Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 55
Fourier Transforms

Welcome to lecture series on Mathematical Methods and its Applications. We have discussed about Fourier series and Fourier integrals. Now, we will come to Fourier transforms. Fourier transforms like we did in we did Laplace transforms, its inverse and its applications, similar way we will see what are Fourier transforms, its inverse, and what are the applications. So, Fourier transform is a integral transform which is similar to Laplace transforms it has wide applications in many areas in engineering, sciences, medicine, etcetera.

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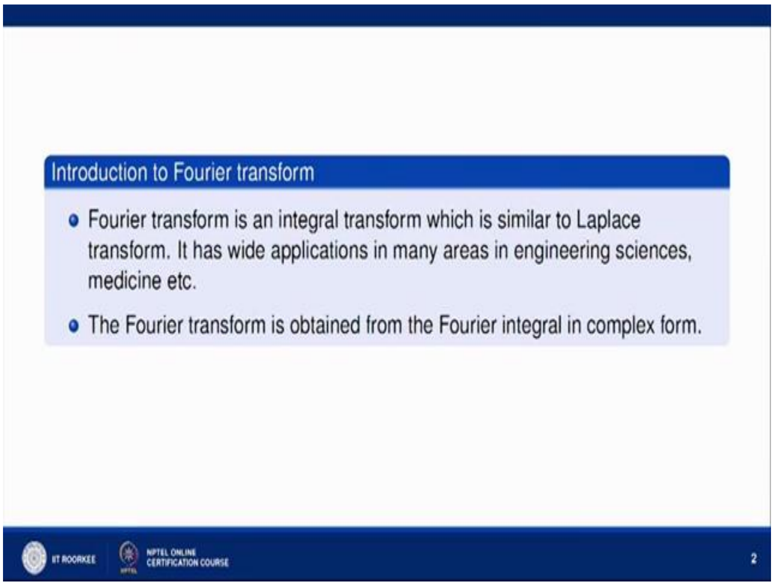

Now, this Fourier transform is obtained from the Fourier integral in complex form. Now, what is it and how it is and how we obtained Fourier transform. Let us see.

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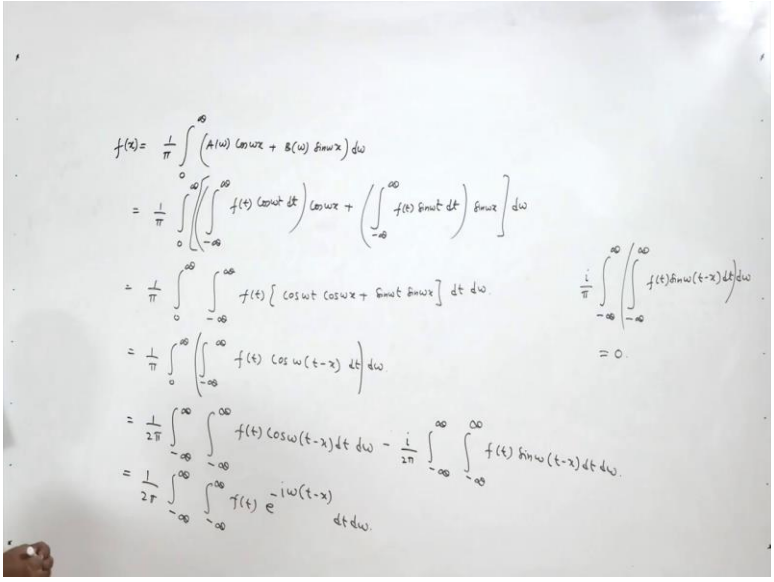

Now, what is a Fourier integral representation of function f x it is nothing but one upon pi times integral 0 to infinity A omega cos omega x plus B omega sin omega x into d omega. So, this is a Fourier integral representation of function f x. Now, substitute the value of A omega and B omega here, it is nothing but 1 upon pi integral 0 to infinity integral minus infinity to plus infinity it is f t, it is cos omega t dt and cos omega x plus integral minus into plus infinity f t sin omega t into dt and sin omega x and this whole with d omega.

