Mathematical methods and its applications
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Lecture - 59
Applications of Fourier transforms to BVP – II

So, welcome to the lecture series on Mathematical Methods and the Applications. So, in the last lecture we have deal applications of Fourier series that how Fourier series we can apply to boundary value problems. Now in this lecture we will see applications of Fourier integrals and Fourier transforms. However, the main heading is applications to Fourier transforms, but in this we will see applications of Fourier series, Fourier transforms and Fourier integrals.

Now, take this equation Heat Equation this is a use of Fourier integrals, consider heat equation this.

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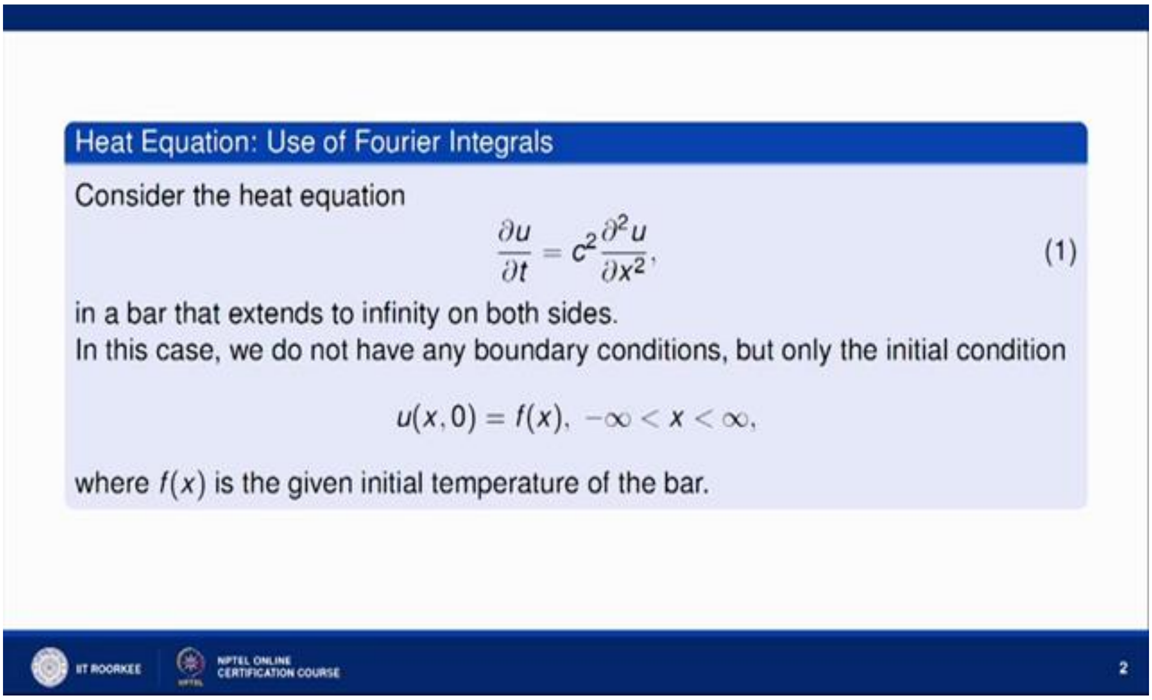

Now, the heat equation is given by del u by del t is equal to c square del square u by del x square. Suppose this is the heat equation.

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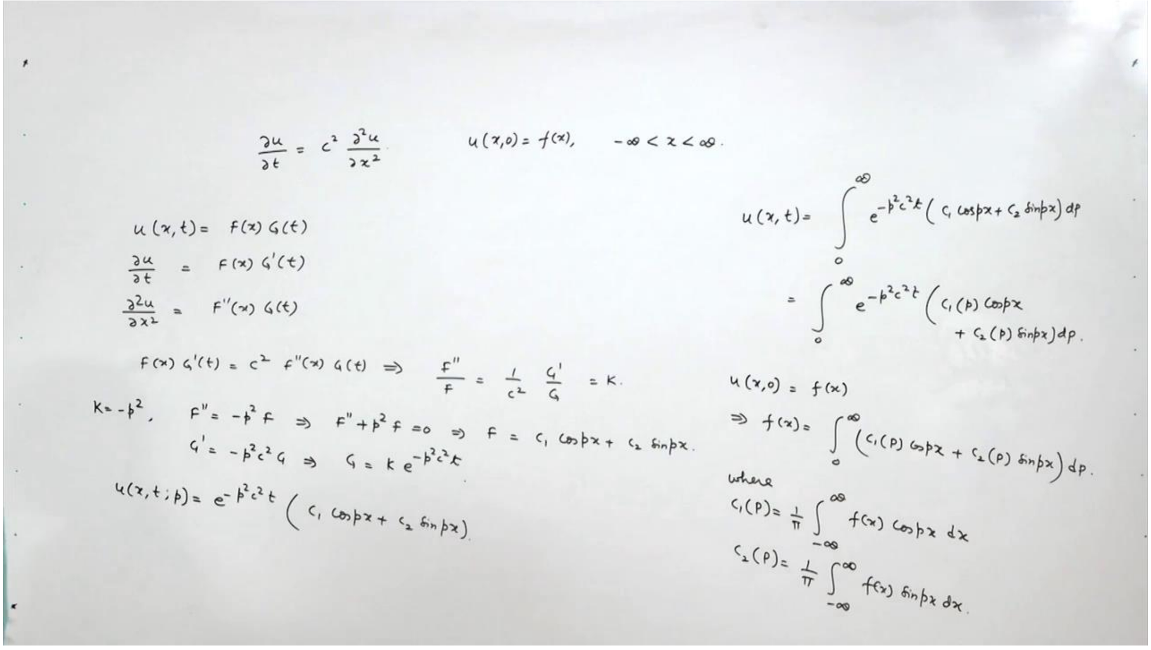

