Mathematical methods and its applications
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Lecture - 58
Applications of Fourier transforms to BVP – I

Welcome to lecture series on Mathematical Methods and it is Applications. So, we have already discussed Fourier series, Fourier integrals and Fourier transforms. Some problem faced on that we are already did. Now let us come to applications of Fourier transforms to bv p. So, in this section we also include applications of Fourier series and Fourier integrals also.

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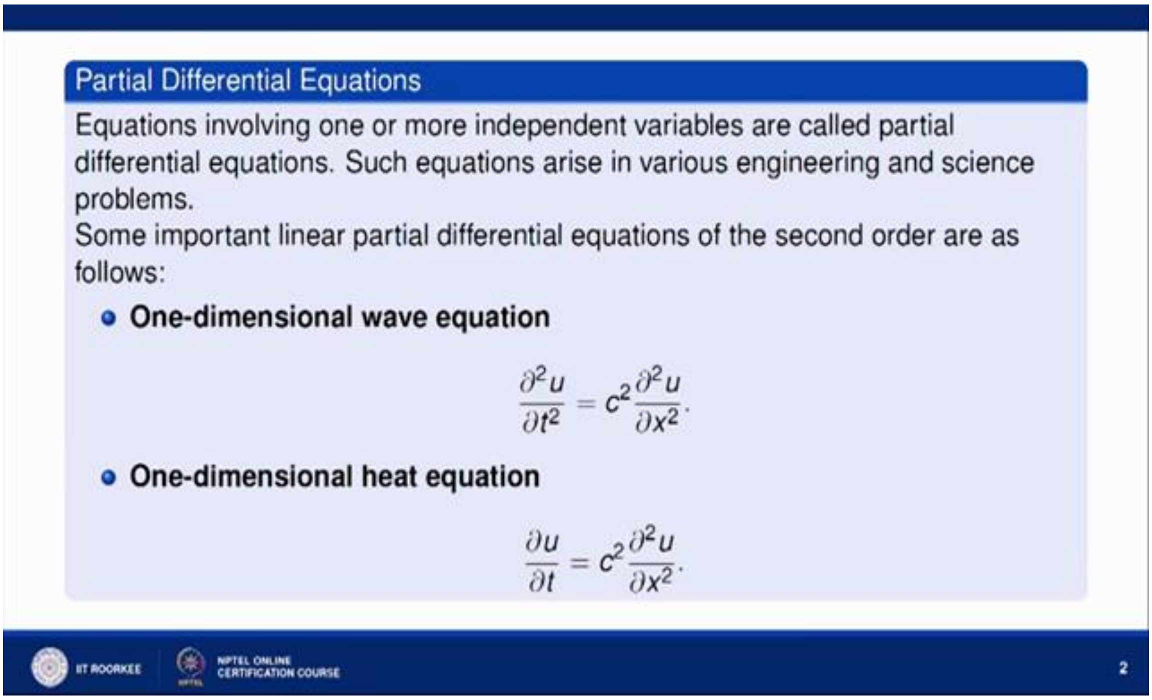

Now, equations involving one or more independent variables are called partial differential equation, this we already know. And such equation frequently arise in varies engineering and science problems. Some important linear partial differential equations of second order are as follows, that we already know that one-dimensional wave equations given by this expression. One-dimensional heat equation is given by this del u by del t, equal to c square, del square u upon del x square, two-dimensional Laplace equation is given by this, del square u by del x square, plus del square u upon del y square equal to

0. Two-dimensional poison equation is given by del square u upon del x square, plus del square u upon del y square equal to f x y.

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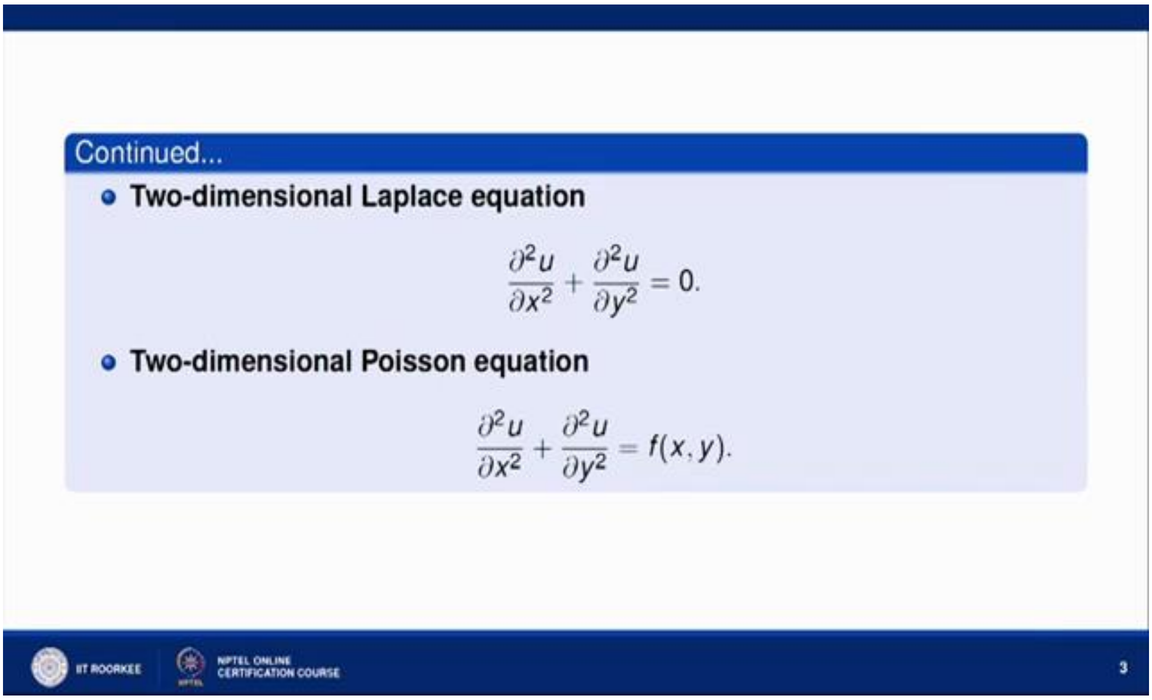

Similarly, two-dimensional wave equations you know this expression and 3 dimensional Laplace equations given by this, expression these are the important partial differential equations. Now can we you solve a given partial differential equation using Fourier series. First of all, we will see using Fourier series. So, we use separation of variables. Now suppose a partial differential equation is given to you any partial differential equation, it may be a one-dimensional wave equation a one-dimensional heat equation or two-dimensional wave equation anything. So, how you solve it?

