write D D f as D square f also. So, this way we can define higher order derivatives of a with respect to x.

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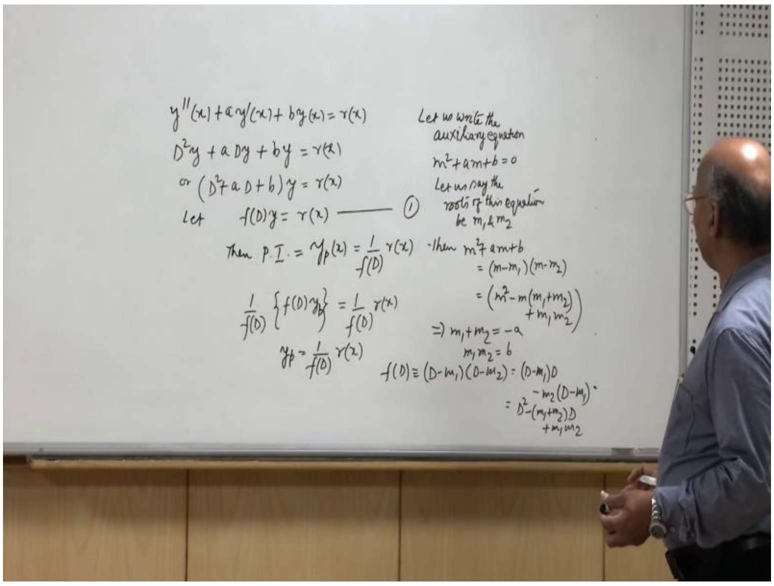

Now, the given equation then y double dash plus a times y dash x plus b times y x equal to r x can be written in a simpler form as D square y plus a times D y plus b times y equal to r x or we can write it as D square plus a D plus b operating on y equal to r x. Now let us say let f d; if you take f D equal to D square plus a D plus b then you can write it as f D y equal to; so f D is a is an operator which when x on y gives you D square y plus a d by plus b y. Now let us see then the particular integral which we also denote by P I; particular integral be in short we also write as P I and we have the notation that we have taken for the particular integral is y p x. So, y p x is given by 1 over f D operating on r x.

Now, let us see how we get this form of the particular integral. Y 1 over f D, we mean that it is the inverse of the operation that is defined by f D that is to say if you operator by 1 over f D on this equation, let me on this equation let us say then when we operate on the equation 1 by 1 over f D then since 1 over f D defines an operation which is inverse to the operation defined by f D, we get the left hand side equal to y and which is equal to 1 over f D, r x since y p solution of the sorry; y p is a particular solution of the differential equation f D y p, I can write it f D y p. So, f D y p equal to this and so y p equal to 1 over f D r x. We have assumed that y p x is a particular solution of the non-homogenous equation. So, f D y p; f D y p will be equal to r x and when we operate on that equation by the inverse operator 1 over f D then what do we get is y p equal to 1 over f D r x.

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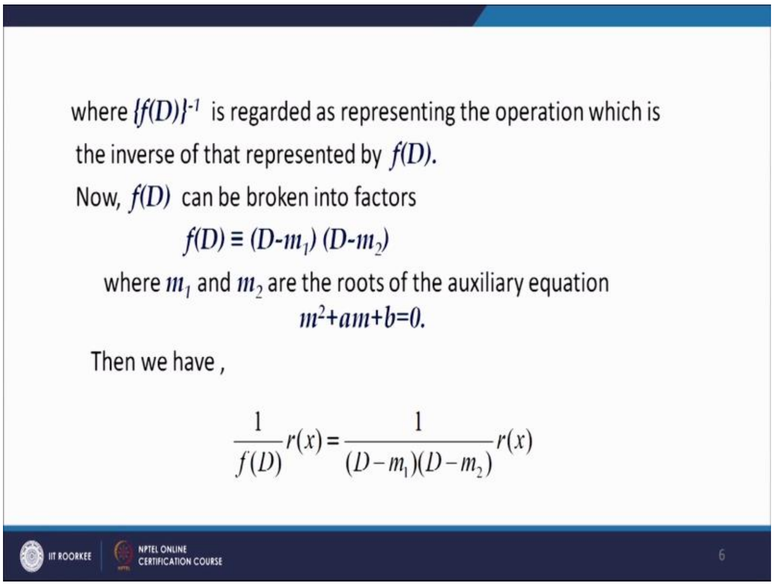

So, y p x; the expression for y p x is given by 1 over f D r x. Now we can see that this f D; f D can be broken into factors which are D minus m 1 into D minus m 2 and m 1, m 2 are at the roots of the equation m square plus m plus b equal to 0. See in this equation; let us write the auxiliary equation, the auxiliary equation for the given differential equation is m square plus a m plus b equal to 0. Let us say the roots of this equation are let us say the roots of this equation be m 1 and m 2 then m square plus a m square plus b is equal to m minus m 1 into m minus m 2 we can write it as m minus m 1 and m minus m 2 now. So, then what we will have if you calculate this m square minus m times m 1 plus m 2 and then plus m 1 m 2. So, cutting the coefficients of the various powers of m both sides we get m 1 plus m 2 equal minus a and m 1 m 2 equal to b.

Now, D is f D is we are writing as D minus m 1 into D minus m two. So, D minus m 1 into D minus m 2 we can operate D minus m 1 on D that gives you m 2 is a constant. We can write it as m 2 times D minus m 1. So, this gives you D square minus m 1 plus m 2 into D plus m 1 m 2.

