Mathematical methods and its applications

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Lecture – 57
Convolution theorem for Fourier transforms

Welcome to lecture series on Mathematical Methods and its Applications. So, we have seen what Fourier transforms are, and also we have seen Fourier sin and cosine transforms. So, how can write Fourier transform function f t?

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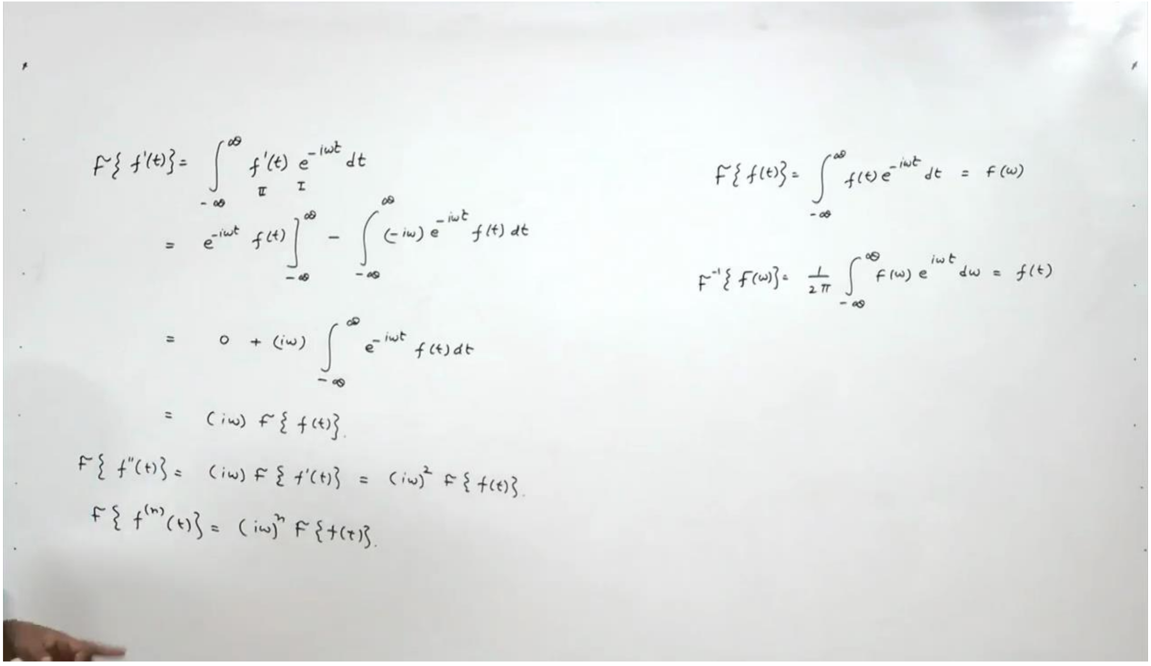

Fourier transform of a function f t is given by minus infinity to plus infinity f t e k power minus iota omega t into d omega into dt, and we are calling it f omega function of omega. So, and the inverse Fourier transform of f omega is f omega is given by it is 1 upon 2 pi integral minus infinity to plus infinity that is f omega e k power iota omega t and it is d omega, so which is nothing but f t. So, this we have already discussed and this come from complex form of Fourier integrals this we have already seen that Fourier transforms f t is nothing but given by this expression; and the inverse Fourier transforms is given by this expression.

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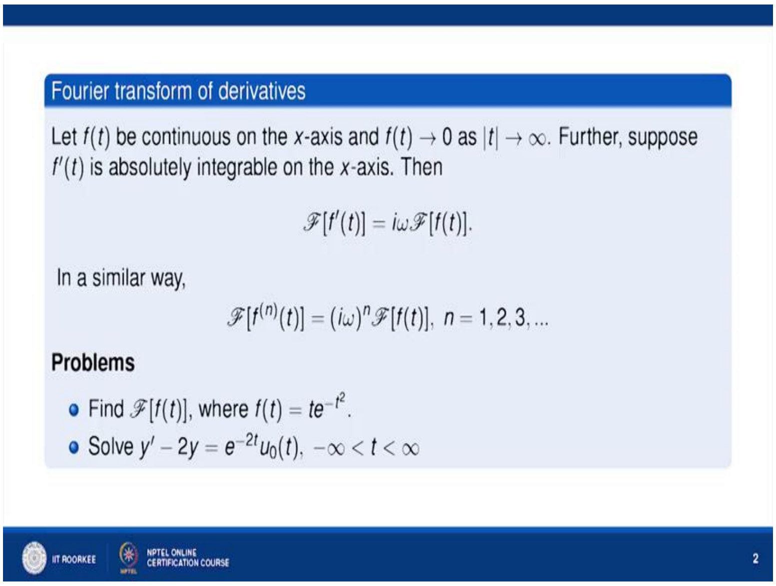

Now, let us come to convolution theorem for Fourier transforms. So, before discussing convolution theorem, let us discuss a more properties of Fourier transforms. First of all Fourier transform of derivates; so let f t be continuous on the x-axis and f t tend into 0 as mod t tend into infinity. Now, further suppose f dash t is absolutely integrable on the x-axis then Fourier transform of f dash t is given by iota omega Fourier transforms of f t.

So, this thing is very easy to proof Fourier transform of f dash t. So, by the definition of Fourier transforms it is nothing but minus infinity to plus infinity f dash t into e k power minus iota omega t into dt. So, this will be nothing but let us suppose it is first function it is second function. So, first as it is integral of second function from minus infinity to plus infinity and minus integral derivate of first is minus iota omega into e k power minus iota omega and integral of second which is f t into dt. So, this is by integration by parts.

Now as we have assumed that as f t as mod t tend to infinity, f t is tend to 0 that is when t is tend to plus or minus infinity, f t is tend to 0. So, from the upper and the lower limit this is tending to 0. So, this is nothing but 0 plus iota omega times minus infinity to plus infinity e k power minus iota omega t into f t dt. Now, what it is it is nothing but Fourier transform of f t. So, it is iota omega times Fourier transforms of f t. So, hence the Fourier transform f dash t is nothing but iota omega times Fourier transform of f dash t.

