Mathematical methods and its applications
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Lecture - 56
Fourier sine and cosine transforms

Welcome to lecture series on Mathematical Methods and its Applications. Now, we will discuss Fourier sine and cosine transforms. We have already seen what Fourier transforms are.

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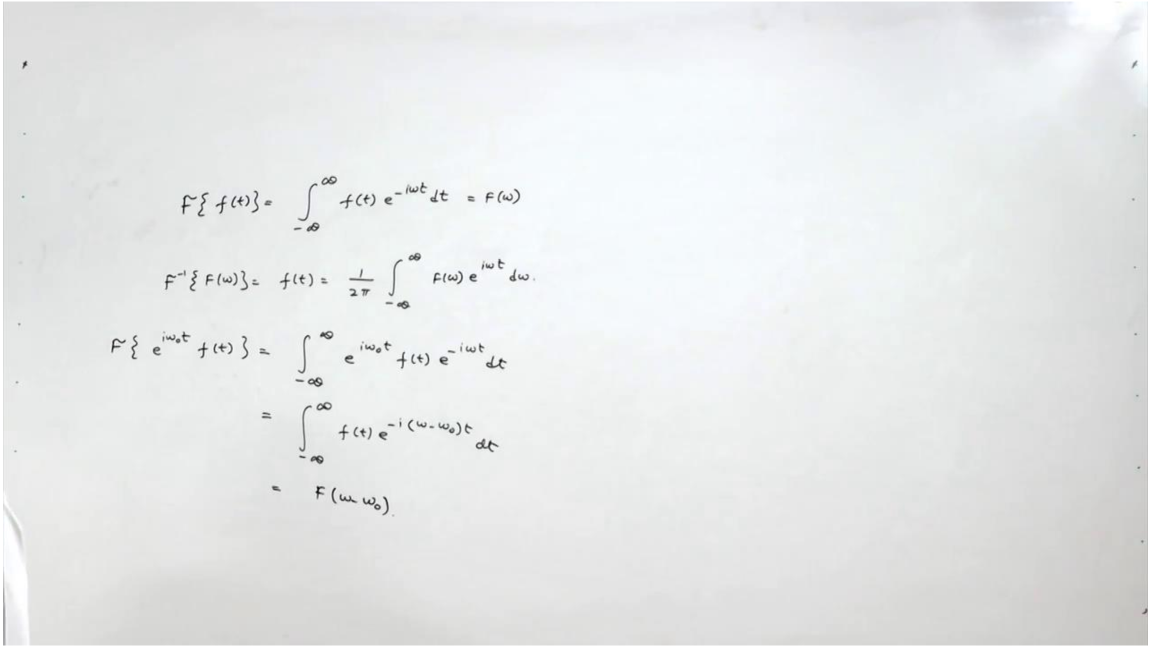

We have seen that Fourier transform is given by Fourier transform of function f t is given by minus into plus infinity f t e k power minus iota omega t dt which is f omega. And its inverse which is f omega inverse of f omega is f t which is given by 1 upon 2 pi integral minus to plus infinity it is a f omega e k power iota omega t t omega, so that is how we can find out inverse Fourier transform and Fourier transform of f t.

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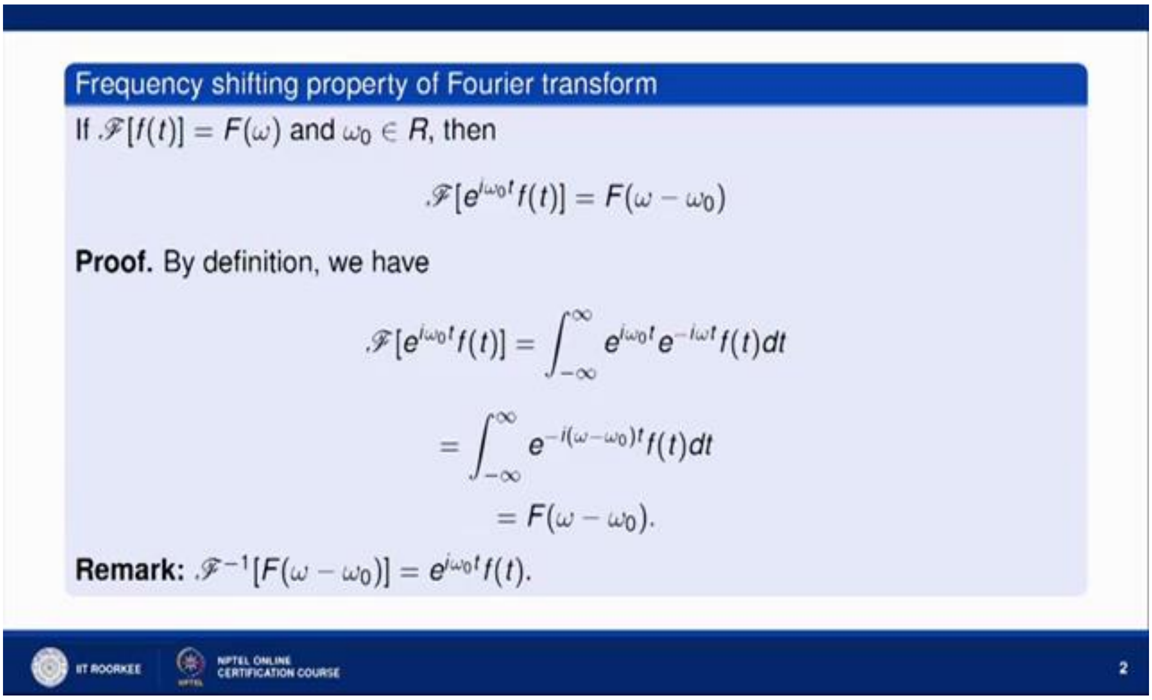