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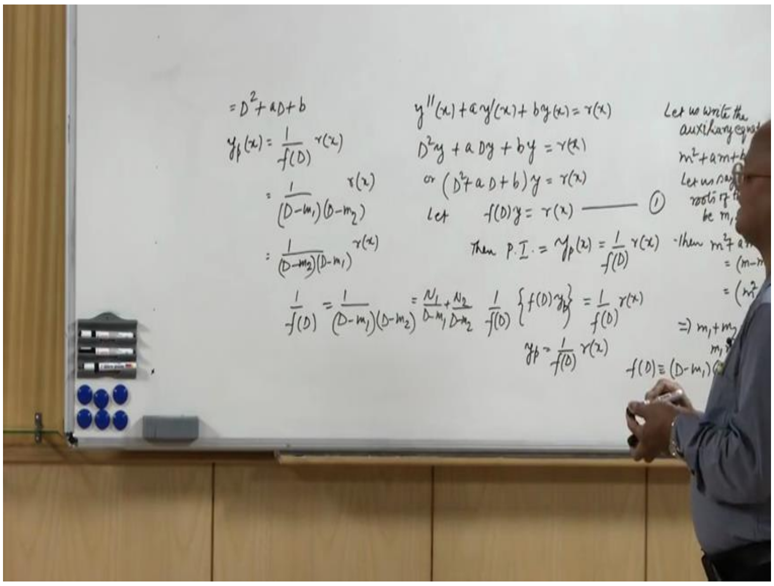

Now making use of m 1 plus m 2 equal to minus a and m 1 m 2 equal to b, what do we get is which is nothing but f d. So, f D can be written also as D minus m 1 D minus m 2. We can write D minus m 2 into D minus m 1 also because that will also give the same thing. So, order of factors here is immaterial. we can write them in any order.

Now, when we want to find the particular integral y p x, we can write 1 over f D as 1 over D minus m 1. Now what we do is we first operate on r x by the operator 1 over D minus m 2 and then whatever is the result of that operation, we operate on that result by the operator 1 over D minus m 1 as I said in my; as I just said, this can also be written as the order of factors here is immaterial. So, we can again; we can also operate 1 over D minus m 1 on r x and then whatever is the result of that operation we can operate on that by the operator 1 over D minus m 2 to get the particular integral in what manner we will have to operate by this operators on r x depends on the function r x which we shall see when we solve some problems on this.

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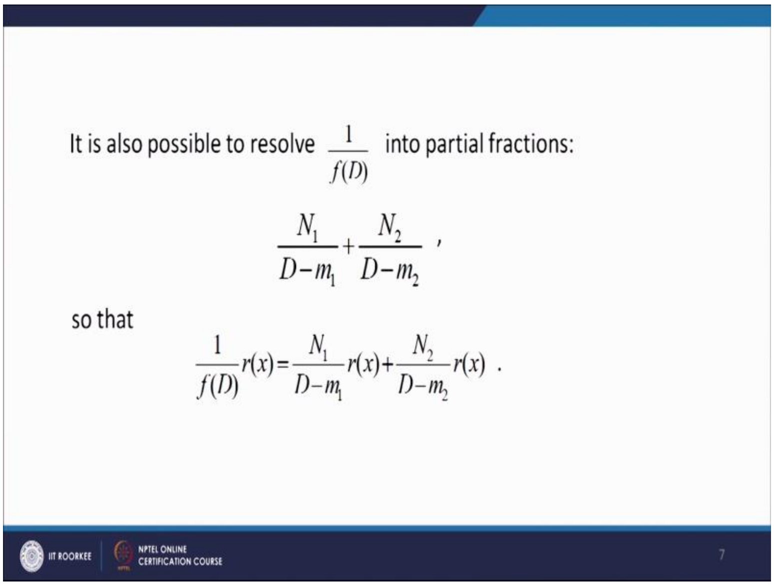

Now, this is one way of finding particular integral y p s where we operate by V operators 1 over D minus m 1 and 1 over D minus m 2. Now there is another method by which we can find the particular integral we can resolve 1 over f D into partial fractions. So, 1 f D is if it is resolved into partial fractions we have 1 over f D as 1 over f D is resolved into partial fractions as N 1 over D minus m 1 and plus N 2 over D minus m 2. So, that 1 over 1 over f D operates on r x, we have the operators N 1 over D minus m 1 operating on r x plus N 2 over D minus m 2 operating on r x.

Now, we can see the following, I mean see if you look at this expression to find 1 over f D r x, you have to operate on r x by the operator 1 over D minus m 1 and then whatever is the result of that you have to multiply that by the constant N 1. Similarly here when you want to find this function of x, you have to operate on r x by the operator 1 over D minus m 2 and then whatever is the result then you have to multiply that y and 2.

So, here the what we are doing is we are operating on r x by an operator of the kind 1 over D minus alpha where alpha is a constant in the previous method here also if you apply this method to find the particular integral we are operating on r x y 1 over D minus m 2 are 1 over D minus m 1 and then we operate by the other operator of the same kind on r x on the result of the operation which we get by operating on r x by one of the operators. So, when operator 1 over y are D minus m 2 on r x whatever function we get we operate on that by 1 over D minus 1 and. So, here also we are operating on r x by an operator of the kind 1 over D minus alpha.

So, in both the methods, what we are doing we are doing in order to find the particular integral we are operating by an operator of the kind 1 over D minus alpha on r x.