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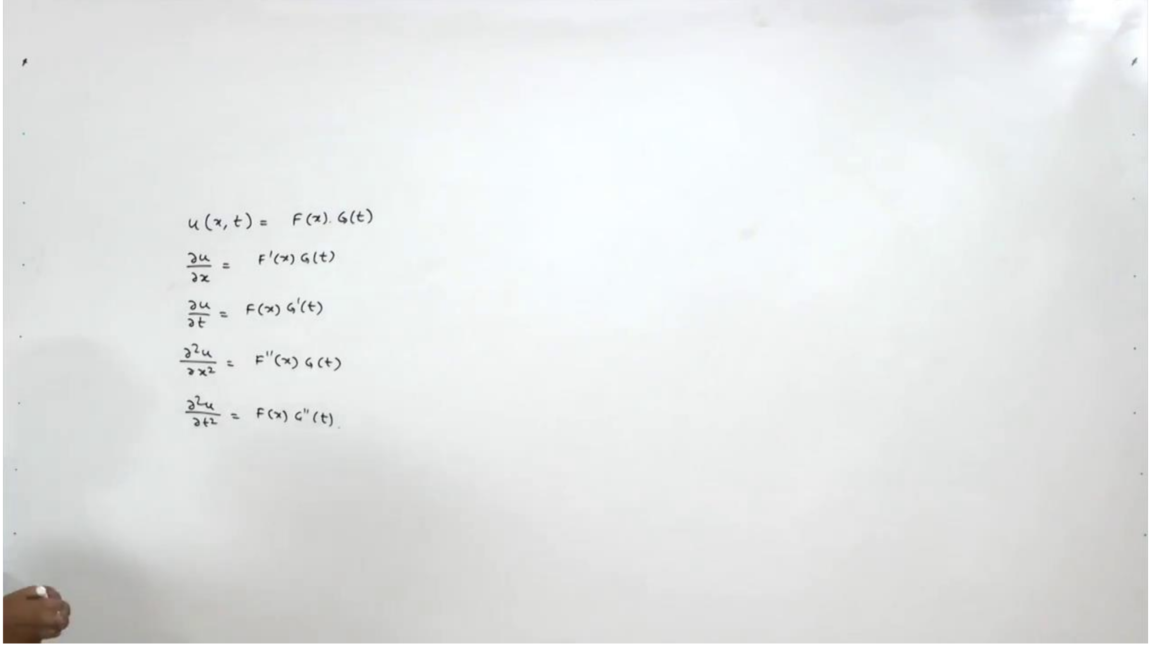

You assume that u x t which is a function of 2 independent variables is simply f x into G x f x, into G t this we assume. So, we basically adopt this method which we call as separation of variables. Now we find del u by del x, what it is it will be nothing but F dash x into G t, because this function is only a function of t. Suppose you find del u by del t. So, that will be nothing but f x into G dash t. So, del square u by del x square, that will be nothing but F double dash x into G t. And similarly del square u by del t square will be nothing but f x into G double dash t.

Now, we substitute this in a given partial differential equation, whatever partial differential equation is given to us, we substitute these values in that equation. Now when you substitute this in that given partial differential equation, we get back 2 ordinary differential equations, which we can easily solve. Now we apply initial and boundary conditions whatever given to us. And hence we can find u x t using Fourier series, then using Fourier series we can find u x t.

So, let us illustrate this method by some examples. So, this is all we have discussed. Now first of all consider one-dimensional wave equation using Fourier series.

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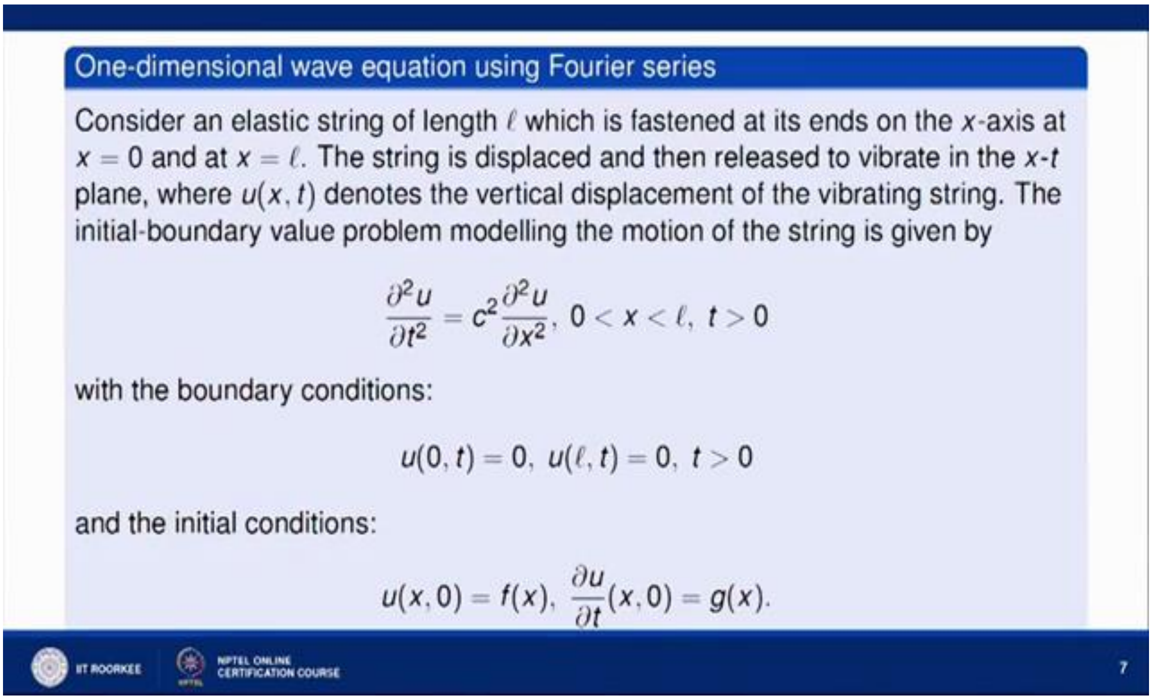

So, consider an elastic string of length l, which is fastened at it is ends on the x axis at x equal to 0, and at x equal to l. The string is displaced and then released to vibrate in the x t plane where u x t denotes the vertical displacement of the vibrating string, u x t is nothing but the vertical displacement of the string. So, the differential equation governing this is given by, what is a differential equation governing this, at is nothing but del square u by del t square, is equal to c square del square u by del x square.