Now, sine and cosine transforms. Before discussing sine and cosine transform, let us discuss shifting property of Fourier transform also. So, suppose Fourier transform f t s of omega and omega naught is belongs to R, then Fourier transform of this is the very simple property e k power iota omega naught t into f t let us find Fourier transform of this. So, Fourier transform of this is nothing but minus into plus infinity, you apply a definition this definition e k power iota omega naught t into f t into e k power minus iota omega t dt. So, this is given by minus to plus infinity f t e k power minus iota omega minus omega naught into t into dt.

So, in this expression, instead of omega, we have omega minus omega naught. So, this is nothing but f of omega minus omega naught. So, this is called a shifting property I mean frequency shifting property of a Fourier transforms. So, this we can use sometime while solving some problems. Now, next is Fourier cosine transform. The Fourier cosine transform function f t is defined as 0 to infinity f t cos omega t dt. Now, which is given as Fourier cosine transform of f t now it comes from basically from Fourier cosine integral representation of function f t.

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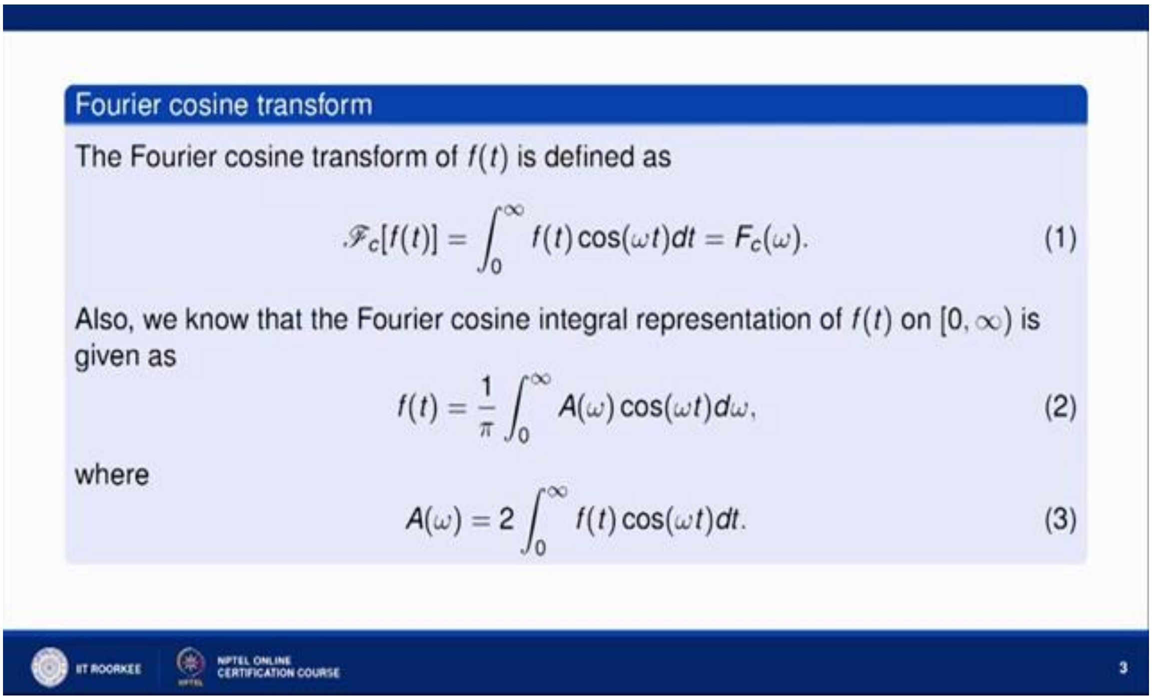

What is the Fourier cosine integral representation of function f t, it is f t is equals to 1 upon pi integral 0 to infinity it is a omega cos omega dt omega, where a omega is given by this expression. Now, if you compare 1 and 3, so a omega is nothing but 2 times f c omega, when you compare these two. So, when you substitute it here, so we obtained f t equal to this which is called inverse Fourier cosine transform of f t.

