Mathematical methods and its applications
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Lecture - 54
Fourier sine and cosine integrals

Welcome to the lectures series on Mathematical Methods and Applications. In the last lecture we have seen what Fourier integrals are, and how we can if we have a Fourier series, I mean of a periodic function. How we can represent a non-periodic function from minus infinity to plus infinity by the Fourier integral that we have seen in the last class.

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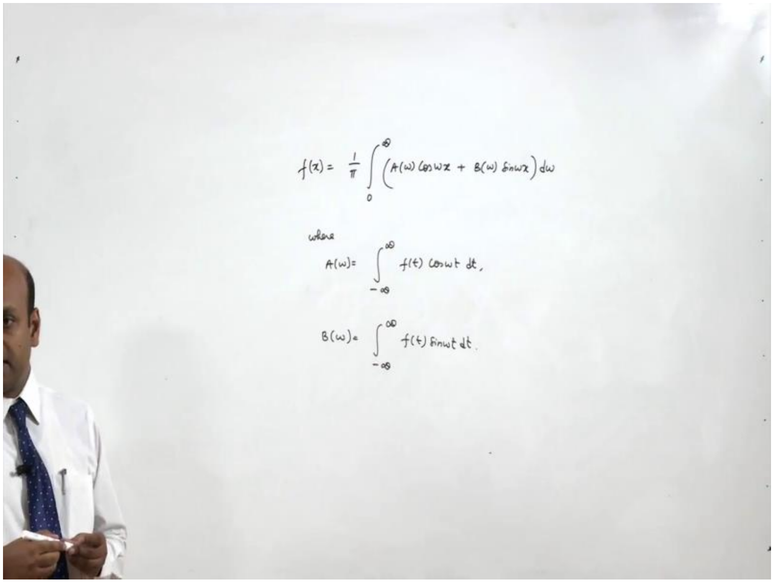

So we have seen that if we have a function f x then the Fourier integral representation of the f x will be nothing but Fourier integral representation will be nothing but 1 upon pi integral 0 to infinity a omega cos omega x plus b omega sin omega x into d omega, and where a omega is given by minus infinity to plus infinity f t, cos omega t d t, and b omega is given by minus infinity to plus infinity, f t sin omega t into d t. So, that we have seen in the last class that, if we have a function f x, then the Fourier integral representation of function f x is given by 1 upon pi integral 0 to infinity a omega cos omega x plus b omega sin omega x d x, d d omega where a omega is given by this expression and b omega is given by this expression.

Now, let us see Fourier sine and cosine integrals. Now it is possible to define Fourier sine and cosine integral representation of function f x on a half real line that is from 0 to infinity.

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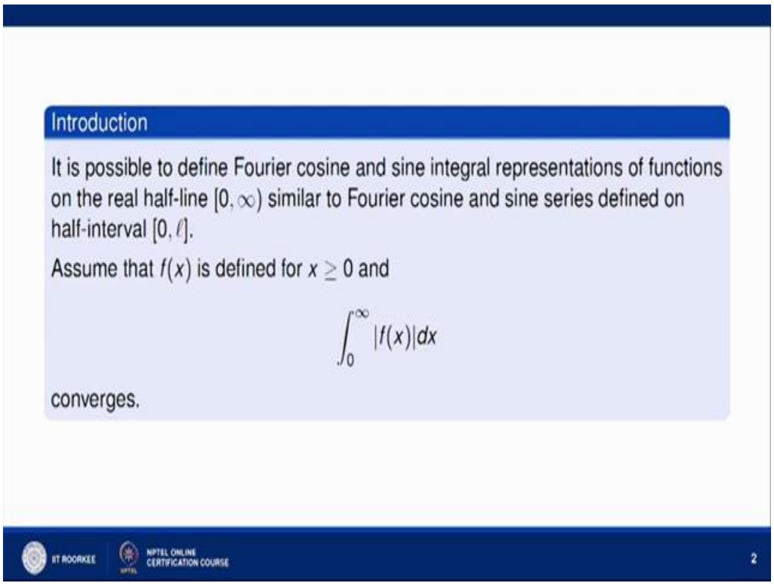

Similar to the Fourier sine and cosine series we find half interval, that we have already discussed if we have a half interval, if an half interval 0 to l how we can represent Fourier sine and cosine series, what we did there, basically we extend if you want Fourier sine series we extend the function assuming the function as an even function. And then we extend the function from minus l to 0 also. And hence we got the function from minus l to plus l, and periodic though that is why we can write it is Fourier series Fourier cosine series expansion.