Now, this can be further written as 1 upon 2 pi integral 0 to infinity integral minus to plus infinity, now this is f t, f t we can take common in both the expressions. So, f t is common, and it is nothing but cos omega t into cos omega x plus sin omega t into sin omega x, and it is dt is first value and dt into d omega. So, this is nothing but 1 upon pi times integral 0 to infinity integral minus to plus infinity, it is f t, it is cos a cos b plus sin a sin b one can easily add up, so it is cos a minus t d t into d omega.

Now, we can easily see that this function is even in omega, this function is even in omega. So, we can easily write this as we can divide and multiply by 2. So, it is 1 upon 2 pi times integral minus infinity to plus infinity minus infinity to plus infinity f t cos omega t minus x dt into d omega, because it is even in omega, so that is why we can multiply divide by 2 and this we can write minus infinity to plus infinity. Now, iota upon if you see this value iota upon pi times integral minus infinity to plus infinity minus infinity to plus infinity f t into sin omega t minus x dt into d omega. So, we can easily see that since it is odd in omega of because of presence of sin omega term only here, it is odd in omega, so this will be 0.

So, if we add 0 here, so the value will not change value will remain same. So, we can add or subtract, so we do it is minus 1 upon 2 iota 2 pi minus into plus infinity of this f t sin omega t minus x dt into d omega, because this value is 0, 1 by 2 of this is also 0. So, 1 by 2 of this is will be also be 0. So, if we subtract 0 from this expression, so the value remains unchanged. So, now this is nothing but we can take 1 upon 2 pi common from both the sides, it is minus into plus infinity minus into plus infinity when we take f t common, so it is cos theta minus iota sin theta. So that will be nothing but e k power minus t iota theta, theta is omega t minus x into dt into d omega. So, this is the complex form of Fourier integral representation.

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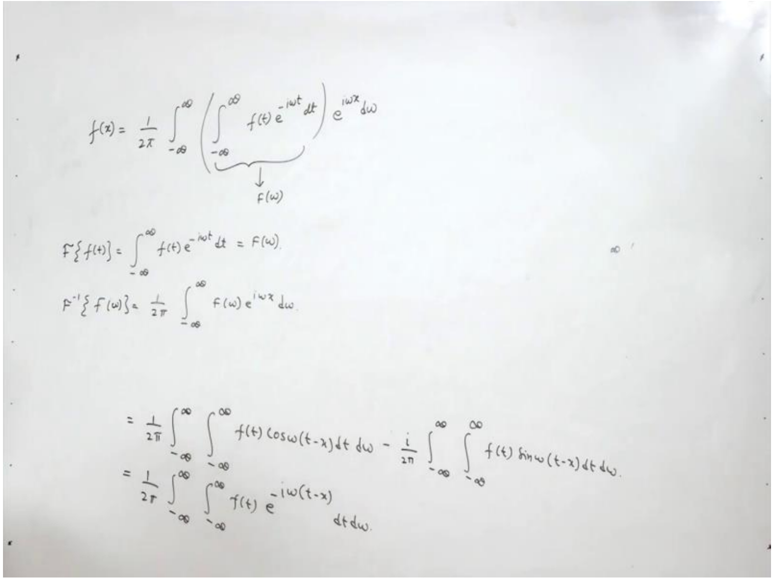

Now, if you simplify it further, so what we obtain. So, we obtain this is f x. So, f x will be nothing but 1 upon 2 pi times integral minus into plus infinity minus to plus infinity it is f t e k power minus iota omega t into dt and whole multiplied by e k power iota omega x d omega. So, this we are calling as this is a function of omega. So, suppose it is f capital F omega. So, this we call as Fourier transform of function f t. So, Fourier transform function f t will be given by minus infinity to plus infinity f t e k power minus iota omega t into dt which we calling as F omega. And the inverse Fourier transform inverse Fourier transform of F omega will be nothing but now we substitute f omega here to get f x, so that will be the inverse Fourier transform it is 1 upon 2 pi times integral minus to plus infinity F omega e k power iota omega x t omega. So, that will be the inverse Fourier transform of F omega which is nothing f x. So, this is how we can define Fourier transform of function f t and its inverse now. So, this is how we can define Fourier transform of function f t and its inverse.