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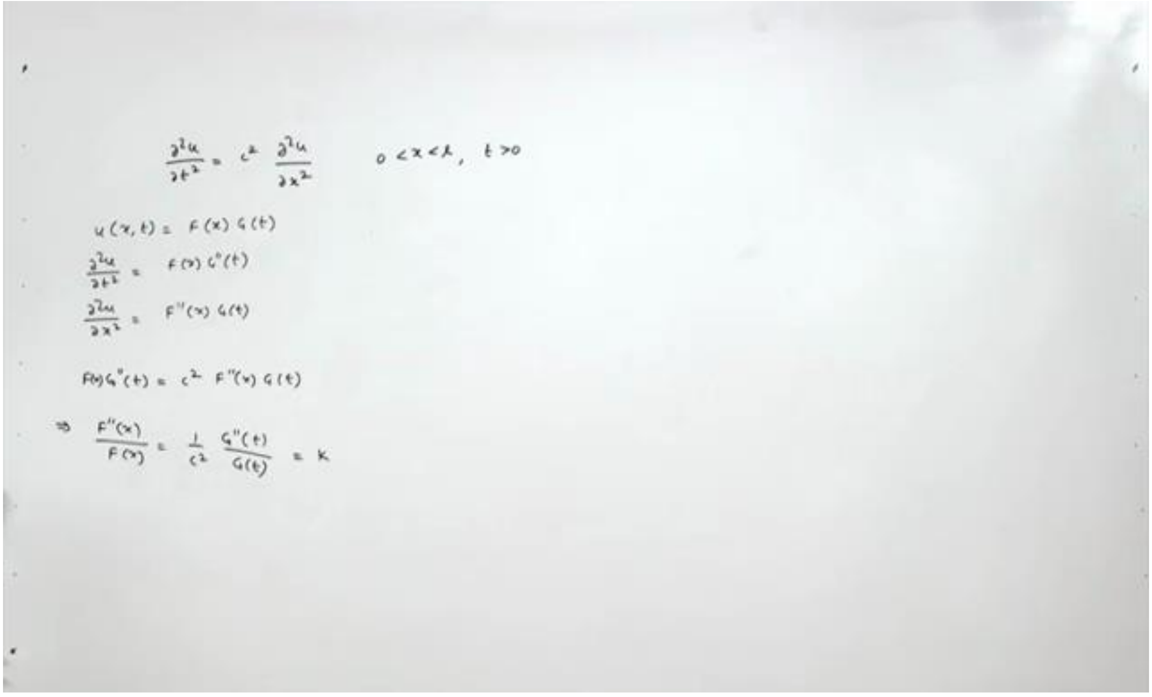

So, it is x is varying from 0 to l and t is greater than 0. So, these are the boundary condition and initial conditions given to us. Now we want to solve this this partial differential equation which is one-dimensional wave equation. How can you solve it using Fourier series? So, we assume that u x t is nothing but a function of 2 variables a function of product of 2 functions, f x into G t. So, what will be del square u upon del t square, it is nothing but f x into G double dash t. And what is del square u upon del x square, it is nothing but F double dash x into G t.

You substitute these things over here. So, what we obtain? It is f into G double dash, f x into G double dash t, which is equals to c square F double dash x into G t. So, this implies F double dash x upon f x is equal to 1 upon c square times G double dash t upon G t. Now this is a function of x alone, and this is a function of t alone and both are equal. So, both will be equal only when it is equal to some constant it is c k, where K is a constant. Because one side we have function of x, and other side we have function of t, and both are equal. So, both will be equal only when it is a constant quantity c k.

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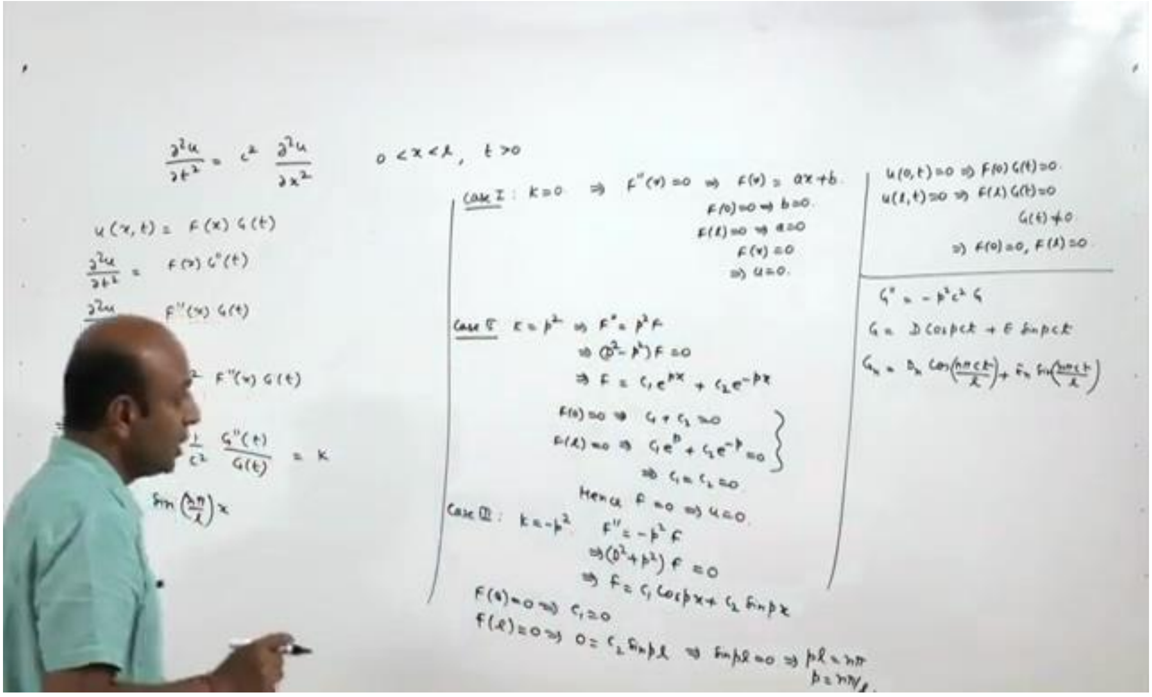

Now, this K has 3 possibilities, either K is 0, or K is positive or K is negative. Now let us take all the 3 cases. Case 1: suppose K is 0. So, F double dash will be 0. And this implies f x will be nothing but a x plus b, where a and b are arbitrary constants. Now using the boundary condition, it is u 0 a t is 0.