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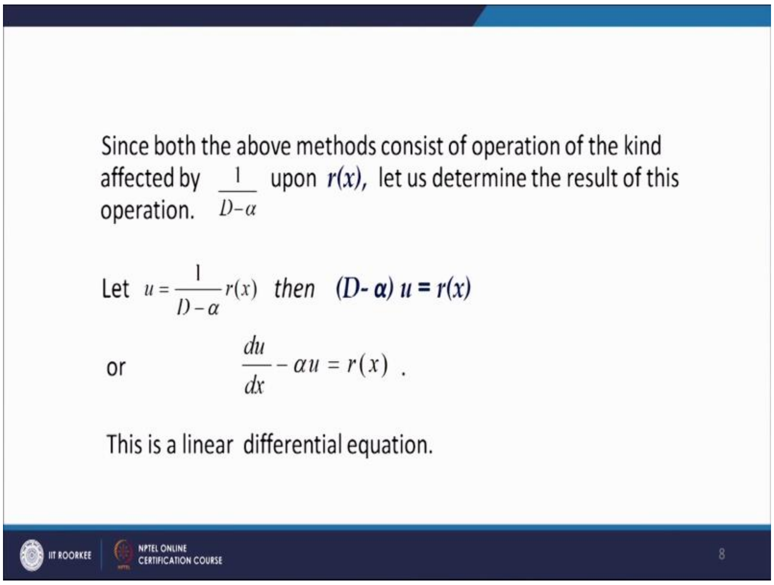

So, let us see how we get the; when we operate on the r x by an operator of the kind 1 over D minus alpha what function of x, we get; see let us see; let us find this. So, let us say that let 1 over D minus alpha when operates on, r x you get a function of x say u x. So, u x equals to 1 over D minus alpha r x.

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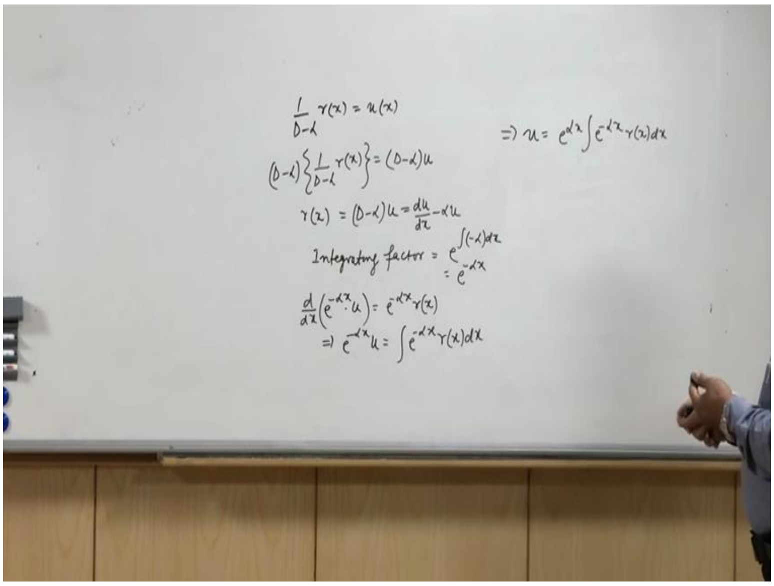

We get a function say u x, alright u is a function of x; now let us operate on this equation by D minus alpha on both sides. So, D minus alpha when op when we operate by D minus alpha on this equation, since D minus alpha and 1 over D minus alpha are inverse to each other, we will get r x equal to D minus alpha u. Now since D is d over dx. So, we get d u by dx minus alpha u. So, what we get is d u by dx minus alpha u equal to r x which is a linear differential equation of the first order with constant coefficient alpha is a constant and we know how to find the solution of this differential equation d u by d alpha d u by dx minus alpha u equal to r x we find the par the integrating factor of this equation.

So, integrating factor is e to the power integral minus alpha dx which is equal to e to the power minus alpha x then the then we know that we when we multiply this equation by the integrating factor e to the power minus alpha x, the left hand side becomes an exact it becomes an exact equation. So, e to the power minus alpha x into u d over dx of that is equal to e to the power minus alpha x into r x after multiplying by the integrating factor the left hand side becomes d over dx of e to the power minus alpha x into u which gives now let us integrate this equation with respect to x.

So, this gives you e to the power minus alpha x into u equal to integral e to the power minus alpha x into r x dx. E to the power minus alpha x is never 0. So, we can write it further as; I am not writing a constant of integration here because the particular integral is free from arbitrary constants. So, 1 over D minus alpha when f acts on r x the function u that we get is given by this formula e to the power alpha x integral e to the power minus alpha x into r x dx, now let us see for example.

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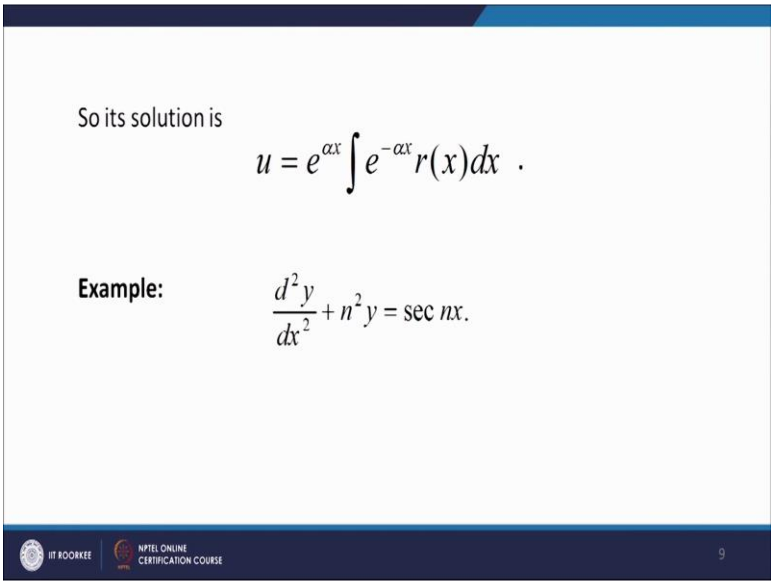