Now, suppose you want to find out Fourier transform of f double dash t, so simply replace f pi f dash in this expression, so that will be nothing but iota omega Fourier transform of f dash t which is equal to Fourier transform of f dash t we already derived is equal to this expression. So, this is nothing but iota omega whole square Fourier transform of f t. So, in the similar way, if we proceed for the nth derivative for the same expression, so we get Fourier transform of nth derivative of t will be nothing but iota omega k to the power n Fourier transform of f t. So, nth derivative of Fourier transform of f will be nothing but iota omega k to the power n Fourier transform of f t. So, this is how we can obtain Fourier transform of derivatives. Now, let us solve these two problems find Fourier transform of f t, where f t equal to this, find Fourier transform of f t actually.

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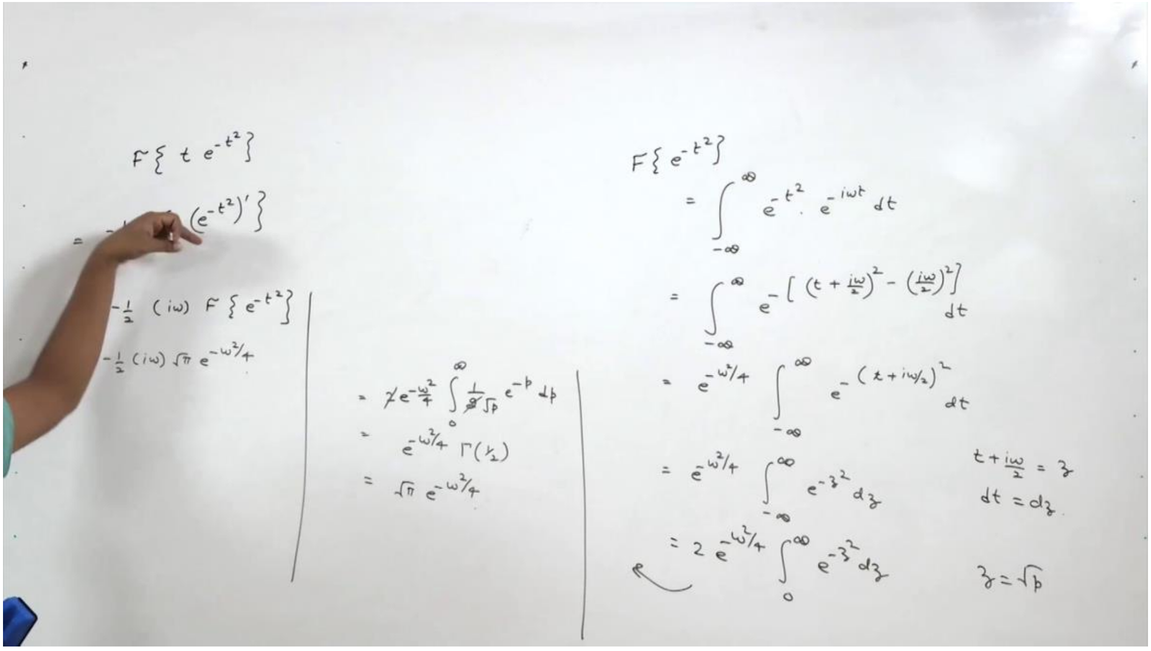

So, we have to find Fourier transform of t e k power minus t square. Now, this is nothing but Fourier transform of e k power minus t square is derivative, is derivative is nothing but minus 2 t times this thing you multiply and divide by this expression. So, this will be nothing but when you simplify this, so you get back this expression. Now, by the Fourier transform derivative, we know that Fourier transfer of f dash t is nothing but iota omega times Fourier transform of f t. So, this is nothing but minus half of iota omega times Fourier transform of e k power minus t square because here f t is e k power minus t square is derivative and its derivative is given by iota omega Fourier transform of f t. So, this is this expression.

Now, let us find Fourier transform of e t square. So, Fourier transform of e t square Fourier transform of e k power minus t square will be given by the minus to plus infinity f t into e k power minus iota omega t dt which is equal to minus to plus infinity e k power minus, to when you make perfect square, it is t plus iota omega by 2 whole square a square plus 2 a b and minus iota omega by 2 whole square into dt. So, which is equal to it is minus minus – plus, it is e k power minus omega square by 4 can come out because it is free from t. And it is integral minus or plus infinity e k power minus t plus iota omega by 2 whole square into dt. Now, you can take t plus iota omega by 2 as suppose under root p or you can take it as suppose you can take it as z first. So, then dt will be nothing but dz. So, it will be e k power minus omega square by 4 minus to plus infinity e k power minus z square into dz, limit will remain the same.

Now, it is an even function. So, this can be written as 2 times e k power minus omega square by 4 0 to infinity e k power minus z square dz. Now, we can let z equal to suppose under root p. So, this will give this integral will give Fourier 2 e k power minus omega square by 4 integral 0 to infinity 1 by 2 under root p e k power minus p into dp, which is nothing but 2, 2 cancel out and it is e k power minus omega square by 4, and it is gamma half by that definition gamma function. This is nothing but under root pi e k power minus omega square by 4.

So, therefore, the value of this is nothing but minus 1 by 2 iota omega under root pi e k power minus omega square 4. So, here we use the concept of derivative of Fourier transform, and to find out Fourier transform of e k power minus t square we go back to the main definition of Fourier transforms they are the simplify and we get the Fourier transform of e k power minus t square.