So, this implies when u x t is f x into G t. So, it is nothing but f 0 into G t, is 0 and 0. Now u l t is also 0. So, this implies f l into G t equal to 0. So, from these 2 conditions we get a either this equal to 0 or this 0. Suppose G t is 0, if G t is 0. So, u will be 0 which is not possible, which is the no interest. Because u must be non 0, G t cannot be 0, because if G t is 0 then u will be 0.

Now, this implies f 0 will be 0, and f l will be 0. Now when f 0 is 0, when you apply this over here f 0 will equal to 0. So, this implies b equal to 0. And f l equal to 0 this implies a equal to 0. So; that means, f equal to 0, and when f equal to 0. Means y equal to 0 again which is of no interest, because we want u as a non 0 function. So, now case 2. So, this case is not possible. Because from if we take a equal to 0 you are taking u equal to 0, which is of no interest.

Now case 2: suppose K is equals to a positive quantity suppose p square. Suppose K is some positive quantity p square, if K equal to p square. So, this implies F double dash will be equal to a K into f, and this implies d square minus K into f K is p square K is p square. So, this is p square. So, this equal to 0, and this implies f will be nothing but c 1 e k power p x plus c 2 e k power minus p x.

Now, again f 0 equal to 0; so this implies c 1 plus c 2 equal to 0 and f l equal to 0. So, this implies c 1 e k power p plus c 2 e k power minus p equal to 0. So, when you simplify these 2 equations on c 1 and c 2. So, this implies c 1 equal to c 2 equal to 0. So, hence f equal to 0. Again this implies u equal to 0. So, this case is also not possible.

So, take the third case. Case 3: when K equals to minus p square. So, these are the only 3 cases. Either K is 0 K is positive or K is negative if K is 0 we are getting u equal to 0 which is not possible. If K equal to p square positive quantity again, we are getting u equal to 0. And if K equal to now let us try for the K equal to minus p square. If K equal to minus p square F double dash will be nothing but minus p square k. So, this implies d square plus p square times f, f will be equal to 0.

So, this implies f will be nothing but c 1. So, it is cos p x plus c 2 sin p x. Now when f 0 is 0: f 0 is 0 implies c 1 is equal to 0. And f l equal to 0 implies 0 equal to c 2 sin p l. Now if c 2 equal to 0 again f will be 0 which in turn give u equal to 0. So, that case is omitting. So, this implies sin p l will be 0. Because it we do not want c 2 to be 0. Because if c 2 is 0 f is 0, and which in turn gives u equal to 0.

So, sin p l equal to 0. So, this implies p l equal to n pi, or p equals to n pi by l, p equal to n pi by l. So, this is the only case possible. And from this case what will be f, what is the f? F will be nothing but c 1 is 0 and we have c 2, c 2 sin p is nothing but n pi by l into x. So, since n is involved. So, we take it F n or we take it c n. Here also we take it c n, because n is involved for different n we get different f. So, we are we are calling it F n.

Now, we now for this p for this K here K equal to minus p square, find G what will be g. So, it is G double dash will be equals to it is minus p square c square g, because K is nothing but minus p square. So, from here again we will get we will get g, as suppose something d, d cos p c t plus e sin p c t, where d and e are arbitrary constant. Now this p is nothing but pi n pi by l.

So, when you replace you call it d n, G n; and d n cos n pi c t by l, plus e n sin n pi c t by l, because n is involved. So, we are calling it d n and e n and here we are calling as G n. So, what will be u now? So, what will be u?

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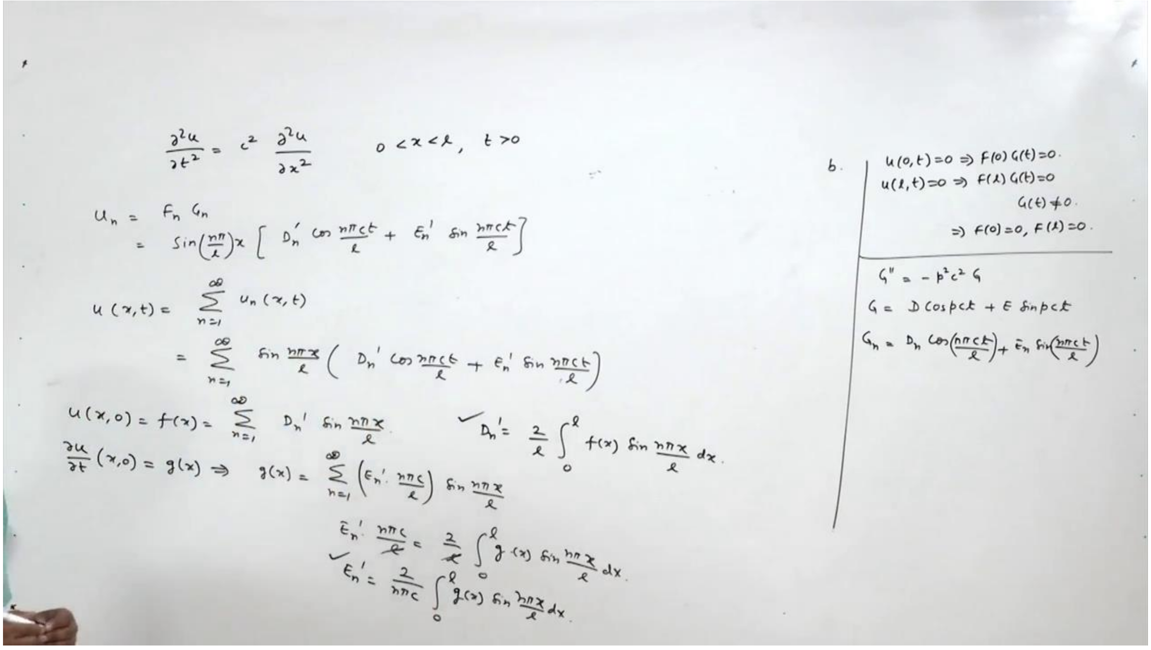

Now, this u is nothing but product of f and G. So, we are calling it u n. So, it is nothing but F n into G n and F n is nothing but c n into c n into this quantity. So, this constant will merge to d n and e n. So, finally, we will get sin n pi by l into x, into d n the new d n supposes d n dash, cos n pi c t by l, plus e n dash sin n pi c t by l

Now, this is n may be one may be 2 may be 3. And all other solution of this differential equation, when you substitute n equal t 1 you get one solution, that is a solution of this differential equation when you substitute n equal to 2 u 1 u 2 u 3 up to so on are all the solution of this given differential equation. So, we know that, but the super position principle if u 1 u 2 u 3 and so, on are the solution of a given differential equation. Then the linear combination of those will also be the solution of the given partial differential equation.