Now this is a bar which extend both to infinite on both the sides, that is bar is not of finite length it is tends to infinite from both the sides. Now in this case we do not have any boundary conditions like we have in finite bar when we have x from 0 to l, in that case we have a boundary condition, but when the bar is tending to infinite from both the sides we do not have any boundary conditions. However, we have the initial condition which is given by u x 0 is equal to f x and of course, x varying from minus infinity to plus infinity ok. So, here f x is a given initial temperature of the bar.

Now if you want to solve this problem, if you want to solve this problem here it is an infinite bar. So, how can we proceed again we will use separation of variables. So, we will take u which is function of x and t as F x into G t. So, what will be del u by del t? It will be nothing, but F x into G dash t and del square u by del x square will be nothing but F double dash x into G t. So, when we substitute these 2 in this equation we will get F double dash x, it is left hand side we have F x into G dash x G dash t will be c square F double dash x into G t. So, this implies F double dash upon F will be 1 by c square G dash upon G. Now, again this is a function of x only F is a function of x only and this is a function of t only and both are equal. So, both are equal only when it is a constant quantity. So, it is equal to K.

Again, as we did in the previous lectures when k is 0 or k is p square. So, we discard both the cases because if suppose here we do not have any boundary condition still if K is a positive quantity. So, F will be nothing, but c 1 e k power p s plus c 2 e k power minus p x which will tends to infinite as x tends to, as x increases, which does not have any practical significance. You see that if you take K equal to p square suppose and F double dash will be nothing, but F double dash will be nothing, but p square f. So, this will be nothing F from here we get e k power p x plus c 2 e k power minus p x and s as x increases, so this increase exponentially because of the presence of this term. So, this has no any, does not have any practical significance. So, we discard the 2 cases when k is 0 or k is equal to p square. So, we take K equal to minus p square.

So, if K equal to minus p square. So, F double dash will be nothing, but minus p square times f which implies F double dash plus p square f equal to 0 and this implies F will be nothing, but c 1 cos p x plus c 2 sin p x; c 1 cos p x and c 2 sin p x. And g will be nothing, but from this condition g dash will be nothing but it is minus p square c square G. So, this implies g will be nothing, but some K times e K power minus p square c square t because when we integrate both the sides we will get G equal to this.

So, what will be u? So, u x t will be nothing but the product of these 2 this is nothing, but e k power minus p square c square t into c 1 cos p x plus c 2 sin p x. This is not; this is u x t, this is u x t with a with a variables p. So, we write it like this with a variables p. So, we obtain u x t p like this.

Now since it is an infinite bar x is varying from minus infinity to plus infinity. So, the solution u x can be obtained and p is positive quantity. So, u x t can be obtained by integrating all the terms 0 to infinite it is e k power minus p square c square t and it is c 1 cos p x plus c 2 sin p x into dp. Now this c 1 c 2 are arbitrary constants here. So, we can take them as a function of p also. So, this c 1 c 2 we can take as a function of p. So, this is integral 0 to infinite e k power minus p square c square t and this is c 1 some function of p cos p x plus c 2 some function of p into sin p x whole with dp. This K will merge with when we take a port of these 2 this K will merge with c 1 and c 2. So, there is no need of writing K in this expression.

Now, u x 0 is given as f x, so this implies f x will be equal to. So, 0 to infinity c 1 p cos p x because substitute t equal to 0 here plus c 2 p sin p x whole with dp. So, this is for a integral, and c 1 p and c 2 p can be obtained by where c 1 p and c 2 p are nothing but; so, c 1 p is nothing, but it will be c 1 p will be given by minus infinity plus infinity; minus plus infinite f x cos p x into dx it is 1 by pi times I think it is 1 by pi times and c 2 p will be nothing but again 1 by pi times minus infinity to plus infinity f x sin p x into dp.

So, let us verify it 1 by pi or what. So, we can simplify like this f x is obtained like this and f 2 and G t is nothing, but this expression.