Similarly, for the Fourier sine series, we have already observed that if we want Fourier sine series to we assume function as an odd function we take the function as an odd function, or we extend the function as an odd function from minus l to 0, and in this way we can extend we can write it is Fourier sine series expansion. In the similar way if we want Fourier sine or cosine integral representation of a function from in the half real line, we similarly assume function to be even function for the Fourier cosine integral representation and for the Fourier sine integral representation; we assume function as an odd function.

So, how we will do that let us see. Now in this we assume that function is defined from for x greater than or equal to 0, and function and integral 0 to infinity mod of f x d x is convergent or converges, that we assume. Now how we can find out Fourier cosine integral representation of function. So, we use even extension of f x to R in order to obtain the Fourier cosine integral representation of f x. So, that is we defined as another function g x. We define function g x in such that g x is an even function. And which is nothing but the even extension of the function f x.

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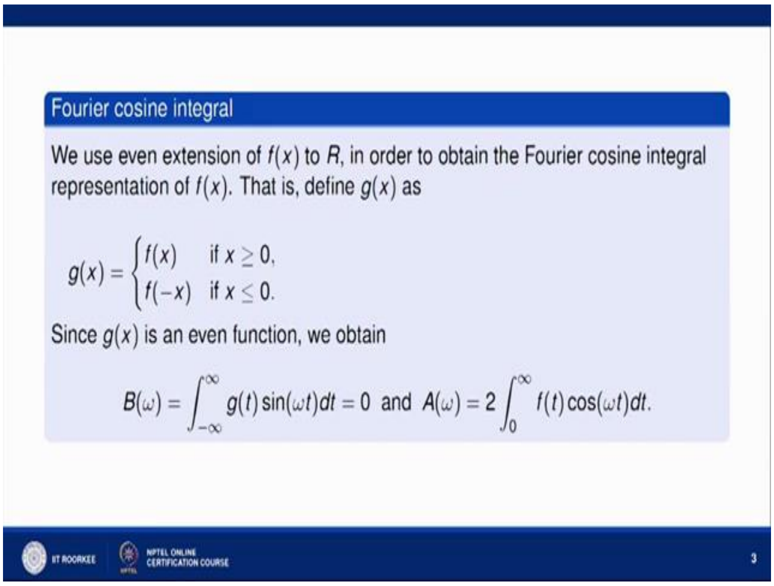

You see if we define function like this which is f x when x is greater than or equal to 0. And it is f of minus x when x is less than or equal to 0. And when we replace x by minus x in g x in this definition what is the definition here.

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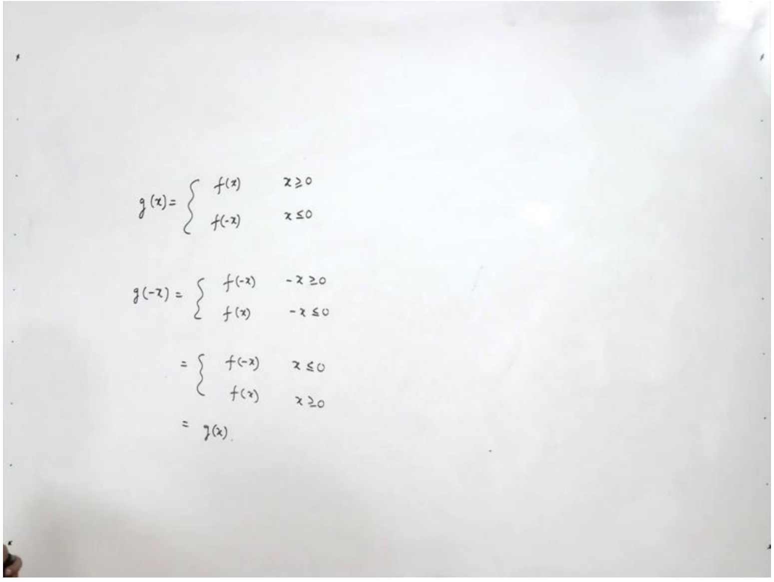

So, definition here is, you can simply see that g x is nothing but f x when x is greater than or equal to 0, and it is f of minus x when x is less than or equal to 0. So, when you find g of minus x it is nothing but f of minus x, when x minus x is greater or than equal to 0. And it is f x when minus x is less than or equal to 0. So, this is equal to f of minus x when x is less than or equal to 0, and f of x when x is greater than or equal to 0. So, this is same as this function. So, this is nothing but g x.

So, therefore, we extend the function f x, in such that, f x is an even such that g x is an even function. So, on the negative side where function where we do not know the value of the function, we assume that function is f minus x so that this function become an even function. Now we can write it is now g x is an even function. Now we can write it is Fourier cosine integral representation. How? Now g t is an even function, here g t is an even function. This we have already defined here g t is an even function. Since g t is an even function and cos sine is an odd function. So, even into odd is an odd function. So, this will be 0 and a omega we know a omega is defined from minus infinity to plus infinity, and this is an even function I mean g t is an even function, and this is an even function. So, even into even is an even. So, this will be 2 times. So, in this way if we write it is Fourier cosine integral representation that will be given by f x will be 1 upon pi integral 0 to infinity omega cos omega x t d x because b omega is 0. And where a omega is given by 2 times 0 to infinity f t cos omega t d t.