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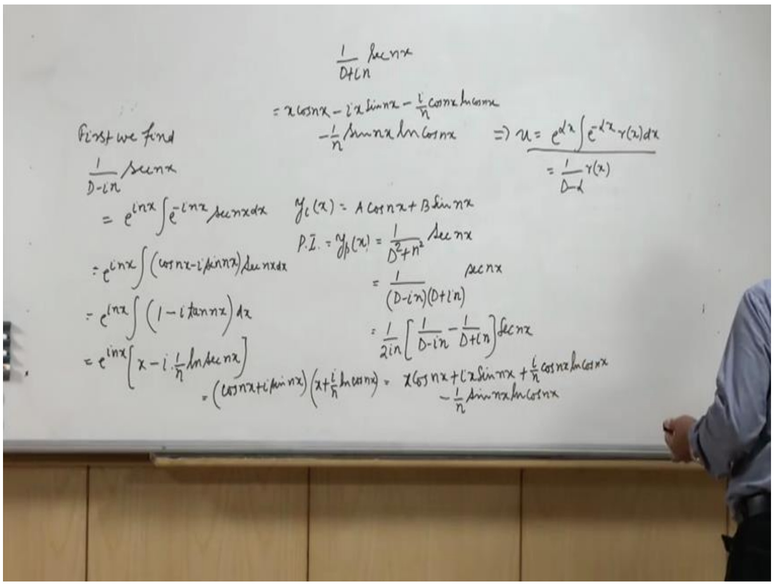

Let us see second-order differential equation with constant coefficient and see how we can find the particular integral. So, D square y over dx square plus n square y, or the auxiliary equation is m square plus n square equal to 0 which give us the complex roots m equal to plus minus I n and so the complimentary function y c x is given by A cos n x plus b sin n x. Now let us find the particular integral here. So, particular integral y p x this is equal to 1 over D square plus N square because the differential equation given differential equation can be written as D square plus n square operating on y equal to sec n x. So, 1 over D square plus f D is D square plus n square so 1 over D square plus square sec n x.

Now, let us write the function f D in terms of its factors 1 over D minus I n, D plus I n. Now as I said we can then find the opera the result of the operation of 1 over D plus I n on sec n x first are the result of the operation of operating by 1 over D minus I n on sec n x first and then we can on the result of what; which we get on that we can operate by the other thing. Now here we can also break it into partial fractions. Let us suppose, we are we wish to bracket into partial fractions then we will write it as 1 over D minus I n minus 1 over D plus I n divided by 2 I n that gives us 1 over 1 minus 1 over D minus I n into 1 over into d plus I n. Now let us find; let us apply this formula this is nothing, but 1 over D minus alpha operating on r x. So, let us make use of this formula. So, let us find; first we find 1 over D minus I n operating on r sec n x. So, alpha is I n here. So, e to the power I n x integral e to the power minus I n x into sec n x dx.

Now, this is nothing, but e to the power I n x integral; let us apply the Euler's formula here, e to the power minus I theta is cos theta minus I sin theta; so cos n x minus I sin n x into sec n x dx. Now this is further; if you solve it, this gives you cos n x into sec n x is 1 minus I tan n x dx. Now we can easily integrate this. This is equal to e to the power I n x integral dx is x minus I times the integral of tan n x is 1 over n; l n sec n x because I n sec n x when you differentiate with respect to x, what you get is 1 over sec n x then sec n x into tan n x into n.

So, we get this, this is what we get and then we can write further this as e to the power I n x is cos n x plus I sin n x and this is to be multiplied by. So, if I write l n 1 over cos x then it will be minus l n cos x. So, x plus I y n l n cos x cos n x so, what we will get? Let us multiply this, this will be x cos n x; x cos n x plus I times x sin n x and then I y m cos n x l n cos n x and then mi I square is minus 1 so minus 1 over n minus 1 over n sin n x l n cos n x. So, this is what we get similarly we can find the value of 1 over d plus I n 1 over d plus I n operating on sec n x if you obtain that in a similar we will get . So, 1 over d plus I n sec n x if we get we simply have x cos n x minus I x sin n x and then minus I by 1 n cos n x l n cos x n x and then plus no sorry, minus 1 by n sin n x l n cos n x. So, then we 1 over D minus I n sec n x we have to subtract 1 over d plus I n sec n x if we; so that and divide 2 I n what do we get?

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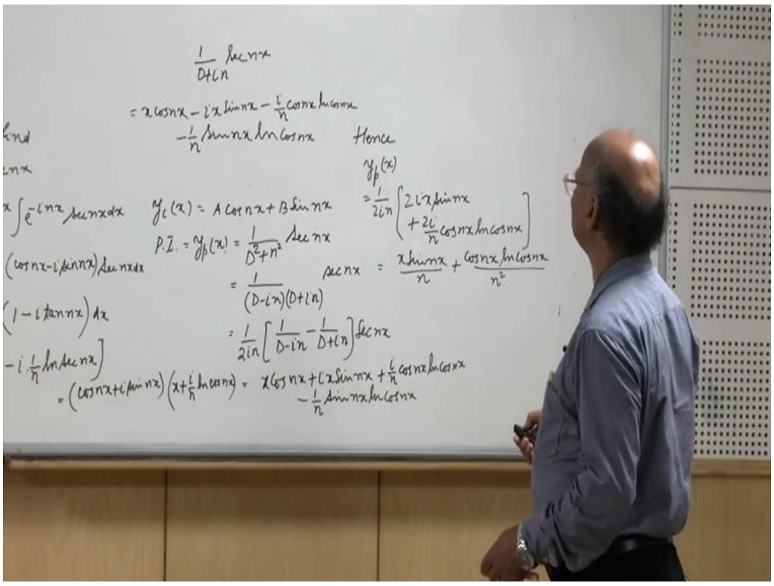

So, hence y p x is 1 over 2 I n, now from this value; from this value we subtract this. So, x cos n x will cancel I x sin n x and I x sin x will add up to I x sin n x, we will get and then I y n cos n x l n cos n x and I y n cos n x l n cos n will add up. So, 2 I y n cos n x l n cos n x we get and then this expression and this expression will cancel each other. So, we have we divide by 2 I n. So, we get x sin n x divided by n and then here we get cos n x.