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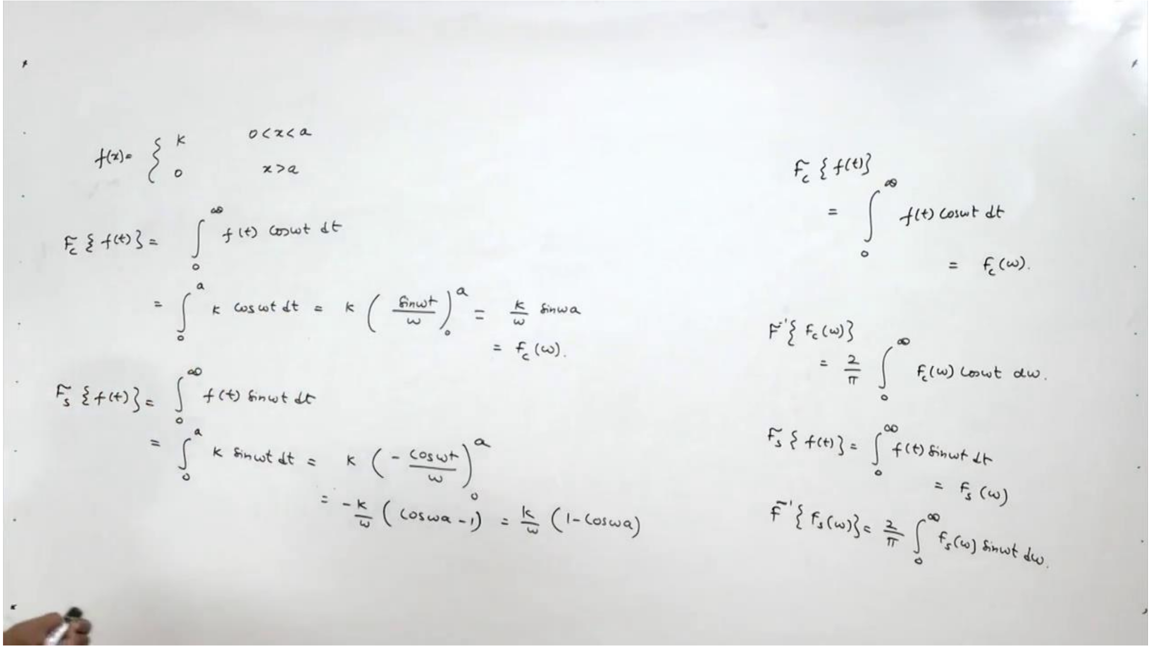

So, what is a Fourier cosine transform, and what is its inverse. So, basically Fourier cosine transform of f t it is given by it is 0 to infinity f t e k power minus iota omega dt which is a Fourier cosine transform of because it is cosine. So, it contains cosine terms. It is f t into cos omega t dt which we are calling as Fourier cosine transform of it is, we are calling as f c omega we are calling it as f c omega. Now, the inverse of this is given by inverse of a Fourier cosine transform is given by it is 2 upon pi integral 0 to infinity f t, it is f c omega here. It is f c omega cos omega t dt or d omega. So, this will be the inverse Fourier transform of inverse Fourier cosine transform of f t.

Now, similarly if we define Fourier sine transform, so Fourier sine transform is given by 0 to infinity f t sin omega t dt which we denote as F s of omega. And it comes on Fourier sine internal representation of f t because we know that Fourier sine integral representation of a function f t is given by this expression where b omega is this.

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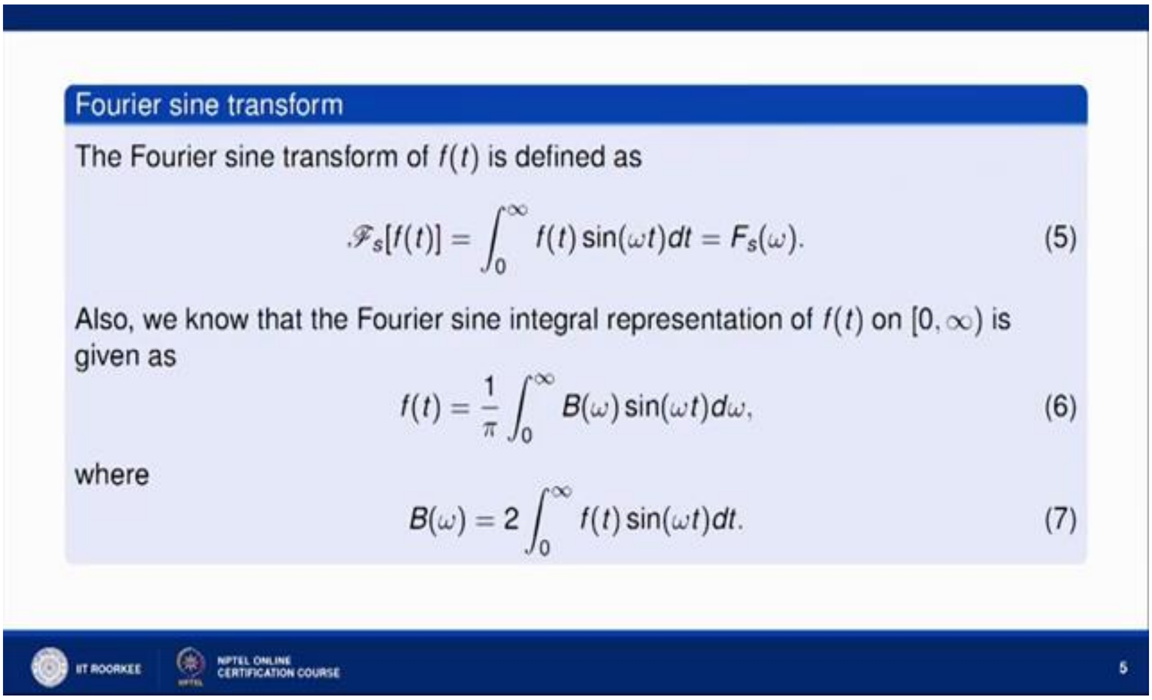