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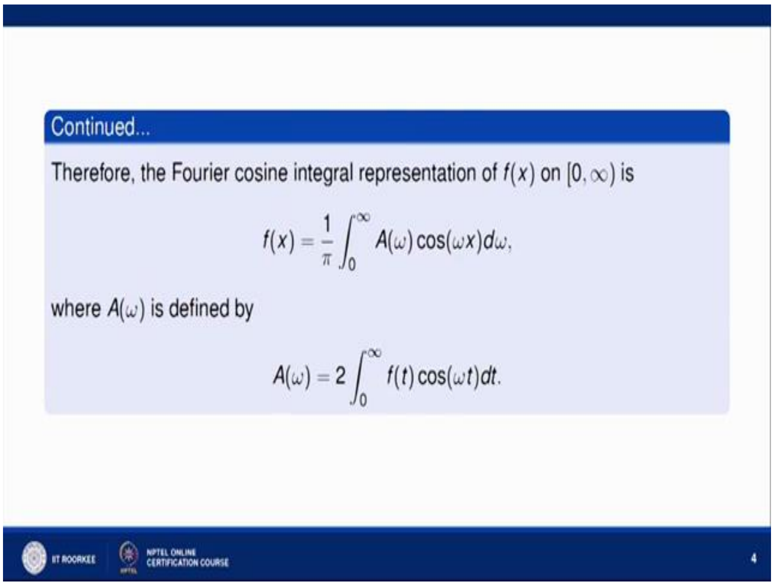

Now, similarly if you want Fourier sine integral representation of a function, which is defined in the half range I mean from 0 to infinity only.

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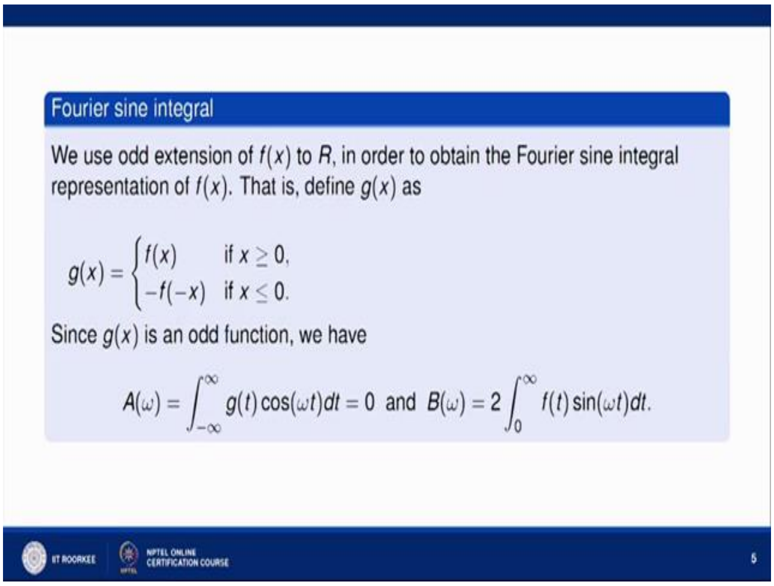

So, we extend the function such that the new function g x become an odd function; that means, we are sustain g x as f x when x greater than equal to 0, and minus of minus, minus of f minus x when x is less than equal to 0. Now here where you take g of minus x, that will be nothing but minus g x; that means, g x is an odd function. Now assuming g x is an odd function because g x is an odd function.

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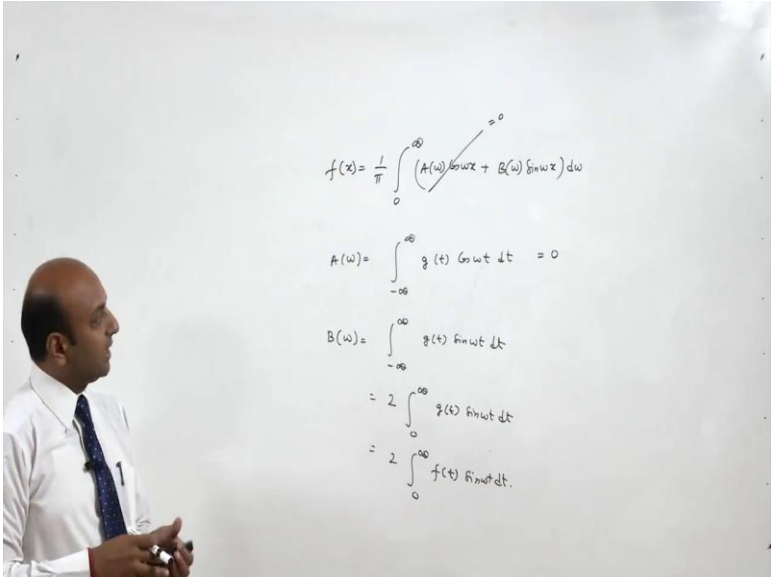