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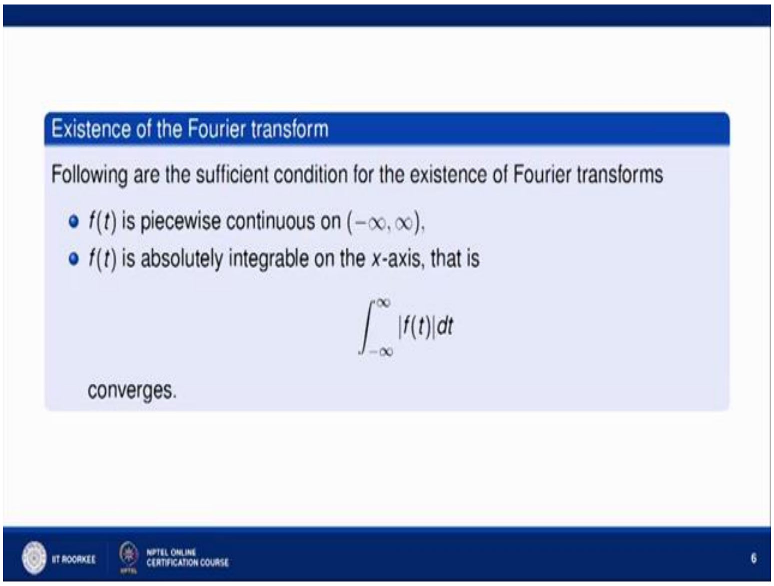

Now, existence conditions for the Fourier transform. So, following are the sufficient condition for the existence Fourier transforms, first is it must be piecewise continuous on minus infinity to plus infinity, and f should be absolutely integrable on the x-axis that is integral minus to plus infinity mod f t dt should converge. So, if you have these two conditions, so that will be the existence conditions for the Fourier transforms.

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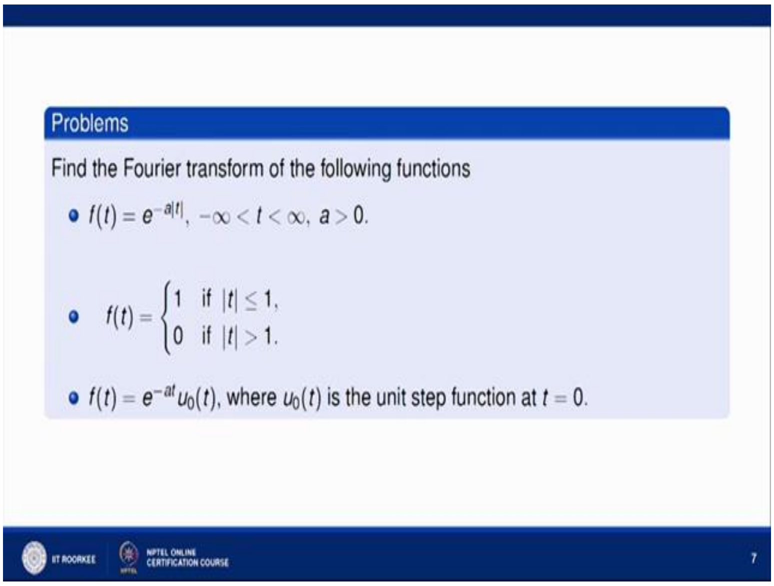

Now, let us find out Fourier transform of these few problems. First problem is f t is e k power minus a mod t when t lying between minus to plus infinity and a is greater than 0.