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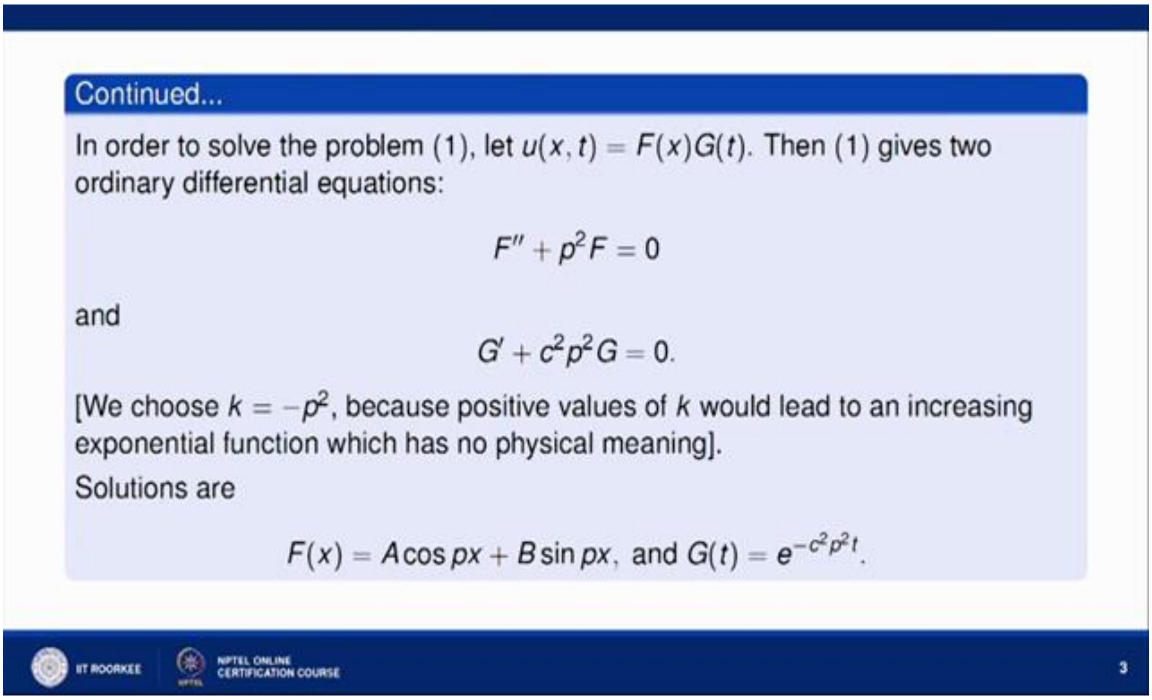

So, when we put it here and integrate over this we obtain this thing. Now u x 0 is this thing it is. So, a p and b p here it is c 1 p and c 2 p. So, c 1 p is this expression and c 2 p is this expression which we obtained here also the same thing. Now we substitute back to this expressions in this u x t. So, what we will obtain?

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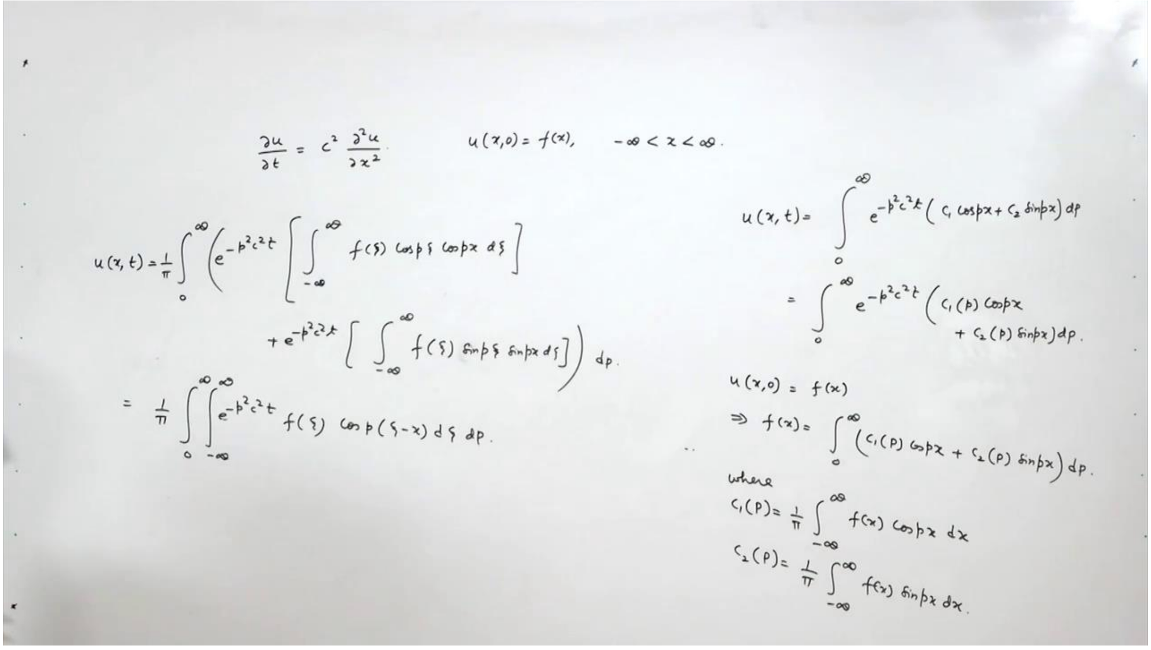

So, u x t will be 0 to infinity e k power minus p square c square t and c 1 is nothing but, c 1 p is nothing but it is integral minus infinity to plus infinity 1 by pi will can come out minus infinity to plus infinity it is f x and it is you can put it f psi or something f psi because x is also here and x is also here. So, it will be a confusion so you can take f psi here f psi cos p psi and it is cos p x d psi this term come here plus e k power minus p square c square t and with this minus infinity to plus infinity again it is f psi cos or sin it is sin p psi and sin p x d psi and whole multiplied by dp.

So, in this way we will get by this expression go on when we simplify this nothing but 1 by pi integral 0 to infinity e k power minus p square square t. So, it is again one more integral minus infinity to plus infinity and it is f psi is common from both the terms and it is cos a cos b plus sin a sin b which is nothing but cos a minus b, cos a minus b or d psi into d p. So, this will be the required solution of this differential equation when we have an infinite bar.

So, in this way we can solve we can solve such type of problems, we can solve such type of problems when the bar is tending to infinity from both the sides using Fourier integrals. Now come to a use of Fourier transforms.