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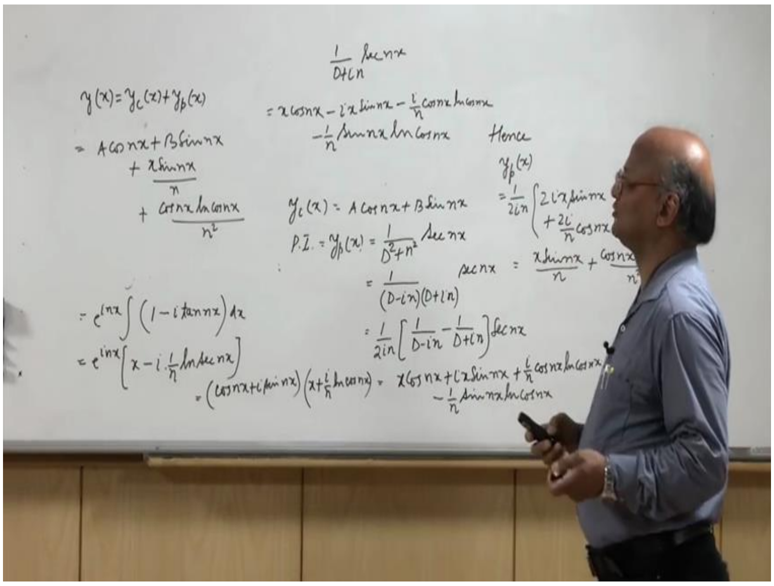

So, hence the general solution of the differential equation D square over by dx square plus n square by a equal to sec n x we can write as y x equal to y c x plus y p x which is equal to a cos n x plus b sin n x plus x sin n x by n plus x sin n x by n plus cos n x l n cos n x divided by n square. So, that the general solution of this differential equation.

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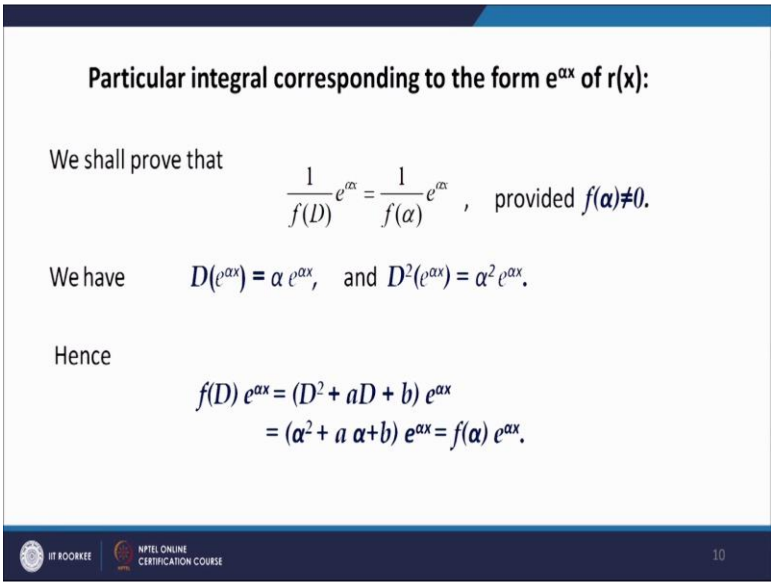

And now we are going to study how to find the particular integral of 1 r x of these some special forms say the first result that we are going to consider is the result when r x is in exponential function. So, suppose r x is e to the power alpha x where alpha e some real number alpha can be any real R complex constant. So, 1 over f D when operates on e to the power alpha x. Let us find the particular integral when r x is e to the power alpha x. So, 1 over f D e to power alpha x gives you 1 over f alpha e to the power alpha x provided f alpha is not equal to 0. We can make use of this formula to determine the particular integral when r x is of the type e to the power alpha x. Now to prove this formula, we can see that when we differentiate e to the power alpha x with respect to x, what we get is alpha times e to the power alpha x if we again differentiate this with respect to D that is we find the second derivative of e to the power alpha x with respect to x which is given by D square e to the power alpha x we get alpha x square e to the power alpha x.

Now, f D when operates on e to the power alpha x gives you f D is the D square plus a D plus b. So, D square plus a D plus b when operated on e to the power alpha x, we get alpha x square e to the power alpha x then a times alpha e to the power alpha x then b into e to the power alpha x. So, we can we get alpha a square plus a alpha plus b into e to the power alpha x. Now this alpha a square plus alpha; a alpha plus b can be written as f alpha.