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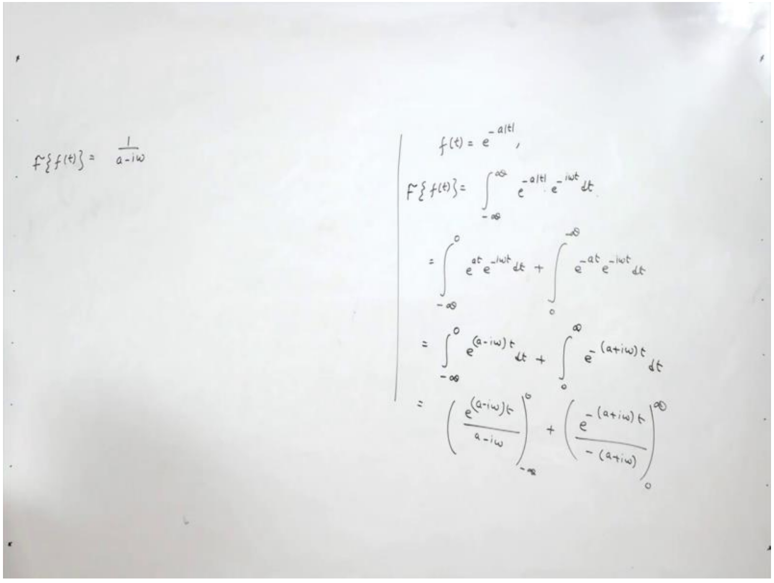

Now, we have to find Fourier transform of this function. So, Fourier transform of this function will be given by minus infinity to plus infinity, you can see in definition it is f t into e k power minus iota omega t into dt, this is by definition of Fourier transforms. Now, this will be equal to when t is negative minus will 0, so it is e k power minus minus - plus a t, and when t is positive, it remains same. Now, it is further equal to minus will 0 e k power a minus iota omega into t dt plus it is 0 to infinity e k power minus a plus iota omega t into dt. So, this can be further be solved e k power a minus iota omega t upon a minus iota omega from minus minus will 0 plus e k power minus a plus iota omega t upon minus a plus iota omega from 0 to infinity.

Now, when you simplify this is what we will get which is the Fourier transform of f t, what we will get it is a 1 upon a minus iota omega, where it is minus infinity tend to 0. Again it is tend to infinity, it is 0; and when it is 0 it will be minus, minus - plus 1 upon a plus iota omega. So, this value will be nothing but 2 a upon a square plus omega square which is f omega. So, this will be the Fourier transform of this function f t.

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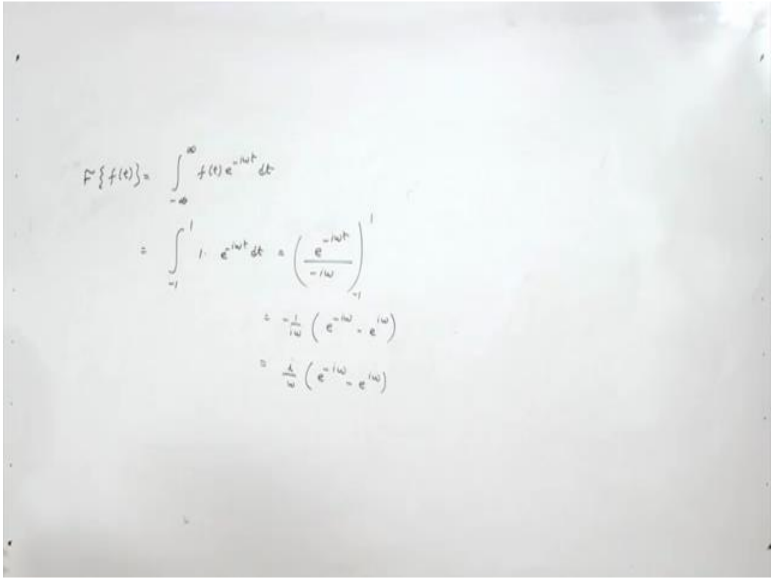