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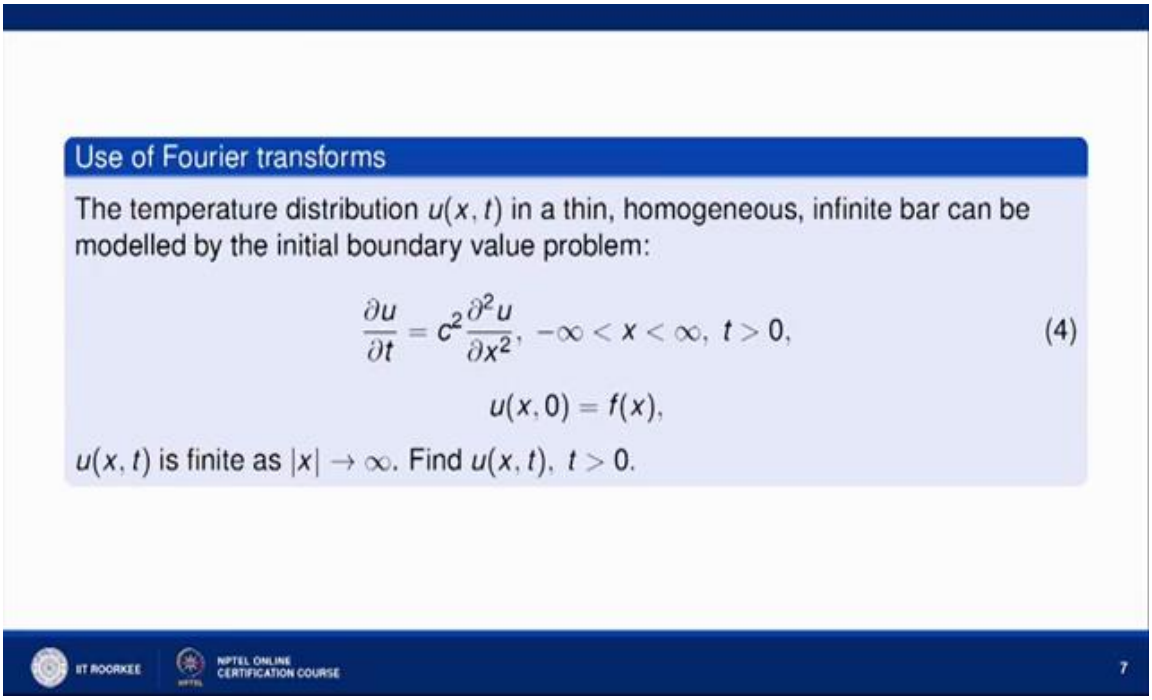

So, let us discuss this problems that temperature distribution u x t in a thin homogeneous infinite bar can be modelled by the initial boundary value problem this. So, now, how can we use Fourier transforms? So, we have already discussed the applications of Fourier series and Fourier integrals that how can we use Fourier series or Fourier integrals.

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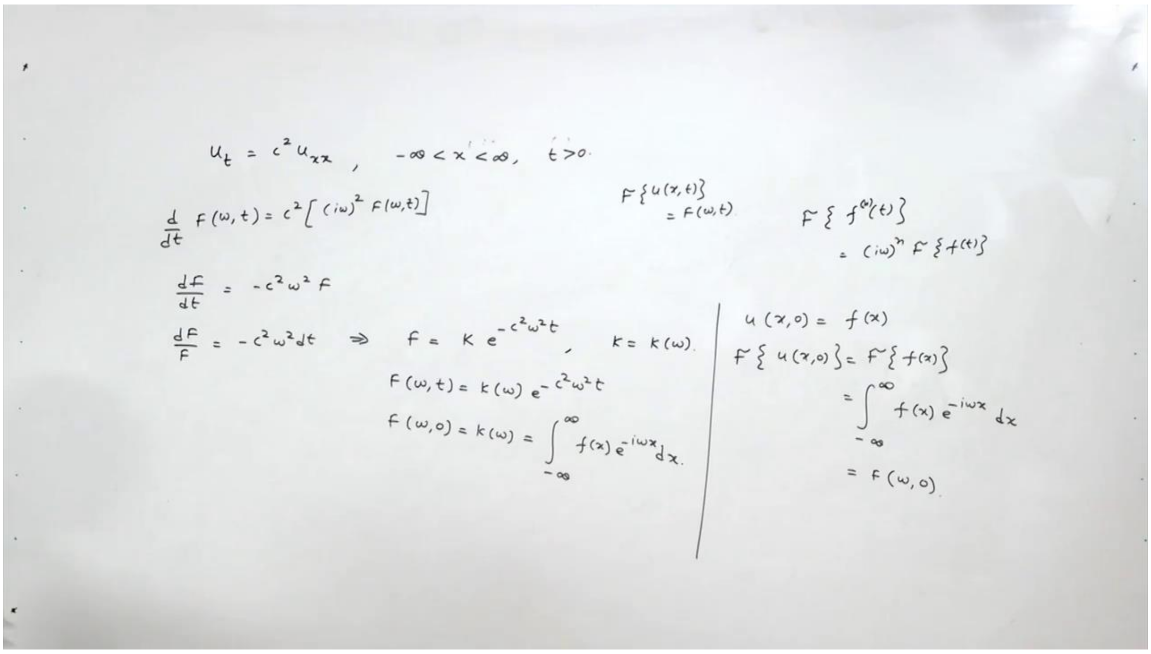

Now, let us discuss Fourier transforms. Now in this problem it is u t it is equal to c square u xx, u t equal to u xx this and minus infinite less than x less than infinite and t greater than 0. So, it is again an infinite bar x is varying from minus infinity to plus infinity and u x 0 is equal to f x it is given to us, it is also given to us that u x t is finite as x tending to plus or minus infinity. So, what will be u x t? u x t is a temperature distribution function so how can you find u x t again.