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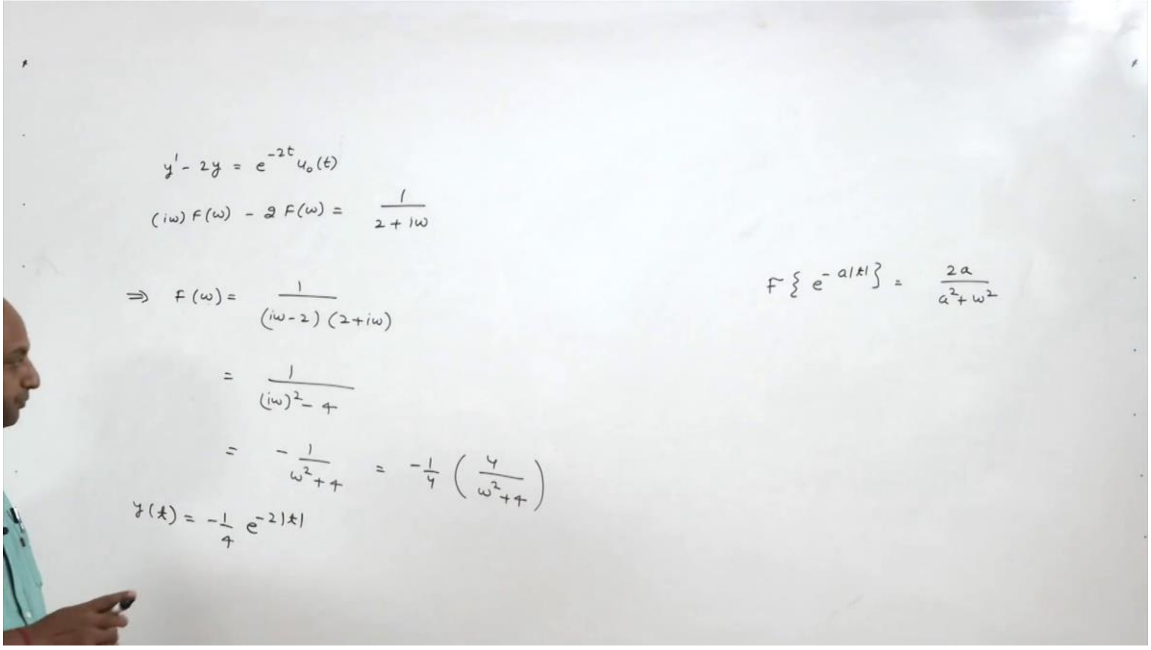

Now, suppose you want to solve this problem the next problem, it is y dash minus 2 y will be equal to e k power minus 2 t u naught t. Take Fourier transform both the sides. What is the Fourier transform y dash, by the formula; it is iota omega f omega. Here f omega is a Fourier transform of y t minus 2 f omega and is equal to we know we already that Fourier transform of e k power minus a t u naught t u naught t is a unit step function at t equal to 0. We already know that the Fourier transform of this is nothing but 1 upon 2 plus iota omega because Fourier transforms of e k power minus a t u naught t is 1 upon a plus iota omega.

So, what we obtain from here we obtain that f omega is nothing but 1 upon iota omega minus 2 into 2 plus iota omega. When they simplify this, so this is nothing but a square minus b square and which is equals to minus of 1 upon omega square plus 4. Now, we have also seen that Fourier transform e k power minus a mod t is nothing but 2 a upon a square plus omega square that we have already derived. So, it is something like 1 upon omega square plus 2 square. So, you can multiply and divide by 4, so minus 4 into 4 upon omega square plus 4.

So, instead of a, we have 2. So, this is nothing but, so if we take inverse Fourier transform both the sides, so y t will be nothing but minus 1 by 4 and inverse of this will be given by that it is e k power minus 2 into mod t, so that will be the solution of this expression this I mean differential equation. So, we take Fourier transform both the sides simplify and the inverse of Fourier transform will give the solution of this differential equation.

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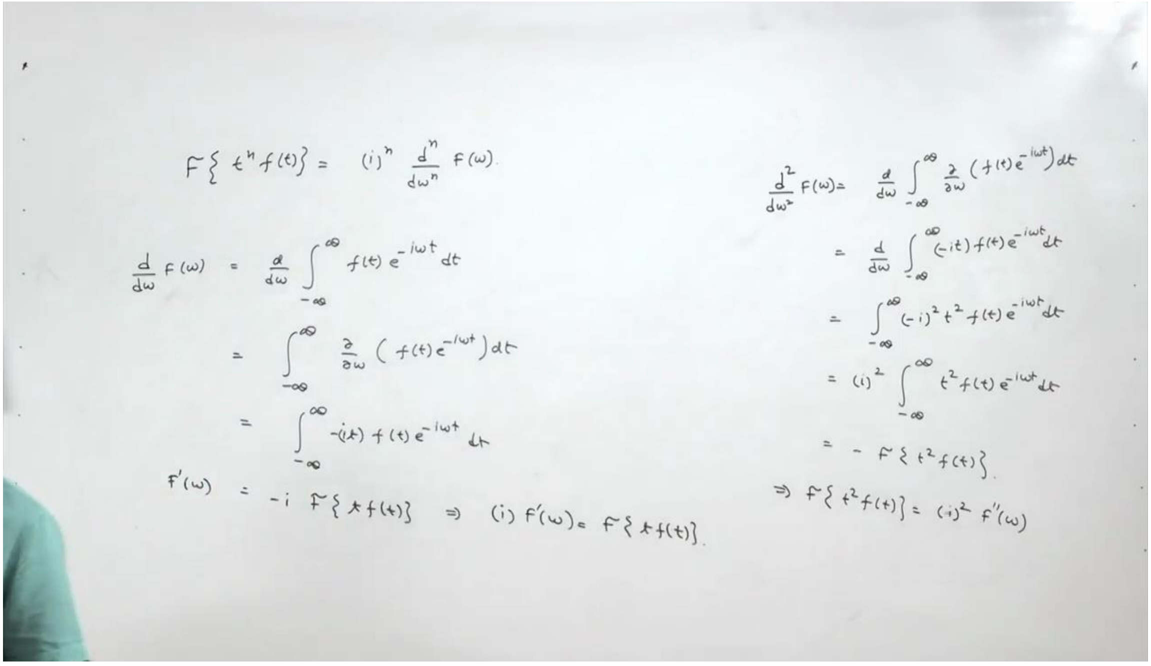

Now, differentiation respect to frequency omega. Let us suppose functions is piecewise continuous on the x-axis and let t k to the power n f t be absolutely integrable on the x-axis, then we have a result for Fourier transforms. Then Fourier transform of t k to the power n f t that is a multiplication of t is nothing but iota k power n and nth derivative respect to omega of f omega nth derivative of omega with f omega.