Now, Fourier transform of next problem when f t is 1. So, Fourier transform of this function by definition Fourier transform, what we have it is minus into plus infinity f t e k power minus iota omega t into dt. So, this is how we define Fourier transform function f t. Now, it is from minus 1 to plus 1 only when it is 1 into e k power minus iota omega t into dt. So, it is very simple to find its value. So, it is minus 1 upon iota omega will come here, it is e k power minus iota omega minus e k power iota omega. So that will be nothing but minus, minus - plus iota upon omega e k power minus iota omega minus e k power minus iota omega, so that will be the Fourier transform of this function f t.

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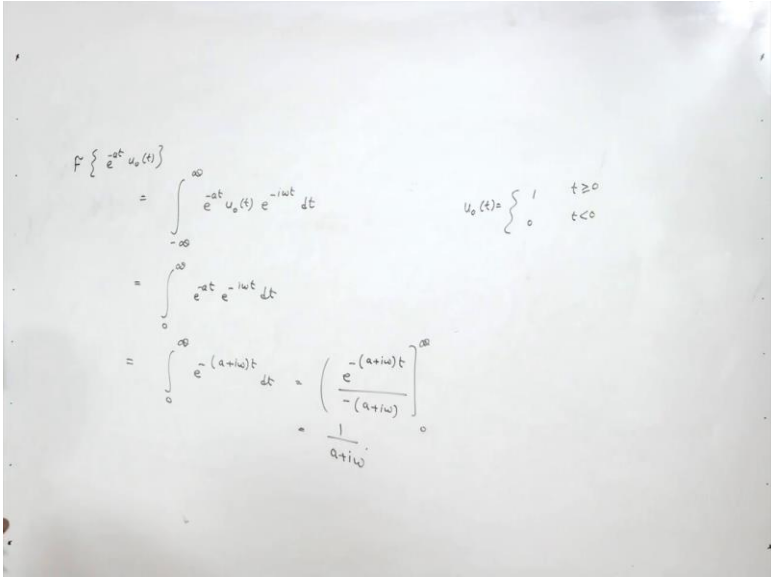

Now, Fourier transform of the next problem. So, it is again easy to find out it has unit step function at t equal to 0. So, Fourier transform of this function that is e k power a t u naught t will be nothing but minus to plus infinity f t e k power minus iota omega t dt, so that will be nothing but. Now unit step function at u equal to 0 is defined as it is one when t is greater than equal to one and 0 when t is less than 0 so that means, it is when 0 to 1, it is only one and e power a t e k power minus iota omega t dt, so that will be nothing but 0 to 1 e k power minus a plus iota omega t into dt. It is minus will come here.

When you integrate this, this is nothing but e k power a t is here, minus a t is here it is e k power minus a t the problem it is e k power minus a t, it is minus, it is minus, so it will be plus. So, when you integrate this, it is e k power minus a plus iota omega t upon minus a plus iota omega from 0 to infinity. Because when t is 0 to infinity, it is 1, so it is 0 to infinity. So, it is 0 to infinity. So, when infinity, it is 0; when 0, it is 1 upon. So, this is the Fourier transform of this function. So, this is how we can find out Fourier transform of function f t using the simple definition of f t.