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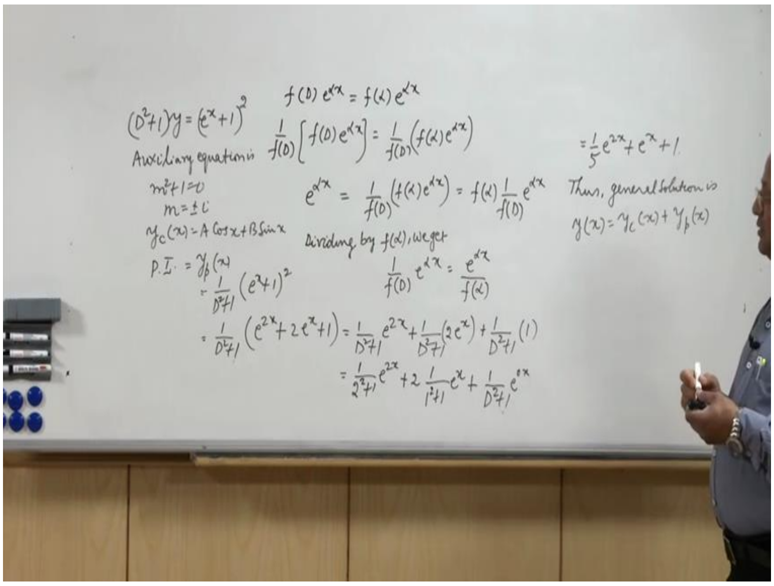

So, we get f alpha into e to the power alpha x. So, f D when operates on e to the power alpha x gives us now let us suppose that f alpha is not equal to 0. So, when f alpha is not equal to 0, let us first operate on both sides by the operator 1 over f D, since 1 over f D and f D are inverse operators we have left hand side as e to the power alpha x and the right hand side is 1 over f D operating on f alpha e to the power alpha x.

Now, f alpha is a non 0 algebraic multiplier, we have assumed alpha to f alpha to be non 0. It is a non 0 algebraic multiplier. So, this is same as f alpha times 1 over f D operating on e to the power alpha x now let us divide this equation by f alpha. So, dividing we get 1 over f D operating on e to the power alpha x equal to e to the power alpha x over f alpha. So, this is the proof and this method fails when it happens that f alpha turns out to be 0 let us apply this method to find the particular integral in the case of the example D square y over dx square plus y equal to e to the power x plus 1 whole square.

So, this differential equation can be written as D square plus 1 y equal to e to the power x plus 1 whole square let us first write the complimentary function. So, the auxiliary equation is m square plus 1 equal to 0 that is m equal to plus minus I. So, complimentary function y c x equal to a cos x plus b sin x let us find the particular integral y p x this is given by 1 over f D; f D is D square plus 1 operating on r x. R x is e to the power x plus 1 whole square now this e to the power r x is e to the power x plus 1 whole square. So, we have to square this and write. Now this is nothing, but we operate by 1 over D square plus 1 on each of the terms of this bracketed expression.

Now this is 1 over D square plus 1 e to the power 2 x can be found from the formula 1 over f D e to the power alpha x equal to e to the power alpha x divided by f alpha because here alpha is equal to 2 and when we replace d y in f D, D square plus 1 when we replace D y alpha there it is f alpha is not equal to 0. So, this is equal to 1 over 2 square plus 1 e to the power 2 x. Now 2 is an algebraic multiplier I can write it 2 times 1 over D square plus 1 operating on e to the power x. So, alpha is one here. So, I write 1 over one square plus 1 e to the power x and here one can be regarded as e to the power 0 x e to the power 0 x. So, alpha can be taken as 0 and we can get the value of this. So, this is equal to further 1 over five e to the power 2 x then 2 over 2. So, e to the power x and then alpha is 0 here. So, 1 over 0 square plus 1, then general solution is y x equal to y c x plus y p x.

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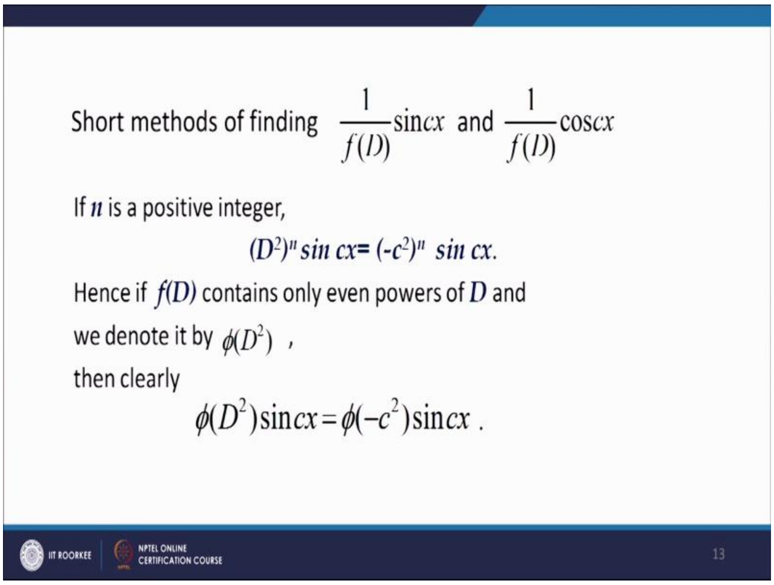

Now, let us go to short methods of finding 1 over f D sin c x and 1 over f D sin cos c x where c is any real number. Now we are finding short methods of finding the result of operating y 1 over f D on sin c x of 1 over f D operating on cos c x. Now if n is a positive integer, notice that D square to the power n sin c x is equal to minus c square to the power n and sin c x, let us prove this when we operate y c on sin c x that is we differentiate sin c x with respect to x we get c times cos c x.