So, when we take Fourier integral representation of function which is given by this expression, 1 upon pi integral 0 to infinity a omega cos omega x plus b omega sin omega x, into d x, into d omega, where a omega is nothing but minus infinity to plus infinity.

Now, here a omega is here f t is nothing but g t, g t cos omega t d t. Because for the full range for the full range minus infinity to plus infinity we have g t and this g t is an odd function odd into even is and odd function this will be 0, and b omega which is given by minus infinity to plus infinity g t sin omega t into d t. So, this will be odd into odd is an odd function. So, this will be this becomes and the multiplication of 2 odd function is even function. So, this is an even function. So, this will be nothing but 2 times 0 to infinity g t into sin omega t d t.

And from 0 to infinity g t is nothing but f t. So, it will be nothing but 2 times 0 to infinity f t sin omega t into d t. So what will be the Fourier sine integral representation of a function? Fourier sine integral representation of a function will be given by now this a omega is 0, this term will become 0. That will be nothing but f x equals to 1 upon pi integral 0 to infinity b omega sin omega is d omega where b omega is given by this expression. So, this will be the Fourier sine integral representation of a function f x. So, that is given in the next slide. So, this will be the Fourier sine integral representation of a function f x.

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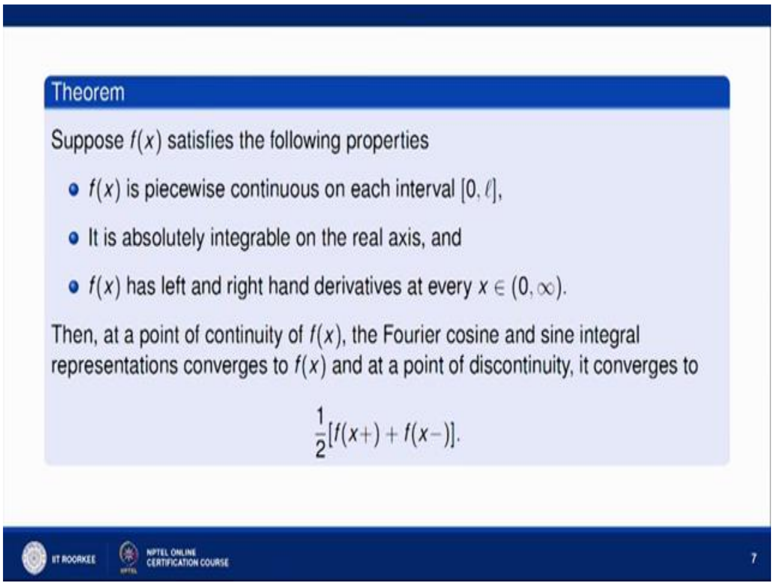

Now, again as we did in Fourier integral representation we assume function is piecewise continuous on interval 0 to l. It is absolutely convergent on the real axis. Function has a left and right hand derivatives at each x belong to 0 to infinity. So, this is the condition which we should assume. So, that the function a Fourier integral representation exists. Then at a point of continuity of f x the Fourier cosine and sine integral representations convergence to f x it has a point of continuity, and at a point of discontinuity it convergence to the average of left and right hand limit at that point. Now let us solve few problems based on this Fourier sine and cosine integral representation for this we solve few problems.

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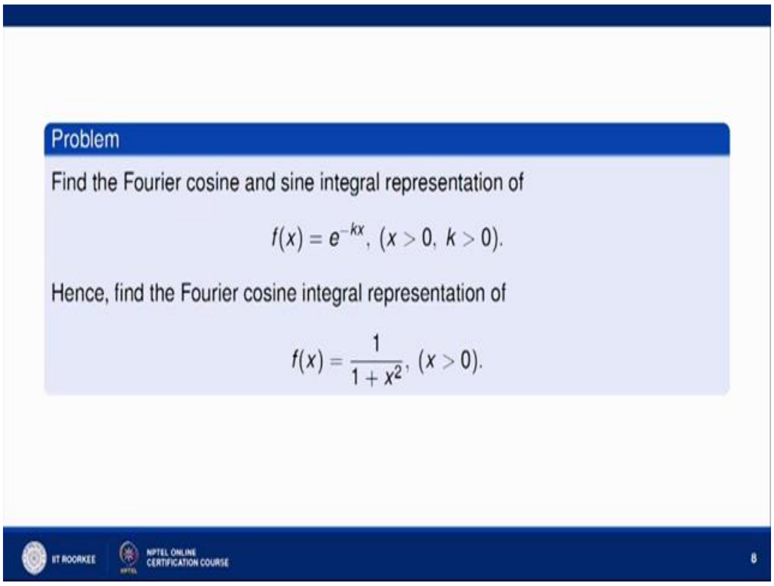