So, we are supposing u x t as summation n 1 fund to infinity u n x t e, which is nothing but summation n from 1 to infinity sin n pi x by l, into d n dash, cos n pi c t by l, plus e n dash sin n pi c t by l. So, this will be the solution of this differential equation by the super position principle. Now let us apply initial conditions. Now u x 0 is f x. So, u x 0 is f x which is equal to summation n from 1 to infinity, when you substitute t equal to 0. So, when you substitute t equal to 0 we get back to this expression.

Now, this is nothing but Fourier sine series. This is nothing but is equal to Fourier sine series half range. Half range Fourier sine series. So, from here what will be d n dash? So, d n dash will be nothing but 2 upon l integral 0 to l, f x sin n pi x upon l into d x. So, this is by the Fourier series. So, so basically d n dash and e n dash is to find out finally. So to find d n dash we substitute we take this boundary condition. From this boundary condition, we take we get this equal to this. This is nothing but Fourier sine series; Fourier sine series half range Fourier sine series. So, from here we get d n dash as 2 upon l 0 2 l f x sin by x to d x.

Now, to get e n dash, we apply another initial condition which is del u by del t, at x comma 0 is G x. So, this implies, now what is del u by del t. Del u by del t is nothing but when you differentiate respect to t. So, this would be sine which is 0 when t equal to 0. And this is cos when which is 1, when t equal to 0. So, we finally, obtain G x is equal to summation n from 1 to infinity, e n dash into n pi c by l, into sin n pi x upon l. Because when you take when you differentiate partial with respect to t and put t equal to 0. So, this is sins which is 0 and this is cos which is 1.

So, only the coefficient will come here. Now this is again Fourier sine series half range Fourier sine series. So, the constant term which is this term the entire term, this implies e n dash into n pi c upon l will be nothing but 2 upon l integral 0 to l f x sin n pi x upon l into d x. So, l, l cancels out. So, from here e n dash will be nothing but 2 upon n pi c integral 0, to l f x sin n pi x by l into d x.

So, these are the values of d n dash and e n dash, which when substitute here you will get back to u x t which as solution of this different partial differential equation. So, hence using Fourier series we can solve such type of problems. Now hence find the solution of one-dimensional wave equation, which is this problem corresponding to a triangular initial deflection. Suppose the u x 0, which is f x is given by this expression, and the initial velocity is 0. Initial velocity 0 means this we are calling as del u by del t which is nothing but the initial velocity at t equal to 0 means, initial velocity is 0, means u x is 0. And f x is given by this expression. So, we can easily find out, if G x is here it is G x sorry because it is equal to G x. So, it is G x sorry it is G x.

Now, if G x equal to 0. So, e n dash will be 0. So, e n dash is 0 and to calculate d n dash. So, we know that f x is given by that expression we simply substitute f x over here, find out d n dash and we substitute d n dash over here, we will get back to the value of u x t. So, this integral we can easily find out. Now come back to next problem, come to next problem. It is a heat equation, again solution by a Fourier series. Consider thin homogeneous bar of wire of length l, let the bar coincide with x axis. We have a bar basically which coincide with x axis.

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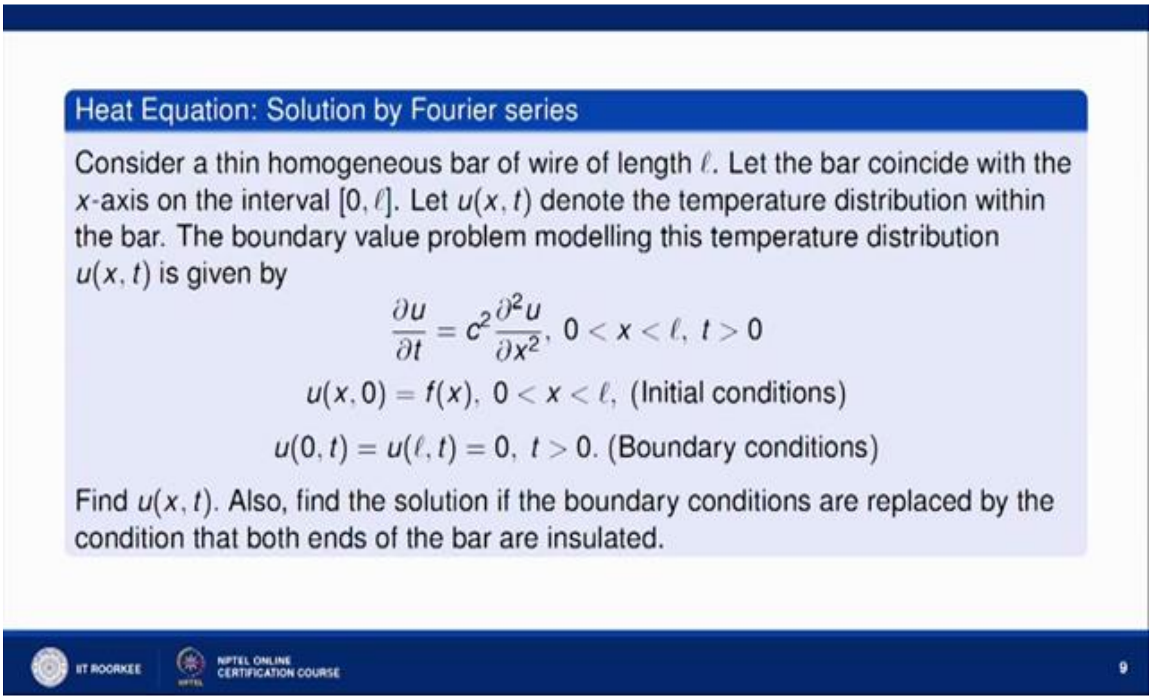