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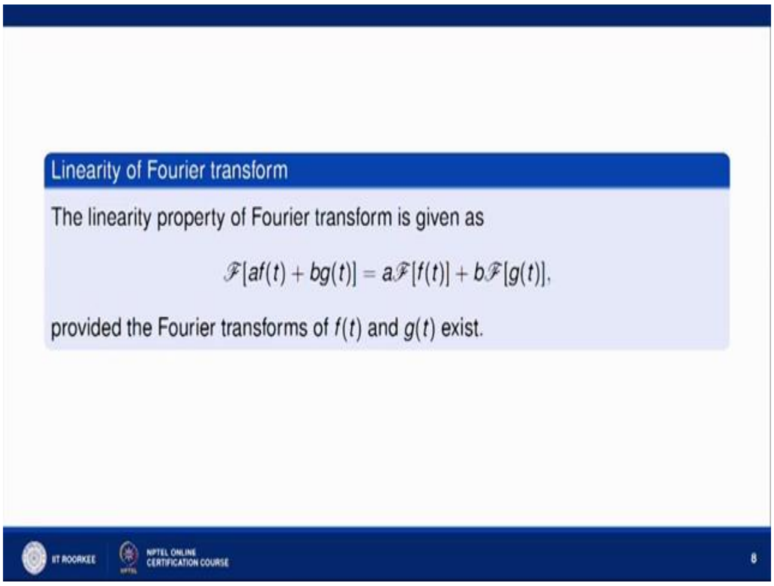

Now, let us see some properties of Fourier transform. The first properties it satisfies linearity property Laplace satisfy linearity property in the same way Fourier transform also satisfy linearity property.

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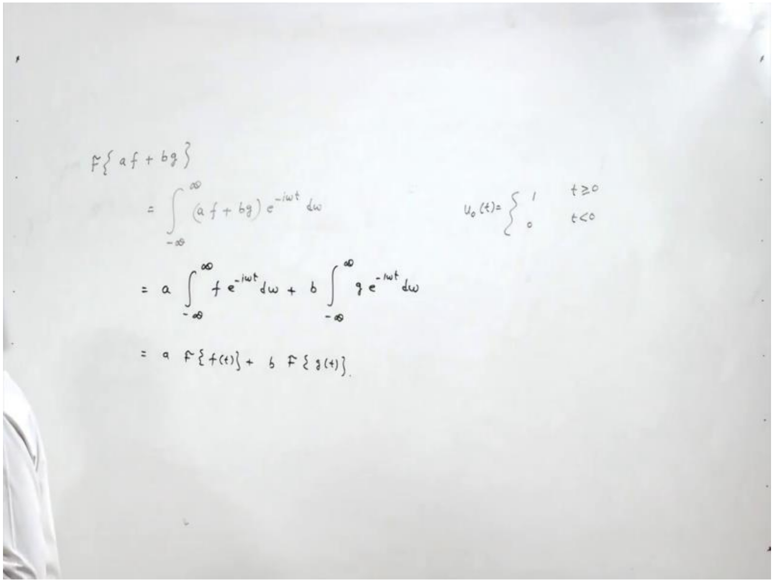

So that we can easily show also because what will be the Fourier transform of a f plus b g, it will be nothing but minus into plus infinity a f t plus b g t into e k power minus iota omega t into d omega by the definition of the Fourier transforms. So, that will be nothing but a times minus infinity to plus infinity f e k power minus iota omega t into d omega plus b times integral minus to plus infinity g e k power minus iota omega t into d omega which is nothing but a times Fourier transform of f t and plus b times Fourier transform of g t. So, this is the linearity property which Fourier transform satisfy number one.

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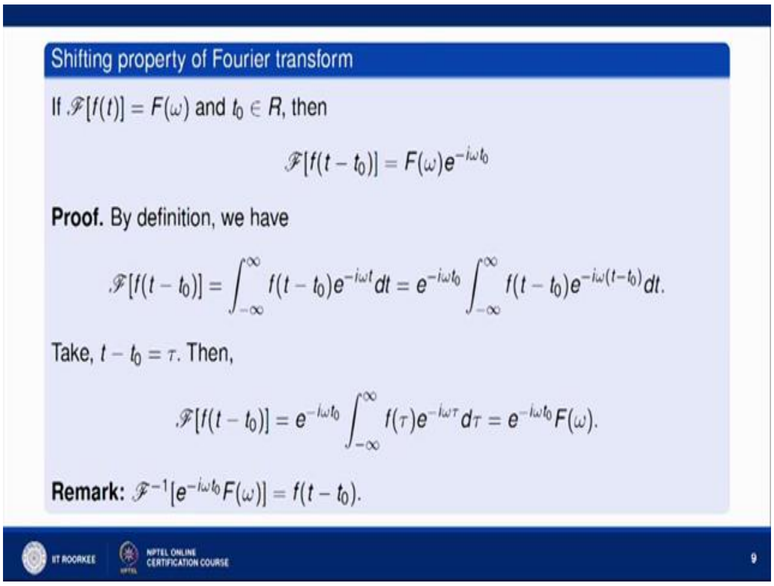