Now, to prove this, the proof is simple you take d by d omega d omega of f omega. Now, f omega is a Fourier transform of f t, but it is nothing but d by d omega of minus to plus infinity f t e k power minus iota omega t into dt, because Fourier transform f t is given by this expression and f omega is nothing but Fourier transform of f t. So, this will be given by the Leibniz theorem, it is nothing but del by del omega of f t into e k power minus iota omega t into dt which is nothing but when you take derivative respect to omega it is nothing but minus iota t into f t e k power minus iota omega t into dt. Now, minus iota will come out and this it is nothing but t f t keep our minus iota omega t that will be nothing but Fourier transform of t f t.

So, therefore, f dash omega will be nothing but this expression. Now, when you multiply with iota both the sides, this is nothing but iota f omega will be equals to Fourier transform of t f t. So, hence Fourier transform of this is nothing but this expression for n equal to 1. So, similarly if you take second derivate, suppose you take second derivate d 2 upon d w t of F f w. So, that will be nothing but by the same by the same concept it is nothing but d by d w of integral minus into plus infinity, first we take first derivate by this expression del by del w of f t e k power minus iota omega t into d t. So, that will be given by d by d w of minus infinity to plus infinity iota t f t e k power minus iota omega t.

Again when you take this d by dw inside and you differentiate with respect to omega, so that will be nothing but minus infinity to plus infinity, it is minus iota square t square f t e k power minus iota omega t into dt, which is iota square times integral minus into plus infinity t square f t e k power minus iota omega t dt. And this is nothing but minus Fourier transform of t square f t, this is nothing but Fourier transform of this.

Now, this implies Fourier transform of t square f t will be nothing but iota square times double derivative of f. Here we have a single derivative, here we have a single derivative and here we have a double derivative. So, that means, similarly if you repeat this process n times, so we will get back this expression that is a Fourier transform of t k power n f t will be nothing but iota k power n f f omega nth derivative of f omega, so in this way we will get by this expression.

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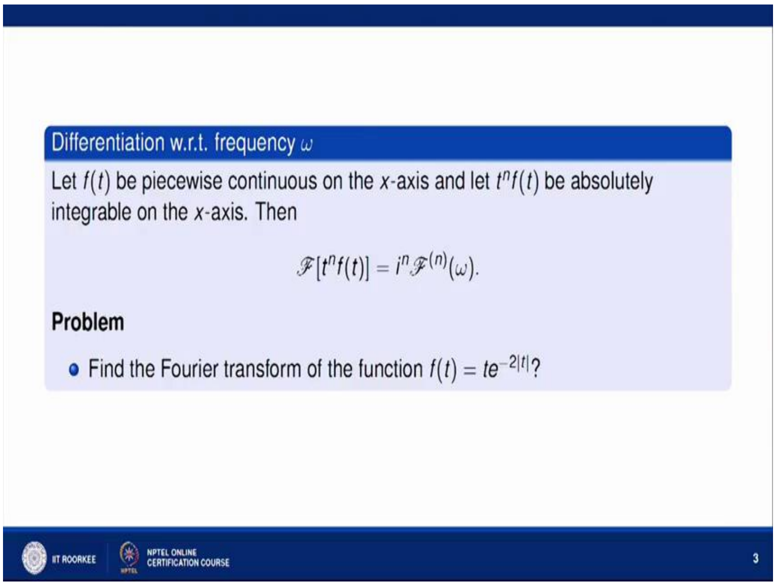

Now, using this suppose you want to find out Fourier transform of t e k power minus 2 mod t so that will be nothing but using this expression it is nothing but iota k power one because n is 1 into Fourier transform of into derivative into d by d omega of f omega.

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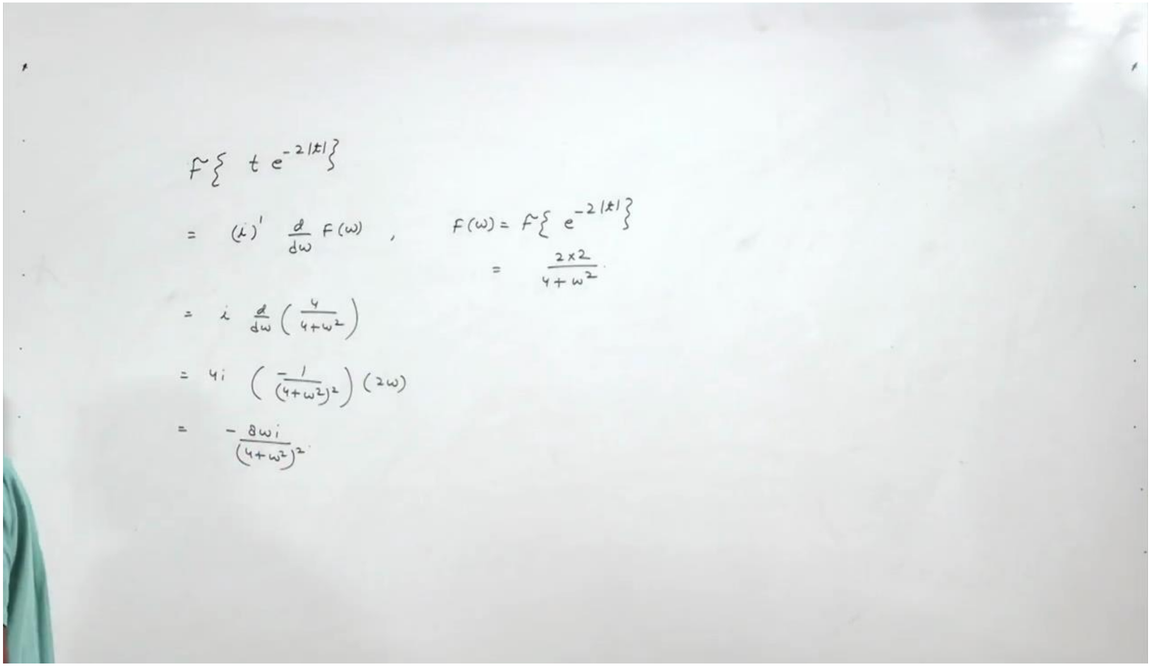