Now, we have to find out the Fourier cosine and sine integral representation of this function e k power minus k x, as greater than 0 means it is defined on the half range from 0 to infinity. And for the Fourier integral representation, we need function defined from minus infinity to plus infinity. So; that means, if it is half range there will be either cosine Fourier integral representation or Fourier sine integral representation of the function. If we need Fourier cosine; that means, we extend the function f x assuming function as an even function. And if we need Fourier sin integral representation we assume we extend the function f x assuming function as an odd function.

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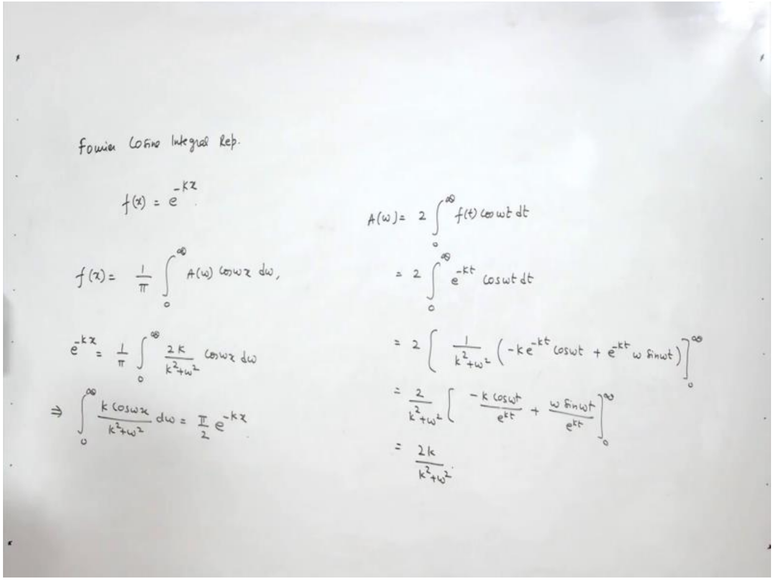

So, suppose we need Fourier cosine integral representation. So, Fourier cosine integral representation of f x is given by what is e k power minus k x. So, we know that Fourier cosine integral representation of function is given by f x equals to 1 upon pi integral 0 to infinity a omega cos omega x into d omega. Because for the Fourier cosine integral representation, we have only the cosine series and the b omega is 0. So, what will be a omega let us find a omega for this, a omega is nothing but 2 times 0 to infinity e k f t cos omega t d t. So, it is 2 times 0 to infinity. What is f t? E k power minus k t and it is cos omega t d t. So, this will be given by 2 times.

Now, let us integrate it. So, you can easily integrate by this integration by parts. So, this will be nothing but 1 upon k square plus omega square this will be minus k into e k power minus k t cos omega t. It is minus e k power minus k t minus minus plus omega sin omega t. And that will be from 0 to infinity. Now it is nothing but 2 upon k square plus omega square it is nothing but minus k cos omega t upon e k power k t, plus omega sin omega t upon e k power k t from 0 to infinity. Now when t tend to infinity this is a value lying between minus 1 and plus 1 upon infinity will be tending to 0 this is also tending to 0. And when k is when t is 0 it is 1 and it is 0. So, this value is nothing but 2 k upon k square plus omega square.

So, therefore, the Fourier cosine integral representation of this function e k power minus k x will be nothing but 1 by pi times integral 0 to infinity. It is 2 k upon k square plus omega square cos omega x d omega. So, from this we get integral 0 to infinity, k cos omega x upon k square omega square d omega will be nothing but pi times pi by 2 times. In fact, e k power minus k x; so substituting different values of k one can get the values of this integral which are given by this expression. Suppose k is 2. So, for k equal to 2 the value of cos omega x upon 4 plus omega square, will be given by this 2 will come here pi by 4 e k power minus 2 x. So, substituting different values of k we will get different values of integrals this integrals can be find out this I want to say. So, this is the Fourier cosine integral representation of this function of this function is this. Basically cosine integral representation is this of e k power minus k x, when we simplify this. So, from here we can find out different values of substitute different values of k, the value of the integral; now the Fourier sine integral representation, of the same function.

