Mathematical methods and its applications
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Lecture – 53
Fourier integrals

Welcome to the lecture series on Mathematical Methods and it is Applications. So, we have already discussed Fourier series. We have seen what Fourier series are and how we can solve some problems based on that. What are properties what are convergence properties of the Fourier series that also we have seen, and also the complex of Fourier series. Now for a integrals what for a integrals are and why they are important. Let us see.

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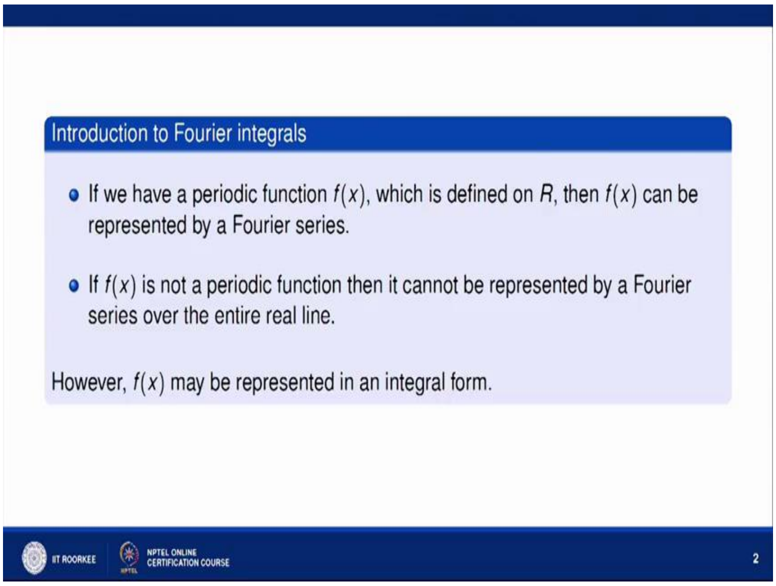

So, if you have a periodic function f x, which is defined on a real line R then f x can be represented it by a Fourier series that we have already seen, which is given by a naught by 2 plus summation n cos n, pi x pi l, plus summation b n, sin n pi x by l, that we have already seen.

If f x is not periodic, suppose f x is not a periodic function. Then it cannot be represented it by a Fourier series over entire real line, is 1 2 express a non-periodic function by a Fourier series then it is not possible. However, f x may be represented it in an integral form. How let us see.

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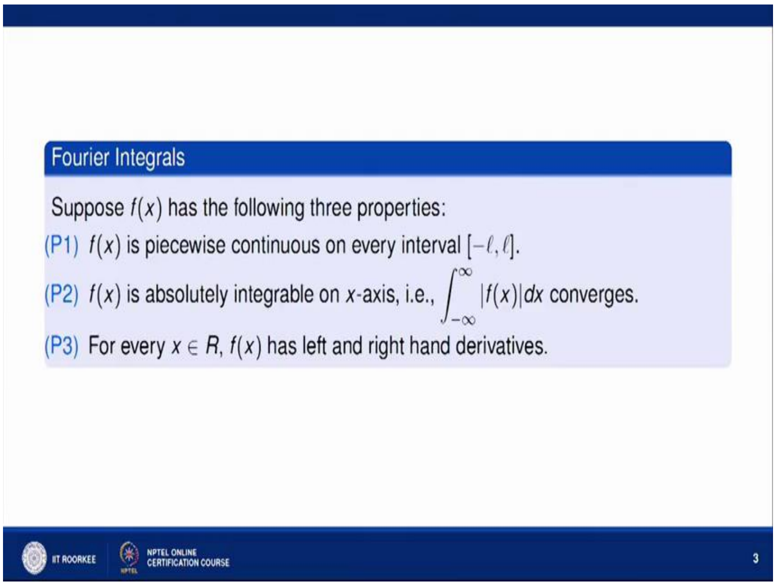

Now, first of all suppose f x have the following 3 properties, f x piecewise continuous on every interval from minus l to plus l, f x is absolutely integrable on x axis; that means, integral from minus infinity to plus infinity, mod of f x converges; that means, this value is finite integral minus to plus infinity, mod f x d x value is finite. And for every x belongs to R, f x has left and right hand derivatives.

So, first of all let us suppose the function f x is periodic and satisfy these 3 properties. Again suppose f x is a periodic function with period 2 will define interval minus l, 2 plus l satisfy the properties p 1 to p 3 which we have discussed. Then it can be specified Fourier series in this way. Basically we derive the Fourier integral from the Fourier series itself. We have the Fourier series like this we already know this f x.

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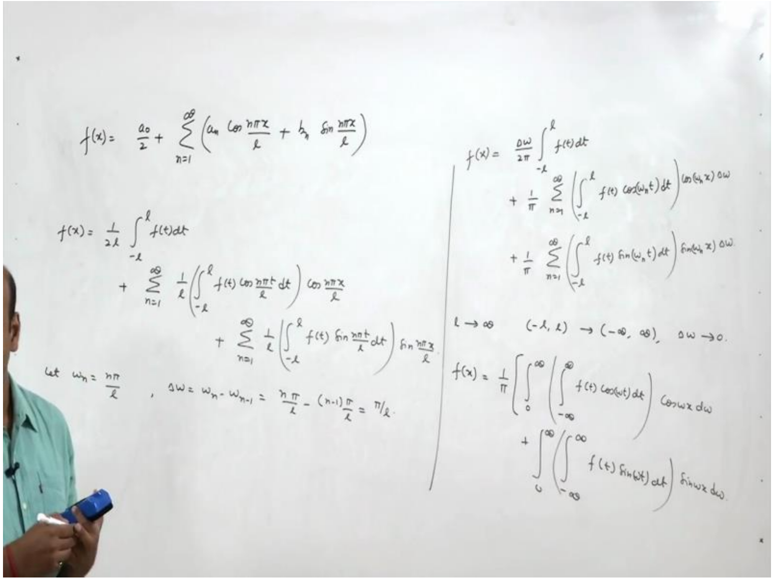