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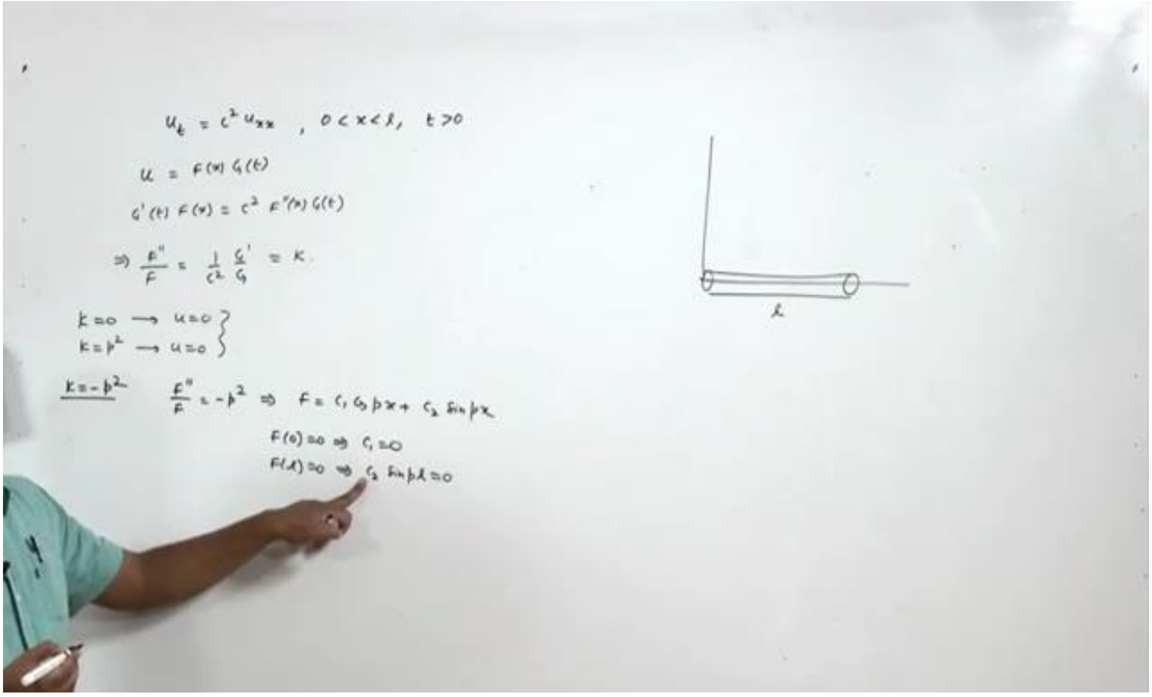

So, this is some bar of length l. Now u x t with a temperature distribution within the bar the boundary value modelling of this temperature distribution u x t is given by this. So, this is nothing but u t equals to c square u x x where this is a finite bar or finite length and t greater than 0. So, these are the initial and boundary conditions given to us, find u x t. Find u x t means find the temperature distribution of the bar, if these are the initial boundary conditions.

So, again we will apply separation of variables. We will take u as f x into G t, we will substitute it here. So, it is G dash t into f x is equals to c square F double dash x into G t, which implies F double dash upon f is equals t 1 by c square, G dash upon t. Again it is a function of x it is a function of t and both are equal. So, both will be equal only when it is a constant quantity say K, now again we will take 3 cases.

So, the boundary conditions are similar to the previous problem. It is 0 at both the ends when x equal to 0 and x equal to n it is 0. So, it is similar to the previous case one-dimensional wave equation. So, again when K is 0 u comes out to be 0 and when K equal to p square, u again come out to be 0. This is boundary condition of this problem and the previous problem. And these problems are same. It is also 0 when x equal to 0, and x equal to l and it is also 0 when you x equal to 0 and e equal to l. So, boundary conditions are same.

So, when K equal to 0 or K equal to p square, u comes out to comes out to be 0. So, we do not take these cases. So, take a equal to minus p square. Solution will exist only when K equal to minus p square. So, when K equal to minus p square F double dash upon f will be equals to minus p square, which is implies f will be equal to c 1 cos p x plus c 2 sin p x. Again f 0 is 0 this implies c 1 is 0, and f l equal to 0. So, this will imply c 2 sin p l equal to 0. And since c 2 can not be 0, because the c 2 equal to 0 again f will be 0, which in turn gives u equal to 0. Because if f equal to 0 then u will be 0 and u can not be 0.

So, this implies p l will be n pi. And this implies p will be n pi by l. So, from here we will get F n equal to sin n, pi l by x. Now we get back to here this expression. So, what will be G dash?

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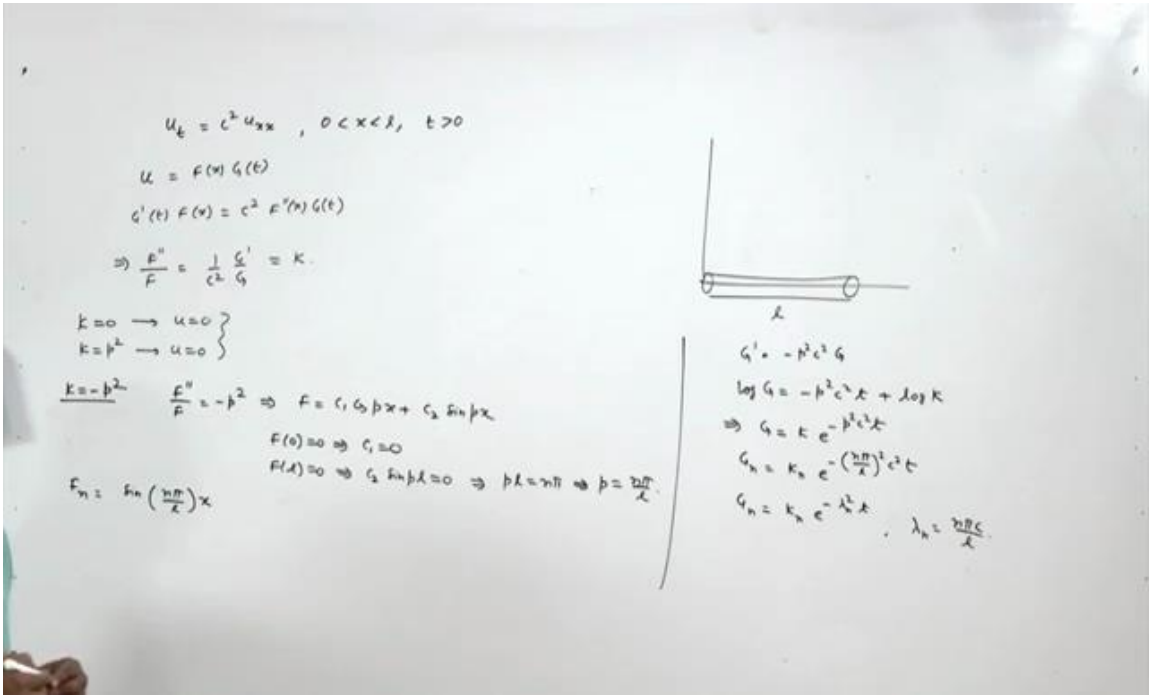