Number two is the shifting property shifting property shares that a Fourier transform f t is f omega and t where belongs to r then Fourier transform of f t minus t naught is nothing but given by this expression. So, this also we can easily show. You see that Fourier transform of f of t minus t naught will be given by again you apply formula minus infinity to plus infinity f t minus t naught e k power minus iota omega t into d t.

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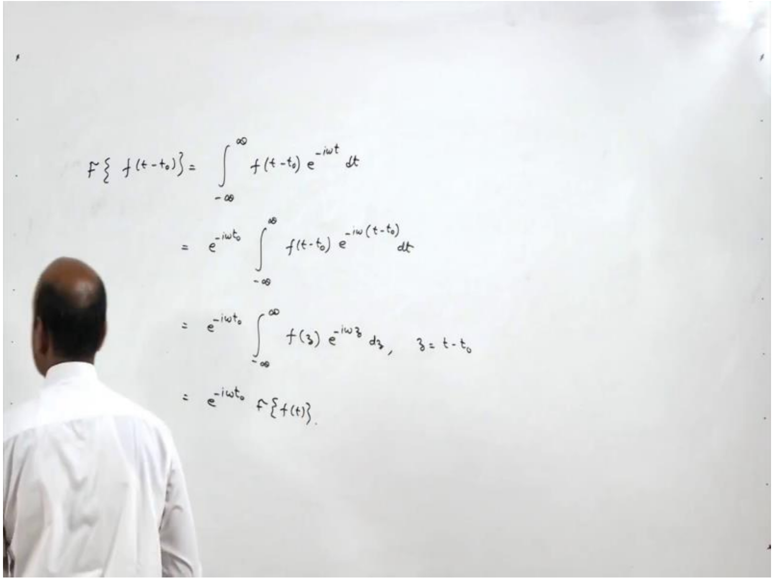

So, this can be written as e k power minus iota omega t naught you multiply and divide by this expression, because this is free from t. Now, if you take t minus t naught as z, so it will be nothing but e k power minus iota t naught minus infinity to plus infinity x z limits will not change e k power minus iota z t z if z is t minus t naught, so that is nothing but Fourier transform of f t. So, it is nothing but e k power minus iota omega t naught and Fourier transform of f t or f z. So, this is how we can obtain this shifting property.

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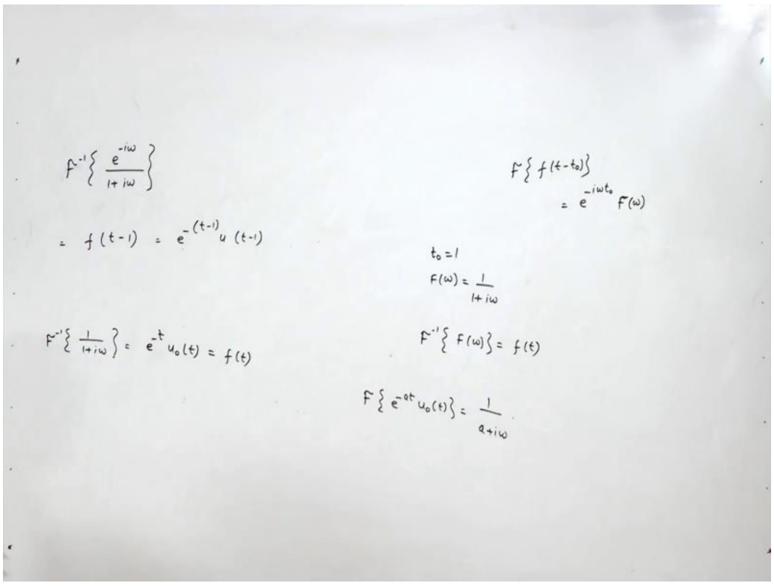