Now, see if compare this with this expression, so B omega is nothing but two times F s omega. So, if you substitute this, this expression B omega 2 time F s omega over here sorry over here. So, it will be 2 upon pi times 0 to infinity F s omega sine omega t dt. So, what is a Fourier sine integral representation of function, Fourier sine integral representation of function f t, it will be nothing but if it is 0 to infinity f t sin omega t dt, which we are given as Fourier sine integral representation denoted by this expression. And the inverse of Fourier sine transform of f t will be nothing about 2 upon pi times integral 0 to infinity f x omega sin omega t into d omega which is nothing but f t.

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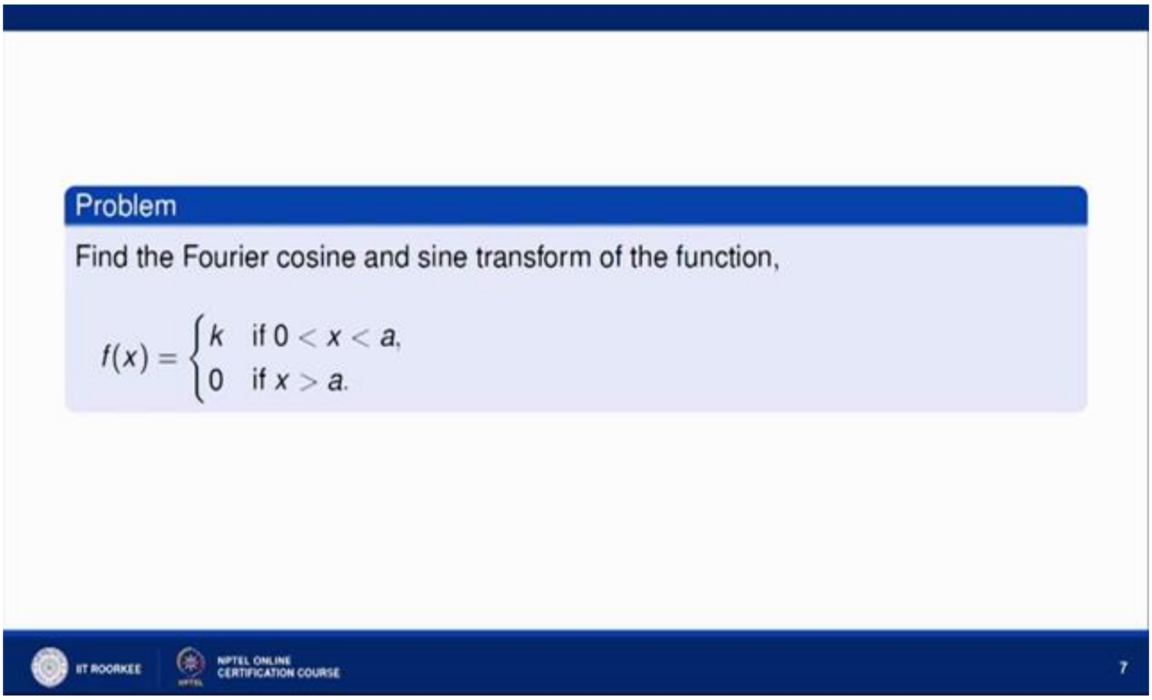

So, now let us find out Fourier sine and cosine transform of this function. So, suppose you find if f x is what f x is given as k, when 0 less than x less than a, and it is 0 when x is greater than a. So, suppose I want to find out Fourier cosine transform this function, so this will be equal to 0 to infinity f t cos omega t dt which is equals to 0 to a it is k into cos omega t dt which is equals to k. Now, when we integrate this is sin omega t upon omega from 0 to a, which is nothing but k upon omega sine omega here. So, this is basically Fourier cosine transform of this function f t.

Now, similarly if you want to find out Fourier sine transform this function, so Fourier sine transform this function f t will be nothing but again 0 to infinity f t sin omega t into dt, which is equals to 0 to k f t f t is k sin omega t into dt which is equal to k can come out it is minus cos omega t upon minus upon omega from 0 to a, it is minus k upon omega times it is cos omega a to minus 1. So, it is k upon omega times 1 minus cos a. So, this will be the Fourier sine transform of this function f t. So, in this way we can find out Fourier cosine or Fourier sine transform of any function f t using these expressions.