Where f omega is the Fourier transform of this f t it is e k power minus 2 mod t. And what is the Fourier transform of this it is 2 a upon 4 plus omega square. So, this will be nothing but iota d by d omega of 4 upon 4 plus omega square which is nothing but 4 iota into minus 1 upon 4 plus omega square the whole square into 2 omega, so that will be nothing but minus 8 omega iota upon 4 plus omega square the whole square. So, this will be the Fourier transform of this expression.

Now, Dirac-delta function, we already Dirac-delta function in Laplace transform that it is given by 1 by epsilon 20 varying from a to a plus epsilon and 0 otherwise, and where epsilon tend to 0. So, roughly speaking Dirac-delta functions is infinity at a point I say t equal to a and 0 otherwise such that the total integral from minus infinity to plus infinity of Dirac-delta function is 1 that we have already discussed in the Laplace transforms.

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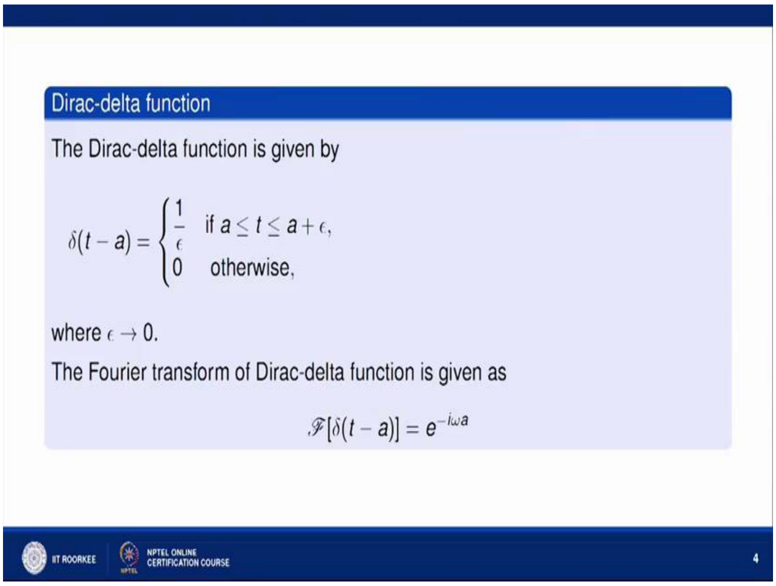

Now, the Fourier transform of Dirac-delta function is given by this. So, this can be derived using the definition of Dirac-delta function, I mean Dirac-delta and Fourier transform.

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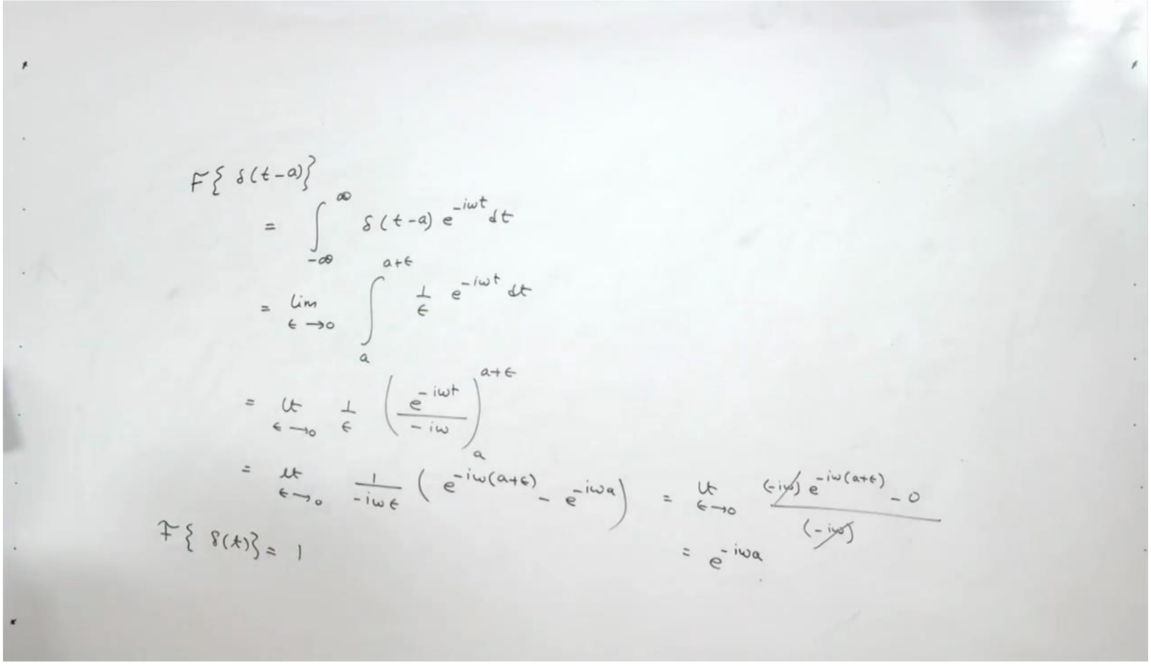

So, to find Fourier transform of Dirac-delta function, it is nothing but minus to plus infinity f t f t is the delta into e k power minus iota omega t into dt. Now, Dirac-delta is given by 1 by epsilon when t varying from a to a plus epsilon, where epsilon tend to 0. So, this we can define like this limit epsilon tend to 0; a to a plus epsilon, it is 1 by epsilon into e k power minus iota omega t into dt. Now, it is limit epsilon tend to 0, 1 by epsilon can come out and the integration of this will be nothing but this term from a to a plus epsilon. Now, it is nothing but limit epsilon tend to 0, 1 upon minus iota omega can come here and it is e k power minus iota omega a plus epsilon minus e k power minus iota omega a by applying upper limit minus lower limit.