Now since x is varying from both the sides it is minus infinity to plus infinity. So, take Fourier transform both the sides. Now here x is varying from minus infinity to plus infinity take Fourier transform respect to x. So, first where we have to identify that with because it contain 2 variable, because u is nothing but function of 2 variable.

So, first we have to decide that respect to which variable Fourier transform is to be taken. So, now, it is x is varying from minus infinity to plus infinity. So, take Fourier transform respect to x. So, when you take Fourier transforms here, Fourier transform of u t which is nothing, but Fourier transform of del u by del t, it is nothing, but minus into plus infinity del u by del t of because we are taking Fourier transform with respect to x.

So, it will be e k power minus eta omega x d omega, eta omega x d x. So, since it is free from t. So, we can take del by del t outside. So, it will be d by d t of minus infinity to plus infinity u into e k minus eta omega x d x which is nothing but d by d t of Fourier transform of u, Fourier transform of u x t. So, that is why if we take Fourier transform of this side because we are taking Fourier transform with respect to x. So, when we take Fourier transform this side this is nothing but d by dt of suppose Fourier transform of u x t is F omega t. Suppose we are Fourier transform of u x t we are taking as F omega t. So, it is F omega t, Fourier transform of u x t is F omega t which is equal to c square times. Now Fourier transform u xx, now Fourier transform we already know that Fourier transform of nth derivative of t is nothing but, nth derivative of t is nothing but eta omega e k power n and Fourier transform of f t.

So, we are taking Fourier transforms respect to x. So, here it will be it is double derivative, so it is eta omega square Fourier transform of u x t which I taking as F omega t. So, when we simplify this, so this is df upon dt where f is a function of omega and t is equal to minus c square omega square into f. So, it is df upon F is equals to minus c square omega square dt. So, when we integrate this it will be x is equals to some K times e k power minus c square omega square t.

So, when we integrate both the sides we get this expression. So, this F is a function of, this F is a function of omega and t. So, this K will be nothing but, here K will be nothing, but function of omega. So, in this way we obtain we obtain F omega t which is nothing, but K omega e k power minus c square omega square t. Now it is given to us that u 0 x, u x 0 is equals to f x. So, u x 0 is equal to f x, now take Fourier transform both the sides. So, Fourier transform of u x 0 will be Fourier transform of f x and will be equal to minus infinity to plus infinity f x e k power e k power minus eta omega x into dx and that will be nothing, but F omega 0 that you can simply take, because you are putting t equal to 0, because Fourier transforms of u x t is F omega t and when we put t equal to 0 both the sides, so F omega 0 will be nothing, but Fourier transform of u x 0. So, Fourier transform of u x 0 will be nothing but F omega 0.

So, when you take F omega 0 over here u get K omega. So, this K omega will be nothing, but integral minus infinity to plus infinity f x e k power minus eta omega x into dx. So, now, take inverse Fourier transforms both the sides in this expression. So, what we are having. So, what we obtained basically?

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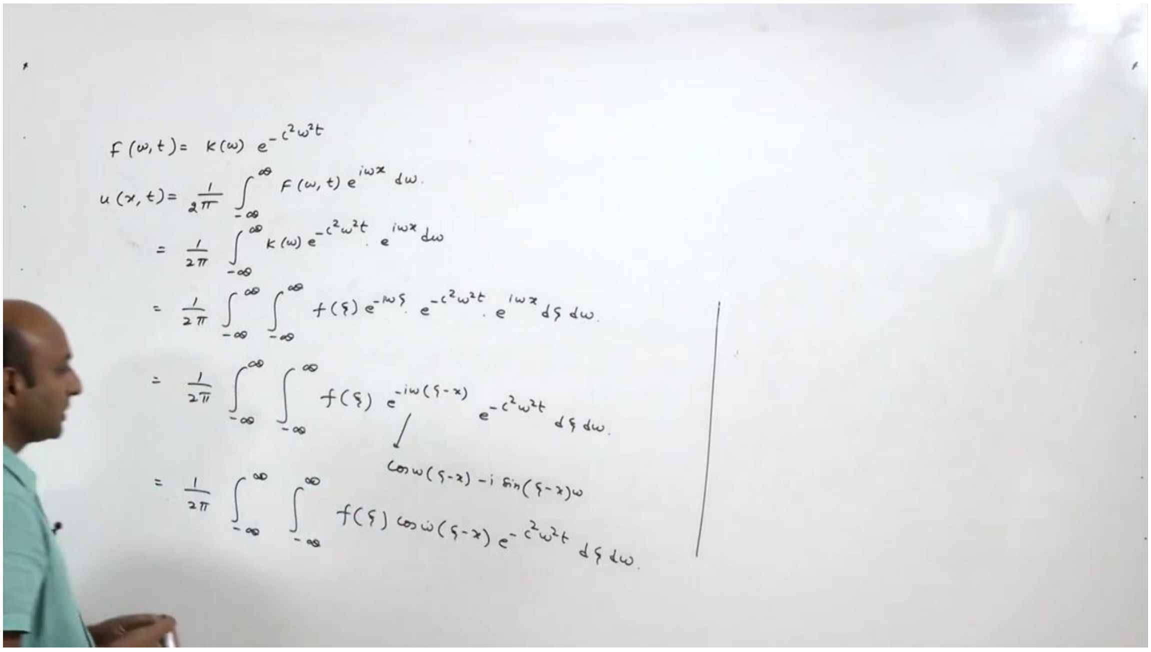