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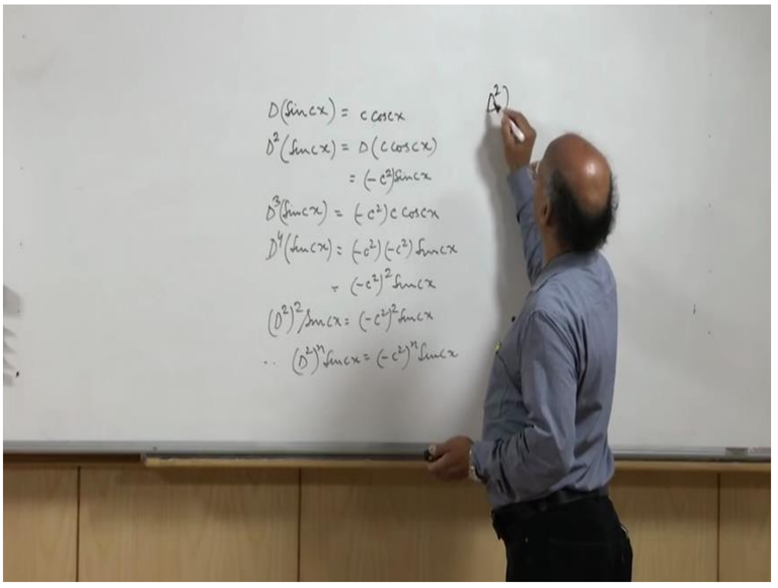

Let us again operate by d on this c cos c x so that is D square 9 c x gives you D of c cos c x which is equal to minus c square sin c x. I can write it as minus c square times c x. Now D cube then sin c x will be minus c square into c cos c x and D to the power 4 sin c x will be equal to minus c square into minus c square into cos c x which can be written as minus c square is square cos sin c x sorry, sin c x. So, minus c square is square sin c x and thus we can say that D square is square sin c x is nothing, but minus c square is square sin c x it can be by mathematical induction we can extend this to and. So, on D square to the power N sin c x is equal to minus c square to the power N sin c x.

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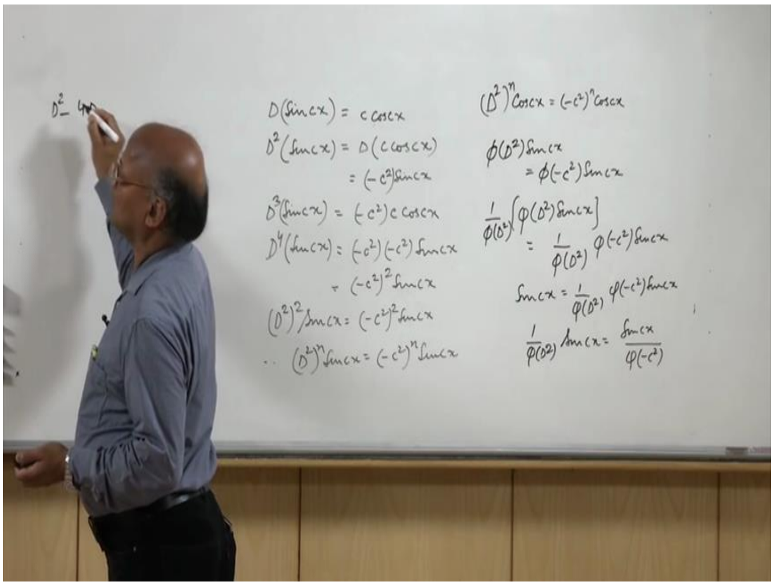

Similarly, we can find D square to the power n cos c x, this will come out to be minus c square to the power n cos x. Now let us see suppose f D the expression of f D contains only even powers of D and we denote it by phi D square then from here we can see that phi D square one x on sin c x the effect of this operation of phi D square on sin c x will be phi minus c square sin c x because here we see that whenever we operate by D square on sin c x we get D square to power N on sin c x we get minus c square to the power N sin c x. So, phi D square when updates on sin dx will get phi minus c square sin c x now, so what we get is when phi D square operates on sin c x we get phi minus square sin c x.

Now, let us assume that phi minus a square is non 0 phi minus c square is non 0 then if we operate by 1 over phi D square on both sides phi D square and 1 more phi D square are inverse to each other will get sin c x equal to 1 over phi D square phi minus c square is a algebraic multiplier non 0, we divide it by minus phi c phi minus c square and get 1 over phi D square operating on sin c x equal to sin c x divided by phi minus c square.

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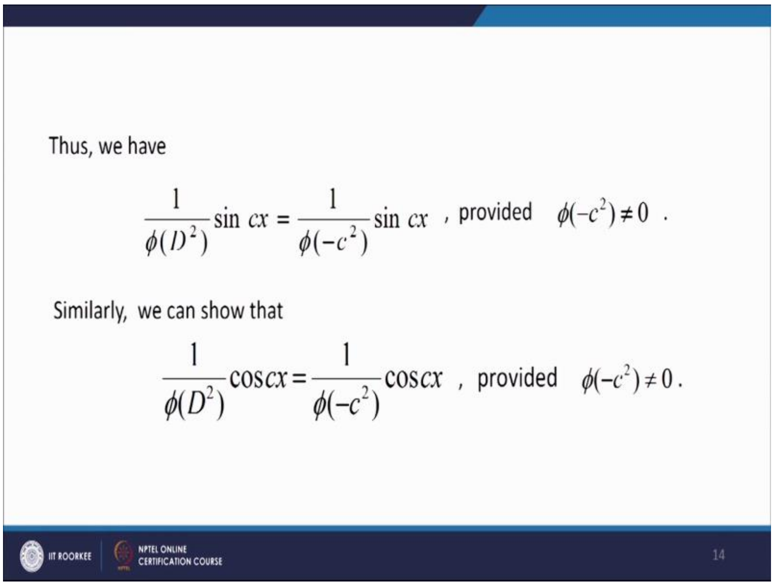

So, similarly we can show that 1 over phi D square one operates on cos c x, we get 1 over c minus c square cos c x provided phi minus c square is non 0 more generally if we have to operate by 1 over phi D square on sin c x plus d we are c and d are com constants then we get 1 over phi minus c square sin c x plus d b can similar they prove this in a similar manner.