So, G dash will be nothing but minus p square c square G when you integrate it is a first derivative only. So, when we integrate both side it is log G which is minus p square c square t, plus log some K, which is which implies G is equals to K e k power minus p square c square d.

Now, when you substitute p equal to pi n by l, it is nothing but we got it G K n, G n and it is K n e k power minus it is a pi, n pi by l whole square into c square t. So, G n will be given as K n e k power, suppose it is minus lambda n square into t, where lambda n is pi this expression. So, what will be u n now? U n will be F n into G n. So, the multiplication of these 2 will be nothing but this gives K n into sin n pi x upon lm into e k power minus lambda n square into t.

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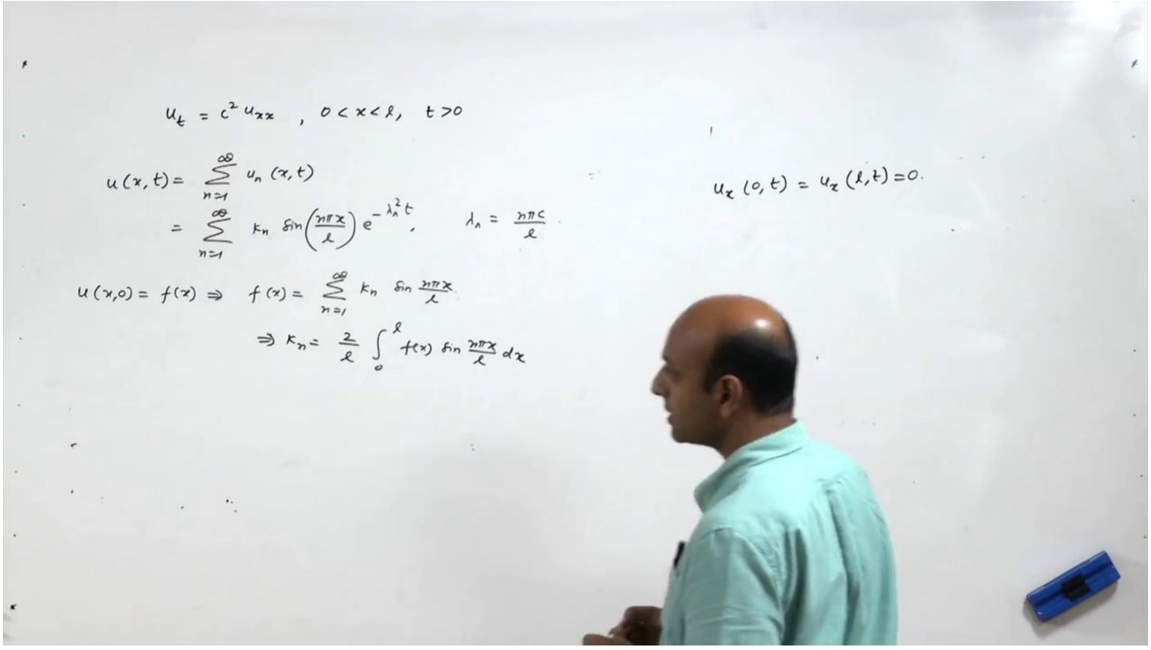

Now, again by the super position principle u x t, which is solution of this differential equation will be nothing but this partial differential equation. So, u x t will be nothing but summation n from 1 to infinity u n x t, which is nothing but summation n from 1 to infinity, u x t is K n sin n pi x pi l, into e k power minus lambda n square into t, where lambda n is n pi c pi l. Now apply the initial condition, that u 0 u x 0 is equals to f x. So, this implies, f x is equal to summation n from 1 to infinity K n sin n pi x by l. Now again this is a Fourier sin series.

So, hence K n will be given by, so this implies K n will be nothing but 2 upon l integral 0 to l f x sin n pi x by l into d x. So, if you substitute this K n over here. So, we will get a the solution u x t of this partial differential equation. Now also find the solution of the boundary condition or replace by the condition that both ends of the bar are insulated.

So, the problem is same. The boundary condition which is a these conditions are replaced by the condition that both ends are insulated; that means, no heat is passing through both the ends. So, what does it means? It means that now the boundary conditions are u x at 0 t, equals to u x at l t, are 0. These are the new boundary condition for these problems. All other conditions are same. So, what will be the solution of this problem under these boundary conditions and initial conditions?

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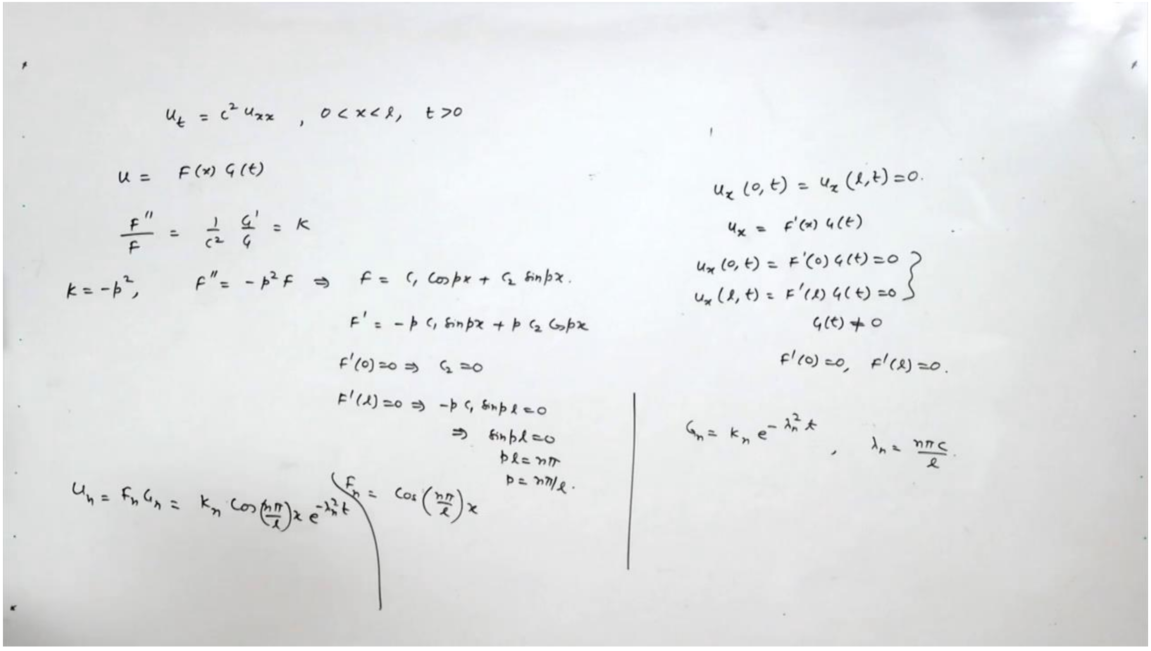