So, now if you take, now, F omega t is nothing, but K omega e k power minus c square omega square t where K omega is given by this expression. Now taking inverse Fourier transform both the side u x t will be given by, u x t will be given by 1 by pi, 1 by 2 pi I think, 1 by 2 pi integral minus infinity to plus infinity it is F omega t, e k power eta omega x into d omega. So, that will be equal to 1 upon 2 pi integral minus infinity to plus infinity it is k omega e k power minus c square omega square t into e k power eta omega x d omega.

Now, K omega is given by this expression to substitute it over here, you substitute it over here. So, what we will obtain? It is nothing, but 1 upon 2 pi integral minus infinity to plus infinity, it is again minus infinity to plus infinity suppose it is f zeta e k power minus eta omega zeta into e k power minus c square omega square t into e k power eta omega x d zeta into d omega.

So, when we simplify this. So, now, we further simplify this. So, 1 upon 2 pi integral minus infinity to plus infinity, minus infinity to plus infinity it is f zeta and e k power minus eta omega it is I minus x e k power minus c square omega square t d psi d omega. Now e k power eta zeta is e k power minus eta zeta is cos theta minus eta sin theta and it will be nothing but cos this expression is nothing but cos omega psi minus x minus eta sin psi minus x omega times.

Now, this is even function omega square even here, so this is even. Now sin is odd in omega it is even, even into odd is odd. So, this will be 0 from minus infinity to plus infinity so we will left with only one term which is 1 upon 2 pi integral minus infinity to plus infinity minus infinity to plus infinity it is f zeta into cos omega psi minus x e k power minus c square omega square t d psi p omega. So, this will be the final solution of, this will be the final solution of this. So, hence we will obtain the solution of the given differential equation, is it Ok.

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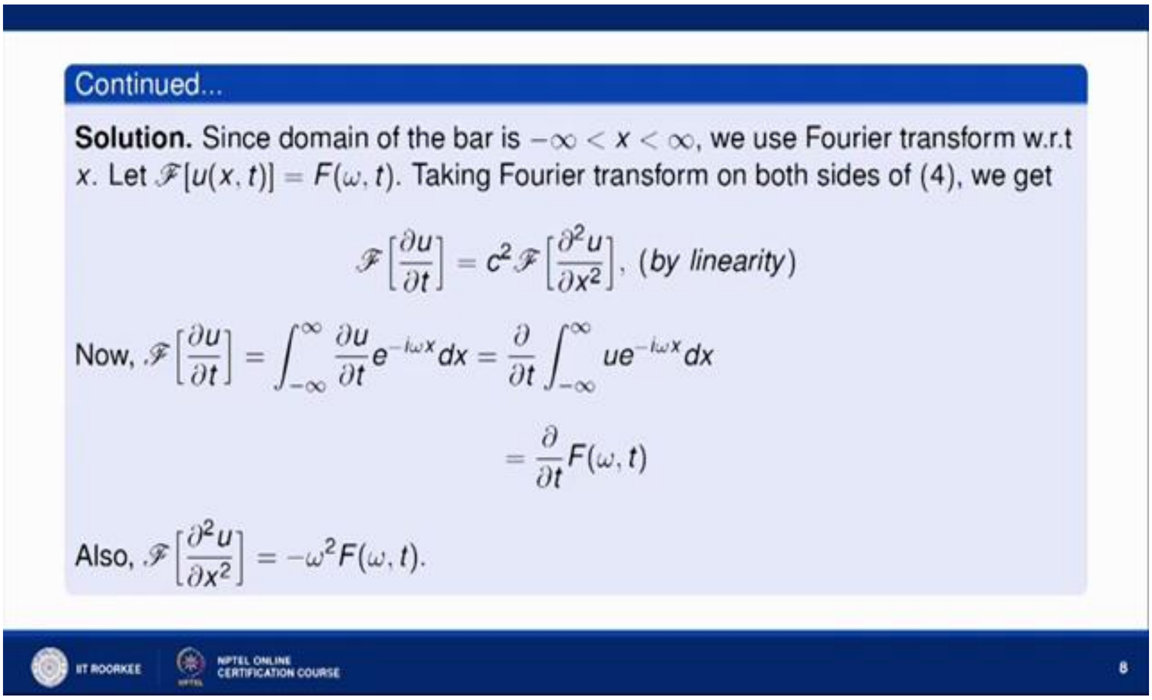

So, here we have solved the same problem, the technique which I have solved. So, when we simplify we obtained the same solution which we obtained over here. So, that is all for this problems now see one more problem using Fourier transforms, this is Laplace equation del square u upon del x square plus del x u upon del y square equal to 0 and x is tending from minus infinity to plus infinity u is finite u is varying from 0 to pi.