Now, when epsilon tends to 0 it is 0 by 0 forms. So, you apply a L'hospital rule to simply this. So, you will take the derivate of the numerator and the denominator. So, limit epsilon tend to 0 the derivate of numerator will be nothing but it is minus iota omega e k power minus iota omega a plus omega minus 0 upon minus iota omega derivate is respect to omega i mean epsilon. Now, these two terms cancels out and when you take epsilon tend to 0, it is nothing but e k power minus iota omega, so which is the same as this expression.

Now, when you take a equal to 0, it is clear that when you take a equal to 0 Fourier transform of Dirac-delta t at a equal to 0 is nothing but 1. So, this is very clear from this expression when you substitute a equal to c.

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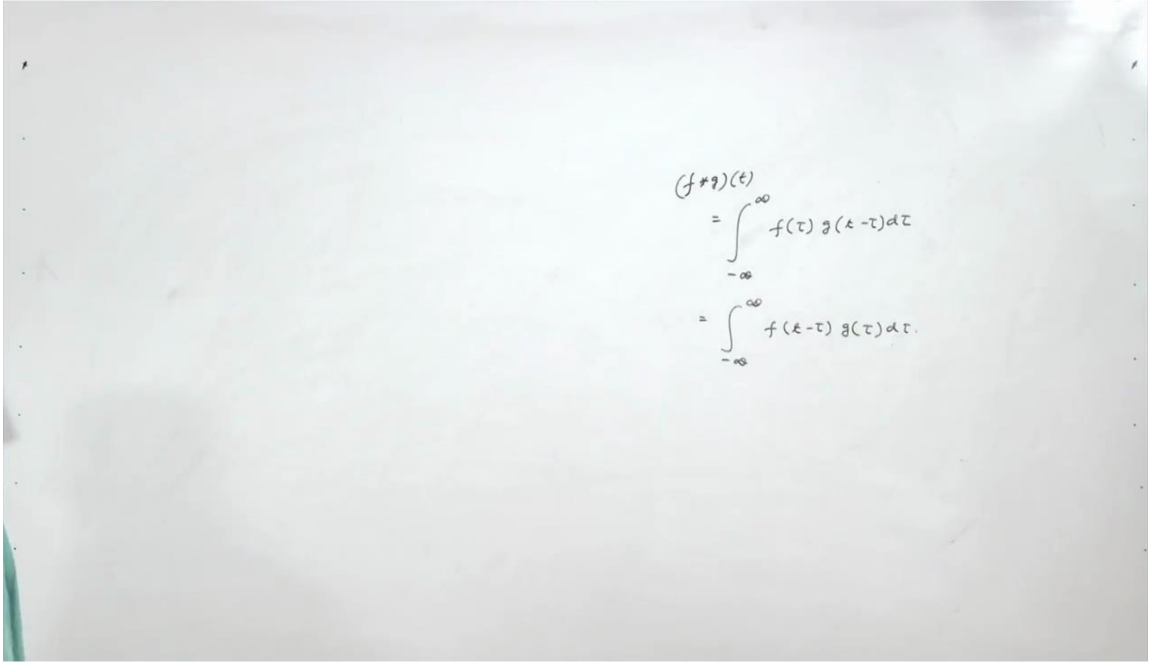

Now, come back to convolution theorem now. The convolution of two function define in a same way as we did in a Laplace transform now here integral is from minus to plus infinity, so convolution of two functions in Fourier transform is given by this. So, convolution of two functions in Fourier transform is given by minus to plus infinity f tau g t minus tau d tau or can be written as minus into plus infinity f t minus tau g tau d tau because this convolution of two functions is commutative.

Now, come to convolution theorem. Now, what it states convolution theorem for Fourier transform. Suppose that f t and g t are piecewise continuous, bound it, and absolutely integrable on the x-axis, then Fourier transform of convolution of two functions f and g will be given by Fourier transform of f into Fourier transform of g which is nothing but f omega into g omega. So, where Fourier transforms of f t is f omega and Fourier transform of g t is g omega.

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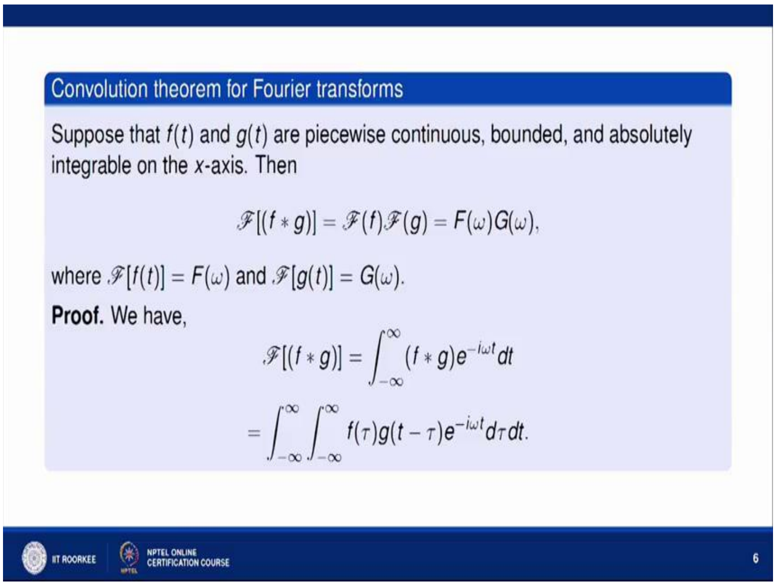

So, now we will see the proof of convolution theorem. So, basically want to find out Fourier transform of f star g. So, by definition it is nothing but minus to plus infinity convolution of this, this function into e k power minus iota omega t dt. This is by the definition of Fourier transform Fourier, where transform of function f t is given by minus to plus infinity f t e k power minus iota omega t d t.