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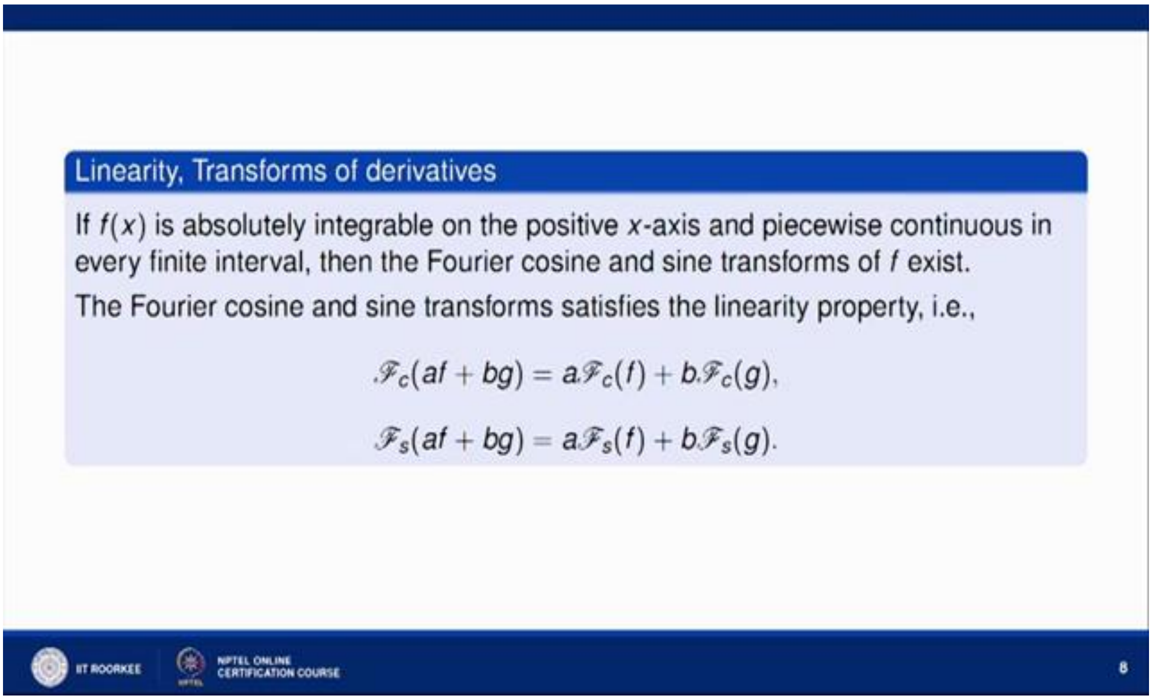

Now, Fourier transforms also satisfy some properties. What are they? Now, first of all the existence condition if f x is absolutely integrable on the positive side of x-axis, and piecewise continuous in every finite interval then the Fourier cosine and sine transform of f exists. So, this is the condition that function must be absolutely integrable on the positive side of x-axis, and it must be piecewise continuous, then Fourier sine and cosine transform of function will exist. Now, the property is the first property is Fourier cosine and sine transform also satisfy linearity property.

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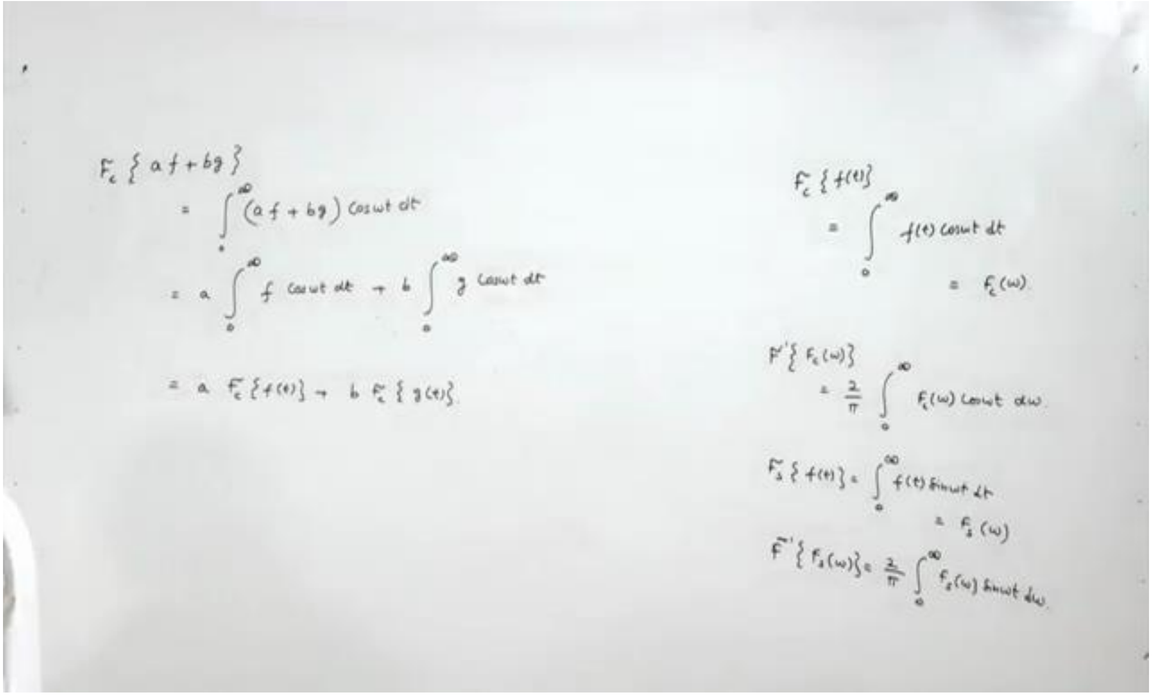

So, this property all satisfy that we can easily see also that is easy to prove. Basically what is Fourier cosine transform of suppose a f plus b g, it is nothing but by a definition it is 0 to infinity a f plus b g into cos omega t dt, so that will be nothing but a times 0 to infinity f into cos omega t dt. This has to be the function of t, g is the function of t plus b times 0 to infinity g cos omega t into dt. So, this is nothing but a times f c of f t and this is plus b times f c of g t. So, hence Fourier cosine transform satisfy linearity properties, similarly we can we can show that a Fourier sine transform also satisfy linearity property, the first thing.