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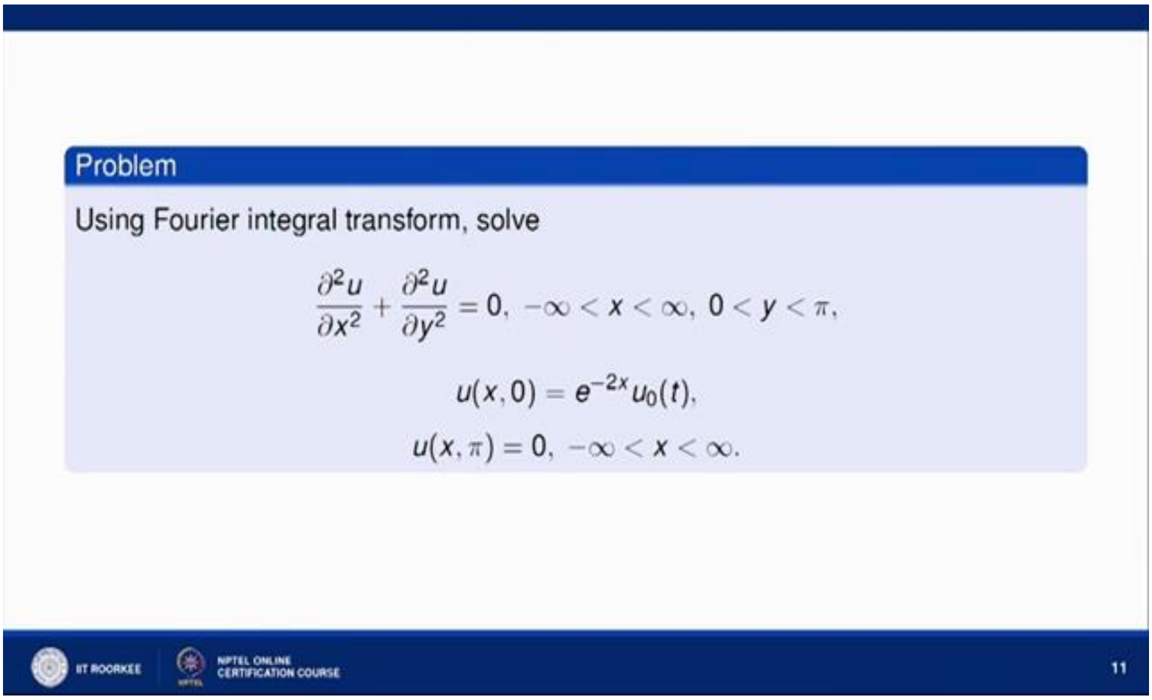

So, we will apply Fourier transforms respect to x. So, let us see how. So, what is a equation given to us.

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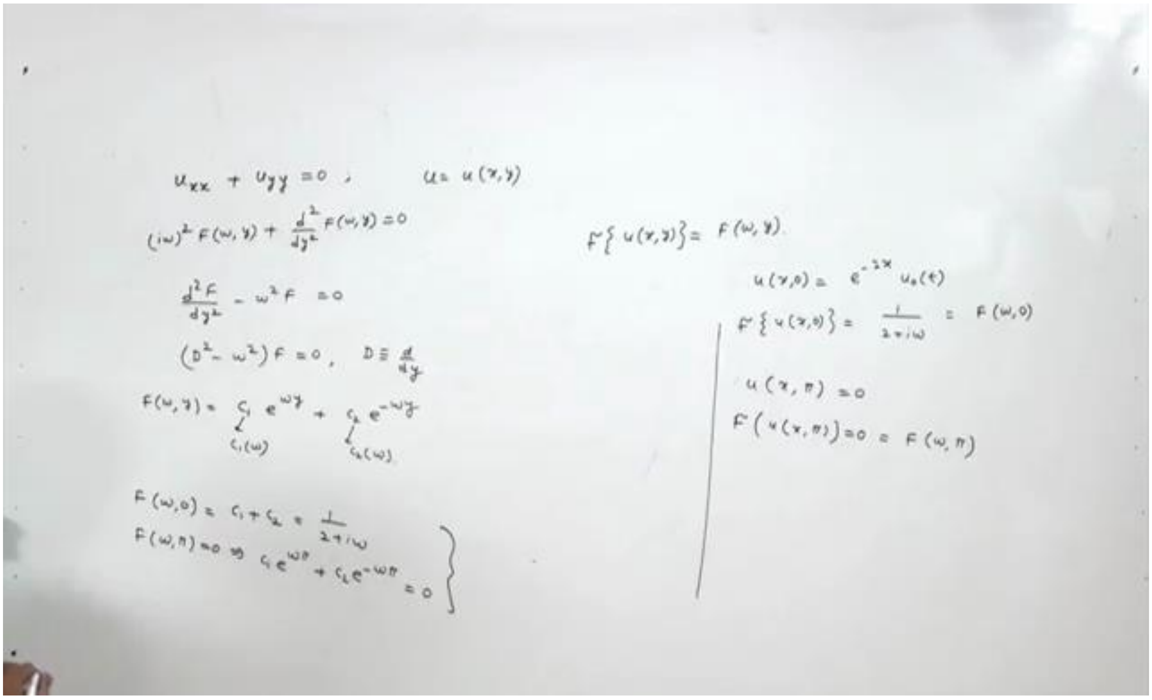

Now, see it is u xx plus u yy equal to 0, basically u is a function of x and y here. Now x is varying from minus infinity to plus infinity as given and u y is varying from 0 to pi, we have to apply Fourier transforms respect to x. So, apply Fourier transform both the sides. So, here you apply Fourier transform it is eta omega whole square Fourier transform of Fourier transform of u x y which I am taking as omega y. So, Fourier transform of u x y because I am applying Fourier transform respect to x.