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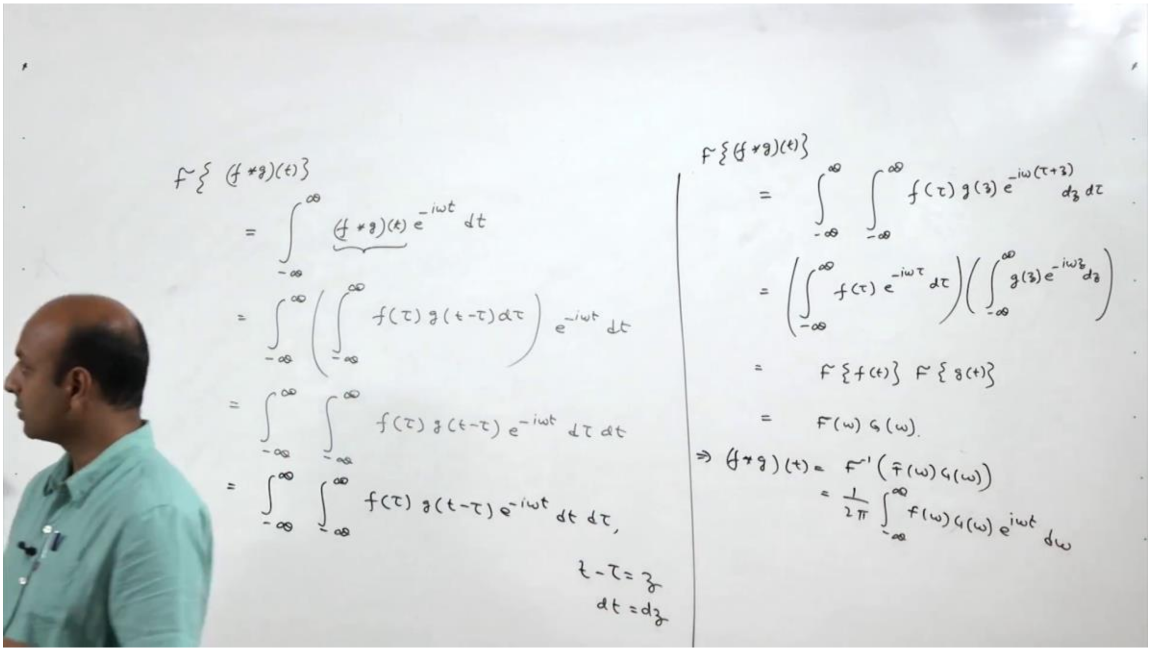

Now, convolution of two function this, this term is given by you replace this term as minus into plus infinity, suppose f tau g t minus tau d tau and whole multiplied by e k power minus iota omega t into dt. So, this can be written as minus to plus infinity minus to plus infinity f tau g t minus tau e k power minus iota omega t d tau into dt. Now, change the order of integration, both the limits are constant minus and plus infinity, so change the order of integration. So, when you change the order of integration is nothing but minus plus infinity minus into plus infinity f tau g t minus tau e k power minus iota omega t dt into d tau.

Now, suppose t minus tau as a new variable suppose z or dt will be dz. So, when you substitute this variable over here in this expression, so what we get. We get so basically we are simplifying Fourier transform of convolution of two functions so that will be equal to this expression. And this will further be equal to minus infinity to plus infinity minus infinity to plus infinity, this is f tau and this is g z e k power minus iota w and t is nothing but tau plus z from this expression and dt is dz, and d tau is d tau.

So, this is nothing but now you can write it like this minus to plus infinity f tau e k power minus iota omega tau t tau because now you can separate two variables, two are separating tau we can separately and z we can write separately. So, we can always do this, and this is nothing but Fourier transform of f t and this is nothing but Fourier transform of g t. So, this is nothing but F omega into G omega. So, hence Fourier transform of convolution of two functions f and g is nothing but F omega into G omega. So, this is convolution theorem, and here is a proof.

Now, from here it is also cleared at when the Fourier transform of f star g is F omega G omega. So, from here we can write that f star g into t is nothing but Fourier inverse of F omega into G omega. Now, inverse Fourier transform is given by 1 upon two pi integral minus infinity to plus infinity the omega function which is F omega into G omega e k power iota omega t into d omega, so that we will already know by definition of inverse Fourier transforms. Now, let us find Fourier inverse Fourier transform of this F omega using convolution theorem.

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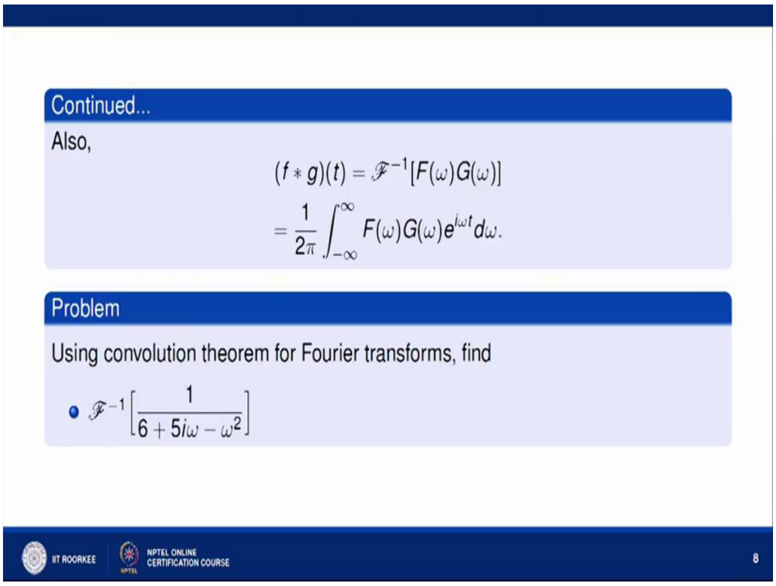

So, how you can do that? So, we will recall convolution theorem again, and using convolution theorem we will try to find out the Fourier inverse Fourier transform of this function. So, what is a function now, what we are find out, Fourier inverse of 1 upon 6 plus 5 iota omega minus omega square. So, this is F omega. So, this is F omega.