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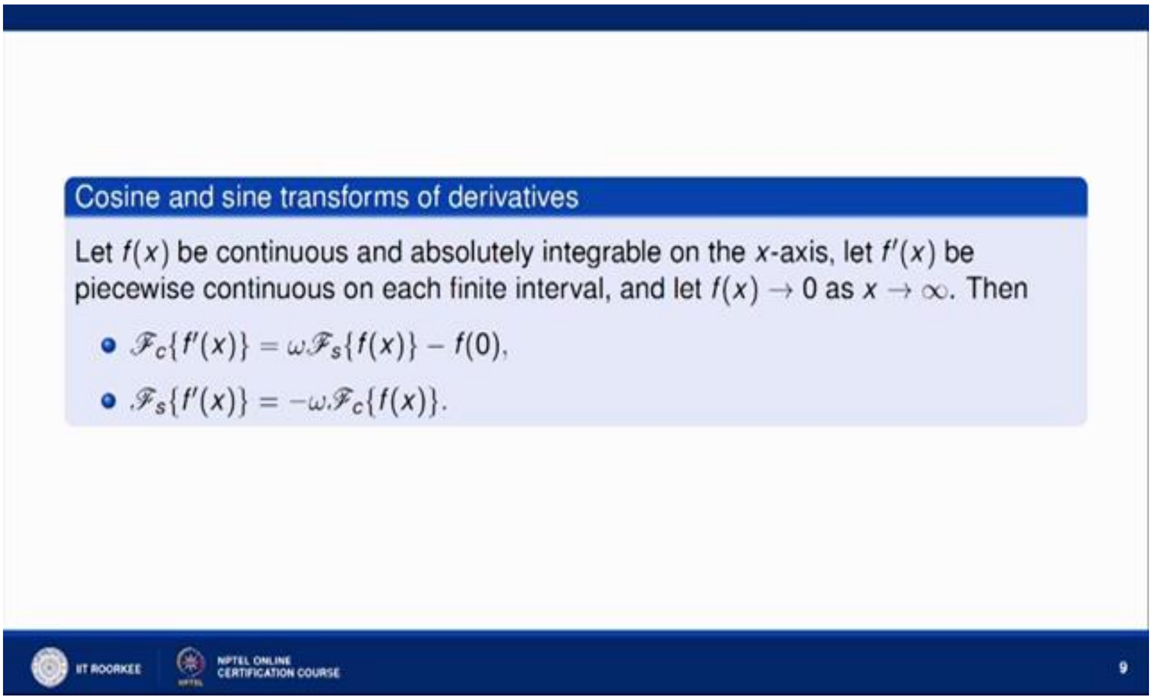

The second thing is cosine and sine transforms derivatives. Now, if function is continuous and absolutely integrable on the positive side of x-axis and f dash is piecewise continuous on each finite interval and suppose f x tends to 0, as x tends to infinity then these two results hold.

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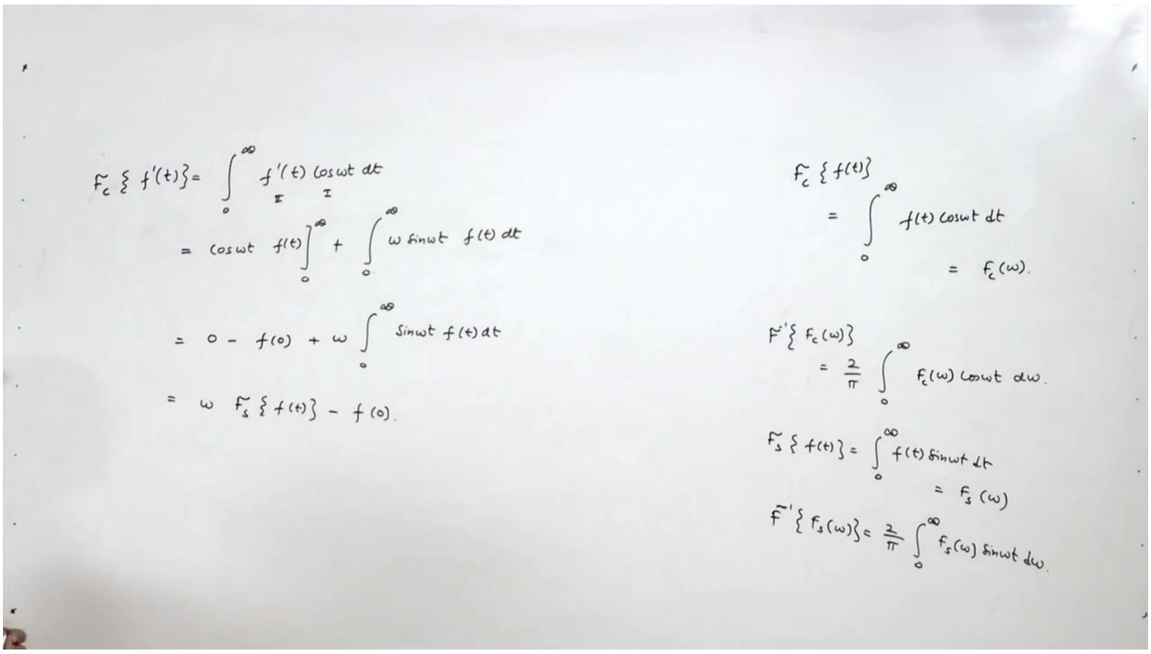