So, it will be nothing, but I am assuming as F omega y plus, now why we are not taking respect to y. So, it remains as it is D square upon d square y of Fourier transform of omega y is equals to 0 Fourier transform of u, u x y, so that will be 0. So, it will be nothing, but D square f upon d y square minus omega square f equal to 0. So, this is D square minus omega square F equal to 0 where D is nothing but d upon d y when you simply, so F omega y comes out to be c 1 e k power omega y plus c 2 e k power minus omega y. Again c 1 and c 2 are arbitrary constant and it is a function of 2 variable omega and y. So, we can take c 1 as a function of omega and c 2 as a function of omega.

Now, we will apply the conditions given to us. Now u x 0 is, u x 0 is given to us as e k power minus 2 x u naught t take Fourier transform of both the sides. So, Fourier transform of u x 0 will be nothing but 1 upon 2 plus eta omega and that is nothing but when you substitute y equal to 0 here. So, that will be nothing but F omega 0. So, when you take F omega 0 here. So, when you take F omega 0 here, so F omega 0 is nothing, but c 1 plus c 2 which will be equals 1 upon 2 plus eta omega.

Now, second condition is u x pi equal to 0. Now u x pi equal to 0, so when you take Fourier transform both the sides u x pi it will be 0 which is equals to F of omega pi because when you substitute y equal to pi both the side we get back to this expression. So, from here when you take F of omega pi which is equal to 0. So, this implies c 1 e k power omega pi plus c 2 e k power minus omega pi will be equal to 0.

Now, solving these 2 equations we can get the values of c 1 and c 2. So, c 1 and c 2 we can obtain from here we can easily solve these 2 equations and find the values of c 1 and c 2 which we can substitute over here. So, what are c 1 and c 2 that we can obtain, I am not solving for c 1 and c 2.

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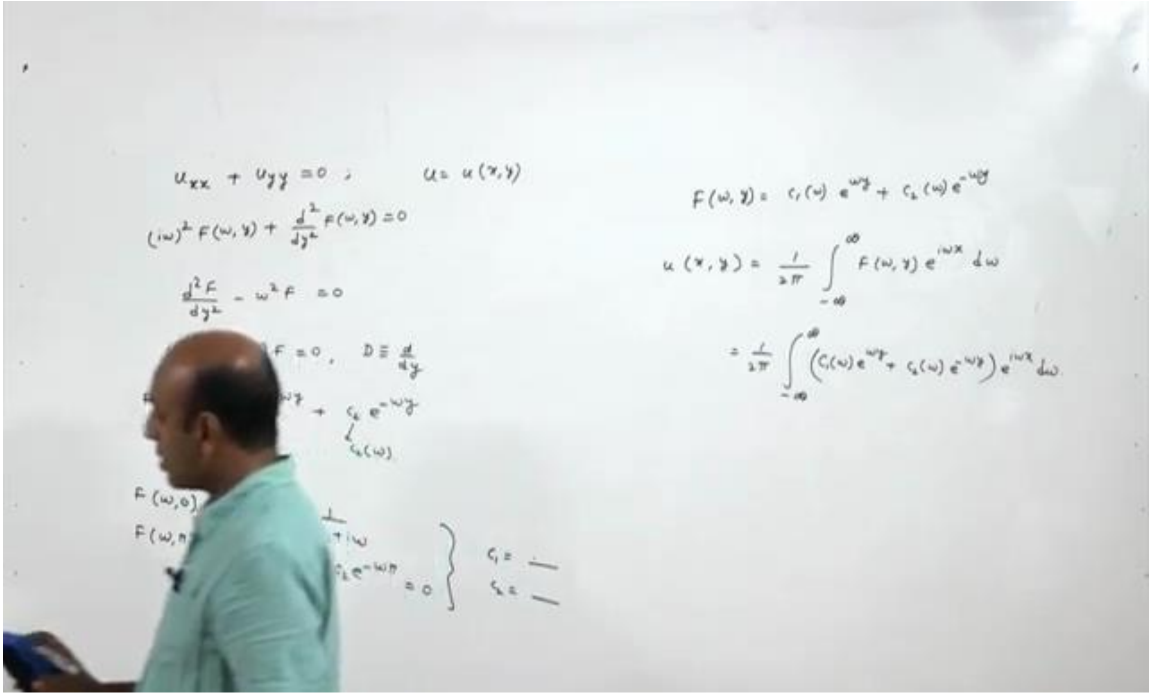

Now f x, F omega y is nothing but c 1 omega e k power omega y plus c 2 omega e k power minus omega y. So, here c 1 and c 2 are governed by these 2 equations we can simply we can solve these 2 equations find the values of c 1 and c 2.

So, the inverse, take the inverse Fourier transforms both the sides. So, it will be u x y will be equal to 1 upon 2 pi times integral minus infinity to plus infinity F omega y e k power eta omega x into d omega. So, it will be 1 upon 2 pi integral minus into plus infinite F omega y is c 1 omega, e k power omega y plus c 2 omega e k power minus omega y into e k power eta omega x d omega. So, whatever c 1 c 2 we obtain from these 2 equations we simply substitute it over here and we get the final solution we get the final answer u x y of this Laplace equation.

So, in this way whenever we have an x or y ranging from minus infinity to plus infinity we can solve those equations using Fourier transforms. So, that is all for this lecture.

Thank you.