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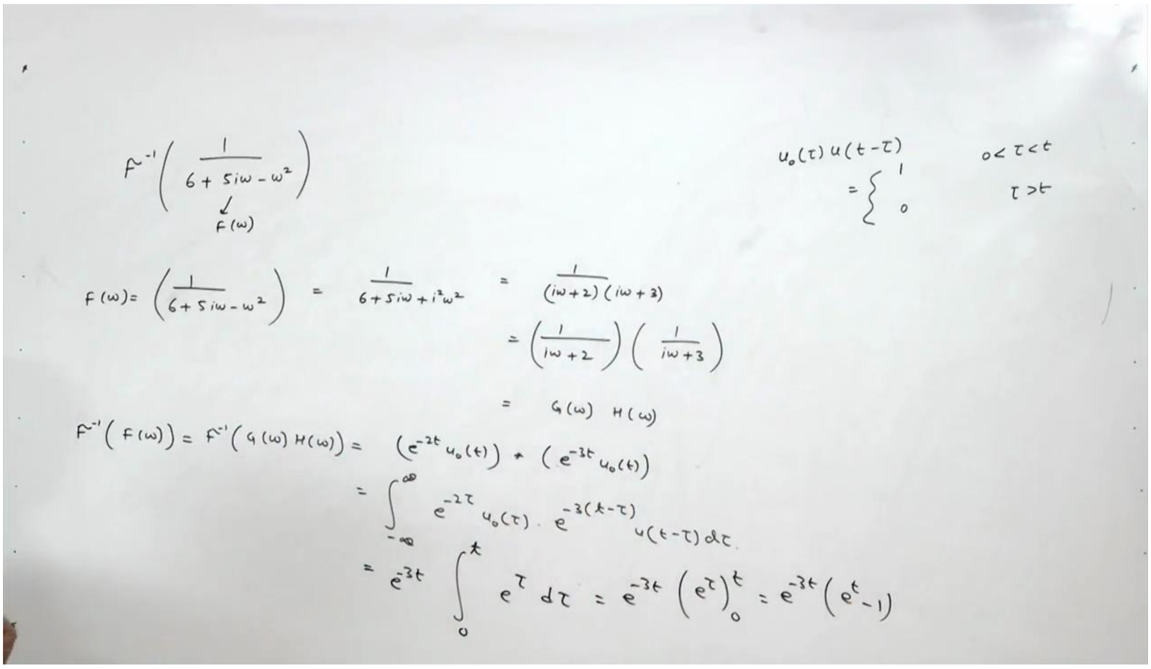

So, F omega is nothing but 1 upon 6 plus 5 iota omega minus omega square. So, in order to apply convolution theorem, we have to write as a product of two omega functions and the convolution I mean Fourier inverse of both the omega functions we should know. So, that convolution theorem we can apply. So, let us try to write this function. So, we can easily write it as 1 upon 6 plus 5 iota omega plus iota square omega square, and this is nothing but we can simplify when we simplify. So, this is a square plus 5 a plus 6. So, this is iota omega plus 2 into iota omega plus 3. So, this is nothing but 1 upon iota omega plus 2 into 1 upon iota omega plus 3. So, this is something like G omega and this is something like H omega. So, we have write this F omega as a product of two omega functions.

Now to find out its inverse using convolution theorem, so F inverse of F omega which is equals to F inverse of G omega into H omega will be equal to, so we will apply convolution theorem it is nothing but convolution of this and this. So, what is Fourier inverse of G omega? So, Fourier inverse of G omega we know that Fourier inverse of this is nothing but e k power minus 2 t u naught t, this we already know. And Fourier inverse of this expression 1 upon Fourier inverse of H omega, H omega is this is nothing but e k power minus 3 t u naught t, this you already know that Fourier transform of e k power minus a t u naught t is 1 upon a plus iota omega.

Fourier transform of F omega which is equal to Fourier inverse transform of F omega into G omega here using convolution theorem will be nothing but convolution of these two function e k power minus 2 t u naught t e star with u e k power minus 3 t into u naught t, the convolution of these two functions. And that will be nothing but minus infinity to plus infinity f tau that is e k power minus 2 tau u naught tau into g t minus tau that is e minus 3 t minus tau u t minus tau and it is dt. This is by the convolution theorem because f tau into g t minus tau.

Now, let us simplify this. Now, here we have two unique step functions here and here. So, let us find what it is. So, it is u naught tau into u t minus tau. So, now we have to see d tau, tau is a variable basically. So, we have to see respect to tau. So, when tau is less than t and suppose greater than 0; so when tau is less than t, this quantity is positive, because t tau is less than t, and this is also positive 1 into 1 is 1, and when tau is greater than t, so this is 0, because this is negative; so this will be 0.

So, when we apply this over here. So, this will be equal to now e k power minus 3 t is free from tau can be come out, and it is e k power minus, minus - plus 3 tau into e k power minus 2 tau is e k power e k power tau. And this product is 1, when tau varying from 0 to t, otherwise it is 0. So, it is from 0 to tau from 0 to t only, and it is d tau. So, this will be equal to e k power minus 3 t e k power tau 0 to t which is equal to e k power minus 3 t e k power t minus 1, so that will be the inverse Fourier transform of this function, this F omega using convolution theorem.

Hence, using convolution theorem, we can find out inverse Fourier transforms also. The main application of convolution theorem is to find out the Fourier inverse. So, sometimes we have to find out Fourier inverse, so it is important to use convolution theorem, so that is all.

Thank you very much.