Now, let us derive these results. So, what is Fourier cosine transform of f dash x that will be nothing but by definition or our f t, f t or f x you can say anything that will be a 0 to infinity f dash t into cos omega t into dt. This is by the definition of Fourier cosine transform. You simply replace f by f dash in this expression, so that will be given by is equal to now you apply integration by parts, you take it first function, you take a second function first as it is. Now, integration of second it is f t from 0 to infinity minus integration derivative of first is omega sin omega t with negative sign, the negative, negative – positive, and it is f t dt.

Now, when you take upper limit, now this we are assumed that as s tending to infinity or t tending to infinity f x tend to 0. So, as t tend to infinity this will tend to 0. So, 0 into some finite quantity will be 0 and that will be it is nothing but 0 minus f 0 because when t is 0, it is f 0 plus omega times integral 0 to infinity, it is a sin omega t into f t dt. So, this is nothing but omega will come here, this omega and this is nothing but Fourier sine transform of f t, this is Fourier sine transform of f t. So, this is Fourier sine transform of f t minus f 0. So, this is the derivation for the first part that the Fourier derivative of Fourier cosine transform of f will be nothing but omega times Fourier sine transform of f t minus f 0.

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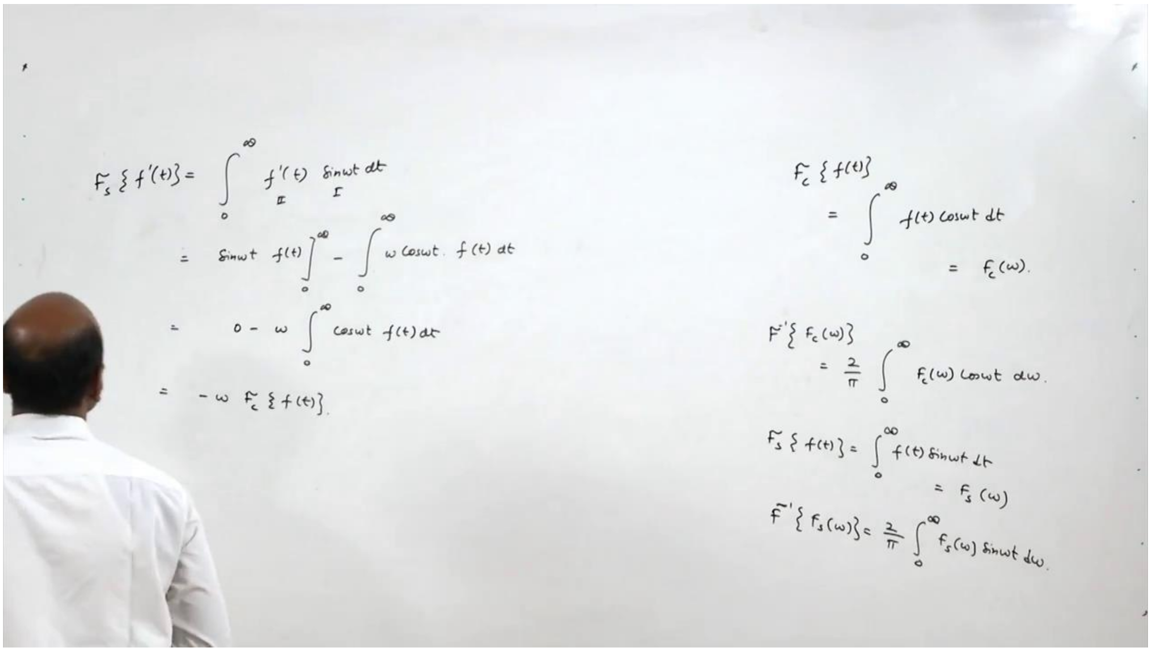

Now, second part, Fourier sine transform of f dash x. So, Fourier sine transform of f dash f for t is nothing but 0 to infinity f dash t into sin omega t dt. This is by a definition of Fourier sine transform. You simply replace f by f dash t in this expression. Now, you again apply integration by parts it is 1, it is 2. Now, it is first as it is integration of second from 0 to infinity minus integral derivative of first is omega cos omega t into f t dt. So, when t tend to 0, when t tend to infinity f t tend to 0, this is by this assumption, so this is tend to 0. Now, when t is 0, sin 0 is 0. So, we have 0 here from both the limits minus omega will come out, it is 0 to infinity cos omega t into f t dt.