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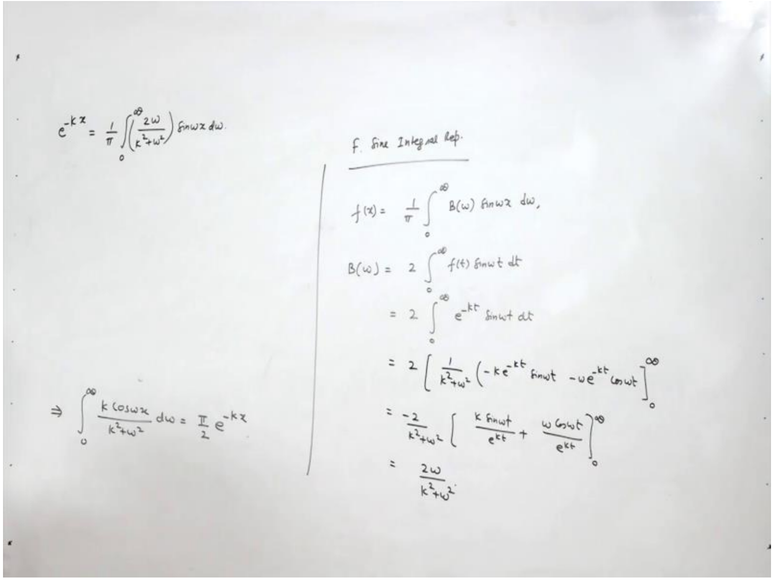

So, what is the Fourier sine integral representation that will given by 1 upon pi integral 0 to infinity, f t, f f x sin omega x d x d omega. And where b omega, it is b omega sorry b omega because a omega is 0. And b omega is given by, 2 times 0 to infinity f t sin omega t into d t. So, that will be nothing but 2 that is integral 0 to infinity, what is f t, f t is e k power minus k t, sin omega t into d t. So, this integral will be equal to again we will integrate by parts. So, this is equals to 1 upon k square plus omega square this is minus k e k power minus omega minus k t, minus k t, it is sin omega t and it is minus e k power minus k t, and it is cos omega t, omega times and 0 to infinity. So, that will be nothing but 2 upon k square plus omega square minus can come out, it is k sin omega t upon e k power k t, plus it is omega cos omega t upon e k power k t 0 to infinity.

Now when you put t equal to infinity t tending to infinity; so this will tend to 0 this will tend to 0 and when t equal to 0 it is 0, and it will be 1 upon this. So, minus minus plus it is 2 omega upon k square omega square. So, what will be the Fourier sine integral representation of this function? The Fourier sine integral representation of this function will nothing but we have a function e k power minus k x, which is equal to 1 upon pi times integral 0 to infinity and b omega is 2 omega upon k square plus omega square, and next is sin omega x d omega. So, this will be the Fourier sine integral representation of this function. Now hence find the Fourier cosine integral representation of this function.

So, this we have to find using the previous result obtained from the previous problem hence. So, we have to find Fourier integral representation of this function. So, let us find Fourier integral representation of this function. So, for a Fourier integral representation of this function is 1 upon 1 plus x square. So, it is given by function is 1 upon 1 plus x square.

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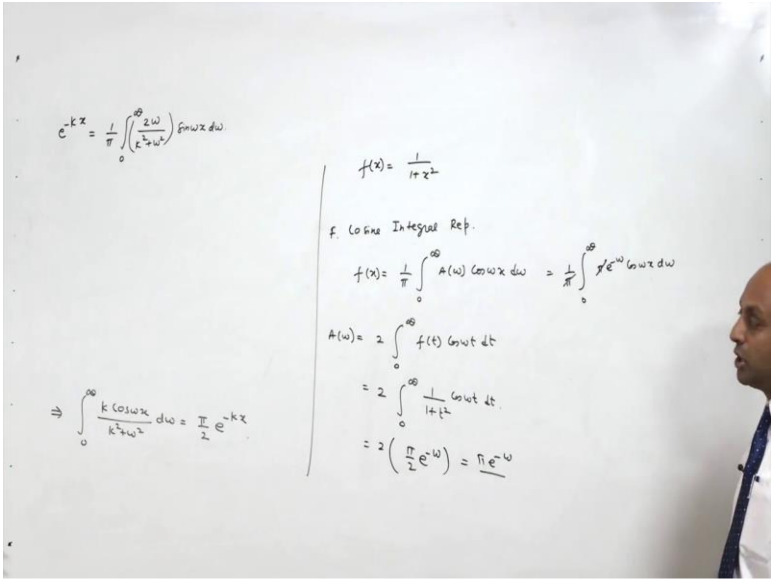

So, Fourier cosine integral representation of this will be nothing but it will be f x will be equals to 1 upon pi times 0 to infinity a omega cos omega, x d omega. Now where a omega is nothing but a omega is 2 times 0 to infinity, f t cos omega t d t. So, this is 2 times 0 to infinity what is f t? 1 upon 1 plus t square and it is cos omega t d t. Now to find out this value we will use this integral which we obtain from the first part of the problem. Now this is cos omega x upon it view 1 plus omega square when k equal to 1.