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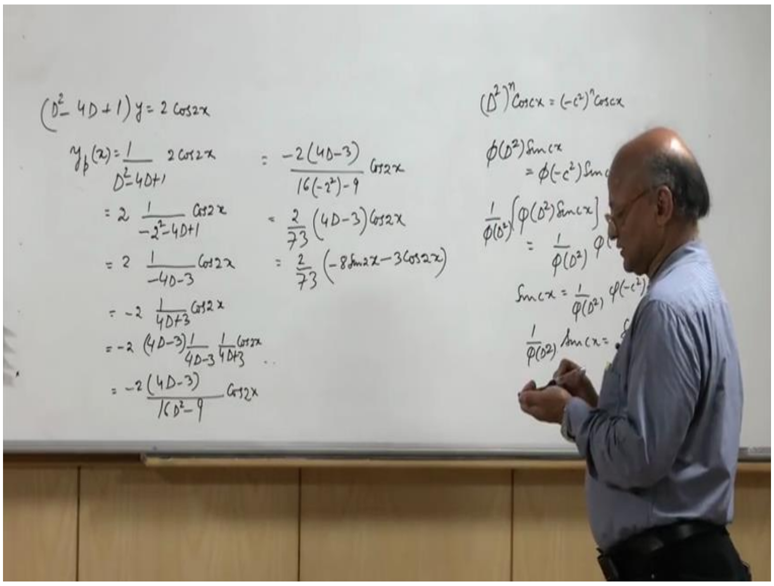

Now, let us take an example suppose we take the example one example one can be written as D square minus 4 d plus 1 operating on y equal to 2 cos 2 x, we know how to find the complement function. So, I will discuss only particular integral. So, y p x equal to 1 over f D which is D square minus 4 d plus 1 operating on 2 cos 2 x 2 is an algebraic multiplier I can write it outside now a here c here is 2. So, we replace D square by minus c square. So, 1 over minus 2 square for D square we write and minus 4 d plus 1 be leave just like that cos 2 x this is 2 times 1 over minus 4 plus one. So, we get minus 4 D minus 3 cos 2 x and this also equal to minus 2 times 1 over 4 D plus 3 operating on; now what we do is to get the operation of one 4 D plus 3 operating on cos 2 x, let us operate by 4 D minus 3 on cos 2 x and 1 over 4 D minus cos 3. So, minus 2 times we write 4 D minus 3 and 1 over 4 D minus 3. They are inverse of each other. So, we get this now this will be equal to minus 2 in the denominator we have sixteen D square minus nine cos 2 x again replace D square by minus a square we shall have. So, minus 2 times D square by minus a square, so 16 times minus 2 square minus nine cos 2 x. So, this is minus 64 minus 9. So, that is minus seventy 3. So, we get 2 by 73 and then 4 D minus 3 operating on cos 2 x.

Now, we have to operate by the operator 4 D minus 3 on cos 2 x. So, we get 2 times 2 over 73 4 D cos 2 x means derivative of cos 2 x with respect to x. So, minus 8 sin 2 x minus 3 cos 2 x, so this is the particular integral in this case ns we add to this the complementary function and write the general solution.

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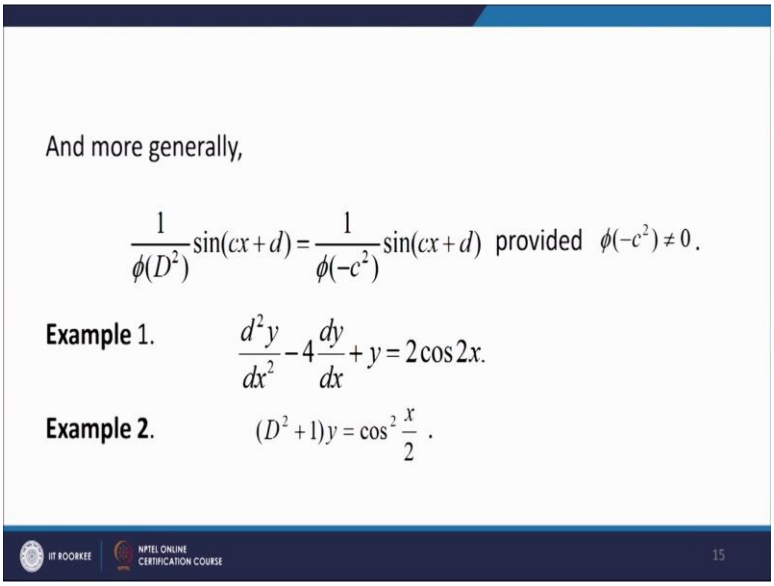

Now, in the case of example 2 D square plus 1 y equal to cos square x by 2, what we do is to find the particular integral we shall write now let us find the particular integral in the case of example 2. So, in the case of example 2 particular integral y p x is equal to 1 over D square plus 1 cos square x by 2. Since this is cos square x by 2, we have to convert it to cosine x function to apply the formula that we have just now proved. So, we shall write it as 1 over D square plus 1; 1 plus cos x by 2 and this will then done by 2; 1 over D square plus 1 operating on one plus cos x.

So, this will be one by 2 1 over D square plus 1 operates on 1, 1 can be regarded as e to the power 0 x. So, we will get 1 over 0 square plus 1 and then 1 over D square plus 1 operates on cos x we shall replace D square one by D square by minus 1 square plus 1, which is not defined. So, how we shall deal with this in example we shall see in our lecture tomorrow because here 1 over D square plus 1 becomes 0 when D square is replaced by minus a square.

So, this example we cannot find a particular example like this will be found this will be tackled in my lecture which I give tomorrow.

Thanks.