Now, let us find out inverse of these two problems Fourier inverse of Fourier transforms. So, whenever we have e k power minus iota omega t naught, so recall shifting property. So, again we recall the shifting property. Shifting property states that Fourier transform of f t minus t naught is nothing but e k power minus iota omega t naught into F omega, where F omega is Fourier transform of f t this is the shifting property. Now, let us apply this property to find out the inverse of this.

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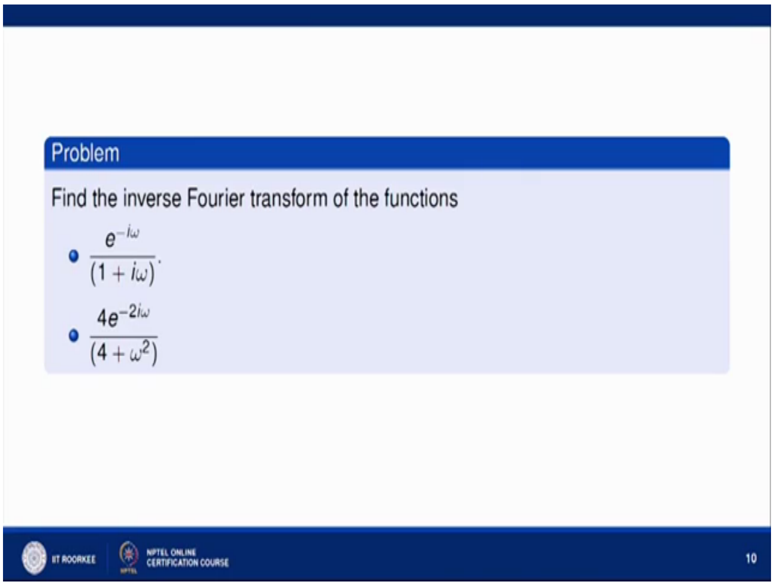

Now, here in the first problem Fourier inverse of e k power minus iota omega upon 1 plus iota omega, is it so? Now here instead of t naught we have 1, and f omega is 1 upon 1 plus iota omega. So, if you compare these two, so t naught is 1 and f omega is 1 plus iota omega. So, the inverse of this will be given by f of t minus 1, where a Fourier inverse of f omega is f t. Now, what will be the Fourier inverse of this? So, we have just find that Fourier transform of e k power minus a t u naught t is nothing but 1 upon a plus iota omega. Let us just find that Fourier transform of e k power minus a t u not t is one upon a plus iota omega.

So, here a is 1 so that means, f t is nothing but that means, Fourier inverse transform of 1 upon 1 plus iota omega will be nothing but 1 upon 1 will be nothing but here a is 1 to substitute a as 1 to e k power minus t u naught t, so this is f t. And what will be f t minus one then you simply replace t by t minus 1, so that will be nothing but e k power minus t minus 1 u naught t minus 1, I mean u t minus 1, so that will be the inverse Fourier transform of this function.

Now, similarly if you want to find out Fourier inverse transform of second function you again apply shifting property shifting property, which is given by this. Here t naught is 2 and the remaining expression 4 upon 4 plus omega square, 4 upon 4 plus omega square, you can take as F omega and you can find out Fourier inverse of 4 upon 4 plus omega using the previous examples which we have discussed. So, in this way we can find out Fourier transform and inverse Fourier transform of few problems.

Thank you very much.