So, simply substitute k equal to on both the sides. So, when you put k equal to 1 you will directly get the value of this expression which is 2 into pi by 2 times e k power minus x. So, this is nothing but pi e k power minus x I mean pi into e k power minus omega, f is the omega function. So, it will omega it is omega. Because instead of x we have omega and this omega we have t. That is why here we are having pi into e k power minus omega. So, what is the Fourier integral representation of this function? This will be nothing but 1 upon pi times integral 0 to infinity a omega is nothing but pi into e k power minus omega and cos omega x d omega, pi pi cancels out. So, this will be the Fourier integral Fourier cosine integral representation of this function 1 upon 1 plus x square. So, in this way we can find out Fourier cosine and Fourier sine integral representation of a function.

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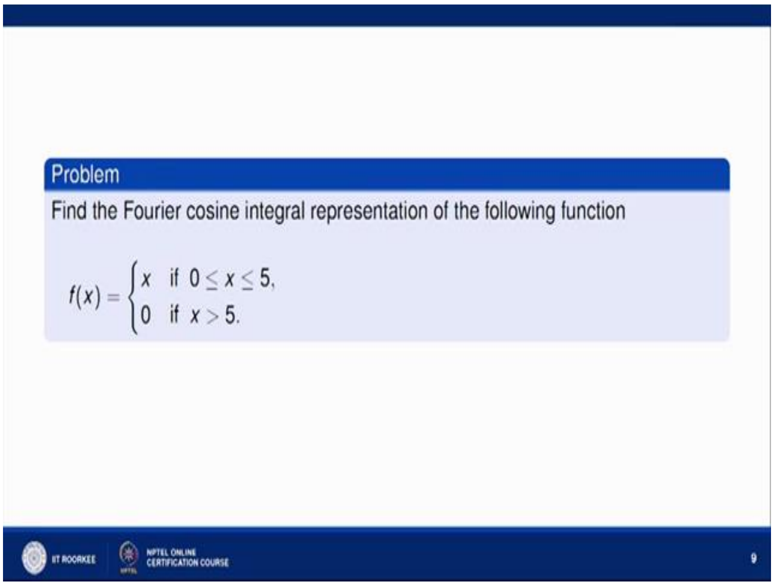

So, now let us try one more problem where we also discuss about discontinuity at a point x equal to 5. Here we have to find out Fourier cosine integral representation of the function. So, we will follow the same methodology which we have discussed here.

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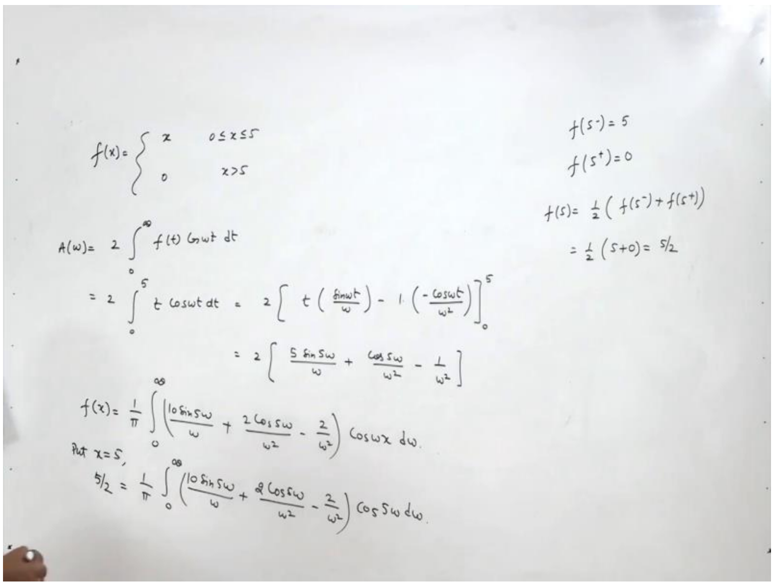

So, function which is defined here is function defined as f x is equals to it is x, when x lying between 0 to 5, and it is 0 as here. So, these are definition of the function. So, it is defined from 0 to infinity, and we want Fourier cosine integral representation of this function. So, for Fourier cosine, we need a omega because b omega is 0. So, a omega is 2 times 0 to infinity f t cos omega t into d t. So, that will be nothing but 2 times here it is 0 to 5 times otherwise it is 0 x cos omega t into d t which is nothing but 2 times, now you can integrate it by parts that will be nothing.