So, again we will substitute u as the same thing F into G F x into G t. So, we will get back to F double dash upon f which is equals to 1 upon c square G dash upon G which is denote as K. K is on the only possibilities is K equal to minus p square that we have already discussed. So, K equal to minus p square. So, when K equal to minus p square, F double dash will be equals to minus p square F which implies F is equals to c 1 cos p x plus c 2 sin p x now.

Now, from here what will be u x, u x will be F dash x into G t. And u x at 0 t means F dash at 0 into G t equal to 0. And u x at l t means F dash l into G t equal to 0. So, G t can not be 0, G t can not be 0 because if G t equal to 0, then u equal to 0 which is not possible. So, G t can not be 0. So, this implies F dash 0 is equal to 0 and F dash l equal to 0.

So, when we apply this thing over here. So, what will be get. So, we will get now F dash, what is a F dash? It is minus p c 1 sin p x plus p c 2 cos p x. And F dash 0 equal to 0 implies c 2 equal to 0. Because p can not be 0, and F dash l equal to 0 implies, minus p c 1 sin p l equal to 0. So, this can not be 0, because p if p is 0 K equal to 0. That K is already omitted, and if c 1 equal to 0 then c 2 is already 0, then F will be 0 this implies u equal to 0 this is also not possible.

So, only the only possibility if sin p l equal to 0. If sin p l equal to 0 the p l will be equal to n pi. So, p will be nothing but n pi by l. So, from here f will be nothing but or f l will be nothing but cos p n pi by l, cos n pi by l into x substitute the value of p. And G n is the same. What will be G n, what G be obtained the last problems, K n e k power minus lambda n square into t, where lambda n is nothing but n pi c by l.

So, what will be u now? U n will be nothing but F n into G n. So, that will be nothing but K n cos n pi l into x, into e k power minus lambda n square into t. So, again by the super position principle what will be the solution?

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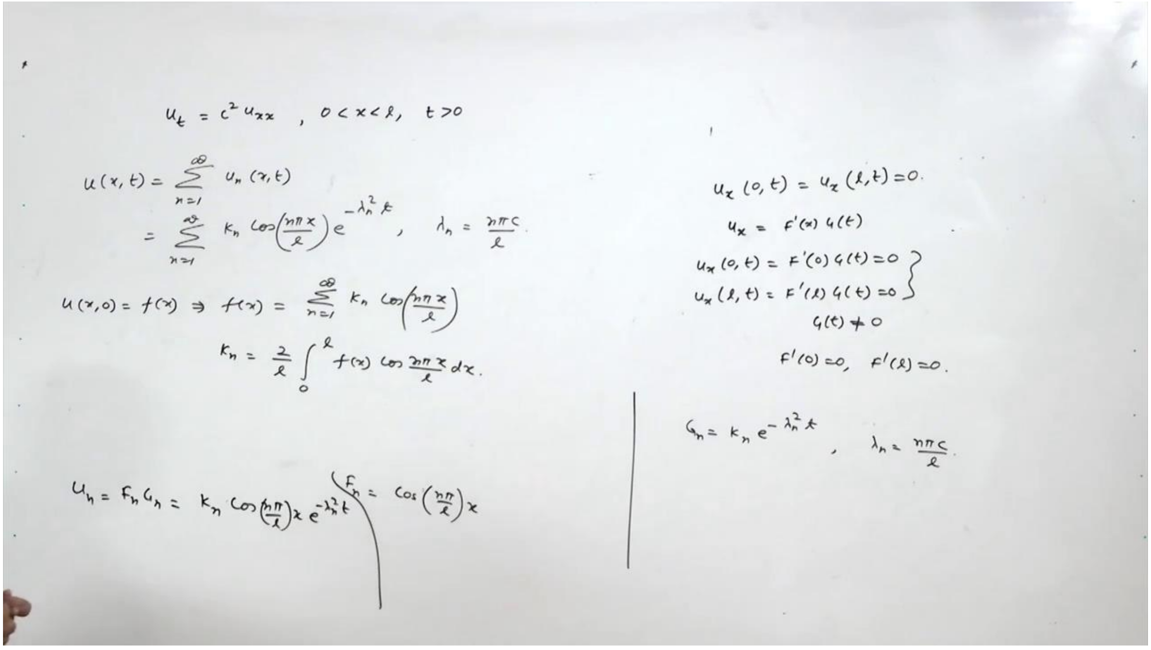

So, the solution will be nothing but u x t will be nothing but summation n from 1 to infinity, u n x t which is nothing but summation n from 1 to infinity it is K n cos n pi x by l, into e k power minus lambda n square into t, where lambda n is n pi c by l.

Now, apply the same initial condition, that u x 0 equal to f x. So, this implies f x will be equal to summation n from 1 to infinity, K n cos n pi x by l. So, now, it is a Fourier sin series, I mean cosine series here is a cosine series. So, though then K n will be nothing but 2 by l, 0 to l, it is f x cos n pi x by l into d x. So, when you substitute this K n over here. So, we will get back to the solution u x t, which is a solution of this partial differential equations.

Hence, the partial differential equation can be solved using Fourier series. So, these are some applications of Fourier series, which we have discussed here. So, that is all for this lecture.

Thank you.