So, this is minus omega times this is nothing but Fourier cosine transform of f t, it is Fourier cosine transform of f t. So, this is how we can obtain the Fourier derivative of Fourier sine transform which is nothing but minus omega time Fourier cosine transform of f t. Now, suppose you want to find out Fourier cosine transform of second derivatives, so that also we can find out in the same result.

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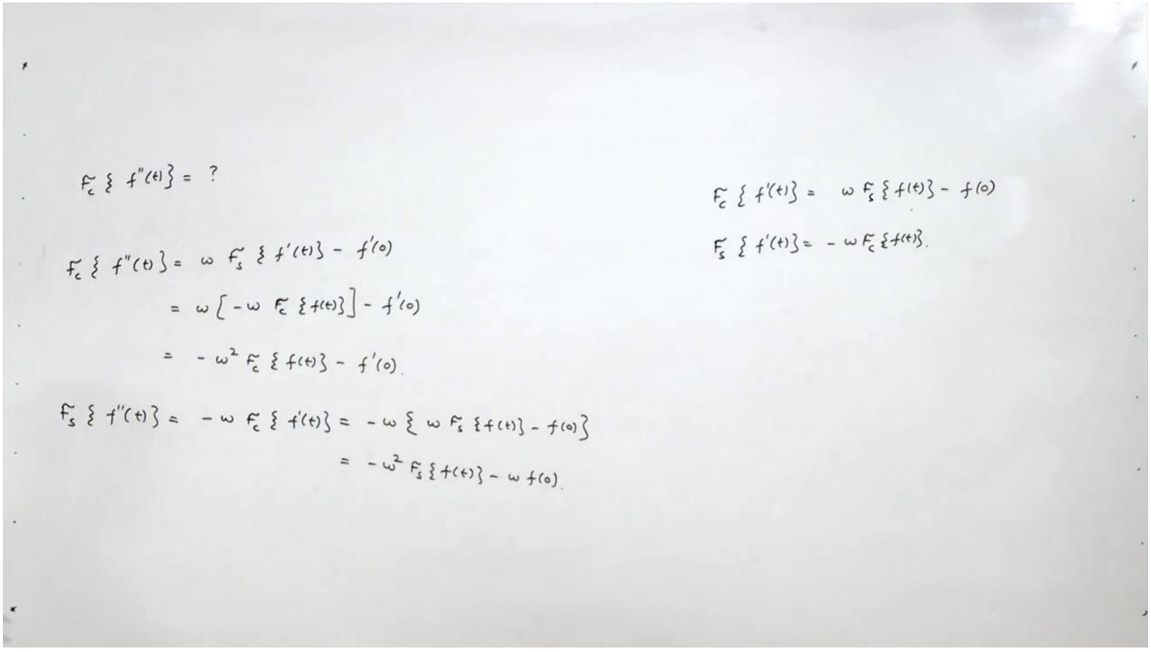

Now, what is Fourier transform I have just Fourier cosine transform f dash x, Fourier cosine transform f dash x is nothing or f dash t is nothing but this is we have derived it is omega times Fourier cosine transform of f dash f x minus f 0, this we have derived. Next is Fourier sine transforms of f dash Fourier sine transform of f dash t is nothing but minus omega Fourier cosine transform of f t. So, these expressions we have just derived.

Suppose, now you want to find out Fourier cosine transform of double derivative of f t. So, this is we want to find out. It is Fourier cosine that must be signed.

Now, suppose I want to find out Fourier cosine transform of second derivative of f t. So, this will be nothing but now you simply replace f by f dash in this expression. So, while you replace s by f dash in this expression. So, Fourier sine transform of f double t will be nothing but omega times Fourier sine transform of f double dash f dash t, we are replacing f by f dash t. So, it is f dash t minus f dash 0 which is equals to omega times. Now, Fourier sine transform of f dash t is we already know it is minus omega Fourier cosine transform of f t minus f dash 0. So, when you simplify, it is nothing but minus omega square Fourier cosine transform of f t minus f dash 0. So, this will be the Fourier cosine transform of second derivative of f.

Now, similarly if we want to find out Fourier sine transform of second derivative of f, so that also you can find out. Now, in this expression, you simply replace f by f dash. When you replace f by f dash in this expression, it is minus omega Fourier cosine transform of f dash t now this equals to minus omega. Now, Fourier cosine transform of f dash t from this expression is nothing you can simply substitute it here, it is omega Fourier sine transform of f t minus f 0, so that will be nothing but minus omega square Fourier sine transform of f t minus omega f 0. So, it is minus omega square into minus, minus – plus, so it will be plus.

So, in this way, we can obtain the Fourier transform of second derivative of f or the Fourier sine transform second derivative of f. So, similarly if we want to obtain the Fourier cosine or sine transform of third derivatives or the higher derivatives, so that also we can obtain using the same concept. So, this is how we can find out Fourier sine or cosine transform of function, if a function is known to us, and how we can find out Fourier sine or cosine derivatives that also we have seen or the higher order derivatives that also we have seen.

So, in the next lecture we will see some more properties of Fourier transform and how the problems can be solved on that that we will see in next class.

So, thank you very much.