But cos into sin omega t upon omega minus derivative of integral of second, which is minus cos omega t upon omega square, and it is from 0 to 5. So, this value is nothing but 2 times when you put t equal to 5 it is nothing but 5 sin 5 omega upon omega, minus minus plus, cos 5 omega upon omega square and when t equal to 0 it is 0. It is nothing but t because f x is x means f t is t. So, it will be this plus this when t equal to 0 to 0, when t equal to this it is 1. So, 1 upon omega square it is minus minus minus is minus 1 upon omega square. So, this will be a omega.

So, what will be the Fourier cosine integral representation of this function? So, that will be given by f x is equals to 1 upon pi times integral 0 to infinity, a omega, a omega is this value it is 10 sin pi omega upon omega, plus 2 cos pi omega upon omega, square minus 2 upon omega square, and whole multiplied by it is cos omega x d omega. So, that will be the Fourier cosine integral representation of this function f x. Now suppose you put x equal to 5 suppose put x equal to 5. So, what will be f 5? Since function is discontinues x equal to 5. So, f 5 to find out the value of f x equal to 5, it will be convergence to mean of or the average of value the function left hand limit and right hand limit at x is 5. So, what is f 5 minus, it will be nothing but 5, from the definition. And what is f 5 plus 5 plus means from the right hand side it is 0.

So, what will be f 5? It will be half of f 5 minus plus f 5 plus. So, that will be nothing but half of 5 plus 0, is equals to 5 by 2. So, the left hand side is 5 by 2. And right hand side is 1 upon pi integral 0 to infinity, when you substitute x equal to 5. It is 10 sin 5 omega upon omega, plus 2 cos 5 omega upon omega square, minus 2 upon omega square, into cos 5 omega d omega. So, if we need this value suppose this integral from 0 to infinity. So, that value is nothing but you can take pi this side. So, that will be nothing but 5 pi by 2. So, in this way we can find out the value of this expression also.

Now, the next problem is Fourier sine integral representation. That also very easy to find out we can easily simplify and find out the Fourier sine integral representation also.

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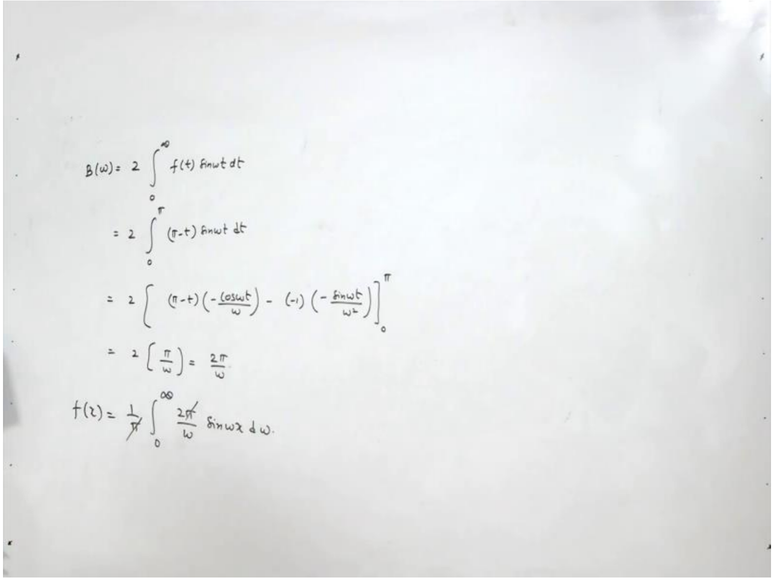

So, again for this problem for finding sine integral representation, we will find b omega which is again given by 2 time integral 0 to infinity, it is f t sin omega t d t. So, that will be 2 times, now it is 0 to pi only define from 0 to pi which is given by pi minus t, into sin omega t d t. So, that is nothing but 2 times now you can integrate it very easily integration by parts. So, pi minus t, it is minus cos omega t upon omega then minus derivative of this and integral of this from 0 to pi. So, this is nothing but when pi it is 0, only when exist from t equal to 0 it is minus minus plus 2 times pi upon omega. So, it is 2 pi upon omega. It is 2 pi upon omega. So, what will be the Fourier sine integral representation of this? So, that will be nothing but f x equals to 1 upon pi integral 0 to infinity, b omega into sin omega x d omega. So, that will be the Fourier sine integral representation of this function.

So, in this way we can find out Fourier cosine or sine integral representation of function, when defined over the half range by the by extending the function assuming the function as an odd function or an even function.

Thank you.