Suppose to a naught upon 2 plus summation a n, cos n pi x upon, l plus b n, sin n pi x by l. So, this is the Fourier series expression, and it is varying from 1 to infinity or function f x. We are a naught is given by, a naught is given by 1 upon l integral minus l 2 plus l f t d t, a n is given by 1 by l integral minus l 2 l f t cos n pi t by l, into d t and b n is given by 1 by l, integral minus l to plus l f t, sin n pi t by l into d t.

So, these are expressions. Now let us substitute these expressions over here. So, what will obtain? F x will be equals to 1 by 2 l integral minus l to plus l f t d t, plus summation n varying from 1 to infinity, a n is nothing but 1 by l, integral minus l to plus l, f t, cos n pi t by l, into d t, into cos n pi x upon l, plus summation n varying from 1 to infinity, again 1 by l integral minus l to plus l, f t sin n pi t by l, into d t and whole multiplied by sin n pi x by l. Now let us suppose let omega n is n pi by l. So, what will be delta omega, delta omega is omega n minus omega n minus 1, which is nothing but n pi by l minus n minus 1 pi by l which is pi by l.

So, let us substitute these terms in this expression. So, what will be effects? What effects we will obtain? So, f x will be nothing but is equal to, now this 1 by l, 1 by l is nothing but delta omega by pi. So, it is delta omega by 2 pi integral minus l to plus l, f t d t from this expression. Now plus again this 1 by l will be nothing but pi upon delta omega upon pi from this expression. So, 1 upon pi will be outside, 1 upon pi will be outside summation n varying from 1 to infinity, minus l to plus l, f t cos and pi n pi l by l is nothing but omega n omega n t, into d t and whole multiplied by cos omega n, into x into delta omega. Delta omega by pi is here instead of this. So, delta omega is here. And similarly plus 1 by pi summation, n varying from 1 to infinity integral minus l to plus l f t sin omega n t, into d t, and whole multiplied by sin omega n x, into delta omega, because pi n pi l by l is nothing but omega n. So, it is omega n x into delta omega.

Now, let l tend to infinity. Because you on to expend the function over the entire real line. We want to expend the function over the entire real line. That is why we are we are tending l to infinity. So, as l tends to infinity of course, minus l to plus l, interval will tend to minus into plus infinity, that is over the entire real line. And delta omega will tend to 0 delta omega is nothing but pi upon l and as l tend to infinity this will tends to 0. So, and function f x is absolutely convergent, that we are already assumed. So, this term, So, this integral when l tends to infinity is finite, and delta omega tend into 0, the first term is tend to 0. Because delta omega tend to 0 and as l tend to infinity.

So, this value since is absolutely convergent absolutely convergence. So, this value is finite into 0 will be 0. So, first term is 0, and this will sum up as remaining sum. Because l tends to infinity l tends to infinity, and this delta omega which tend 0, so this sum up as remaining sum. So, this will be nothing but 1 upon pi integral 0 to infinity, 0 to infinity, integral minus infinity, plus infinity f t cos omega t into d t, whole with cos omega x into d omega, plus integral 0 to infinity integral minus to plus infinity, f t it is sin omega t, into d t and whole multiplied by sin omega x into d omega. Because l is tend to infinity, and delta omega tend to 0.

So, this will be sum up as remaining sum which is given by 1 upon pi integral 0 to infinity minus to plus infinity this term plus this term. So, that is what we have given the ppt, also that this presumed as a remaining sum of the different integral as l tend to infinity. So, this will give this value.

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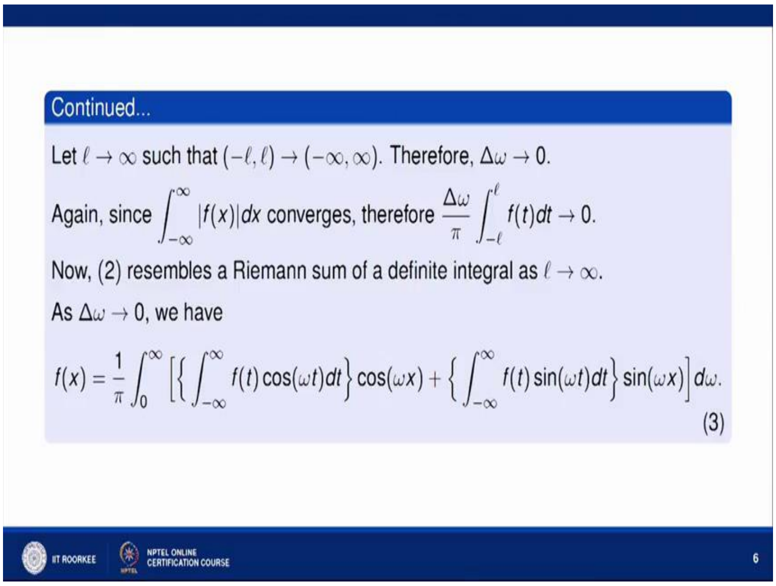

Now, here if you take this as a omega, because this is a function of omega, and this as b omega.

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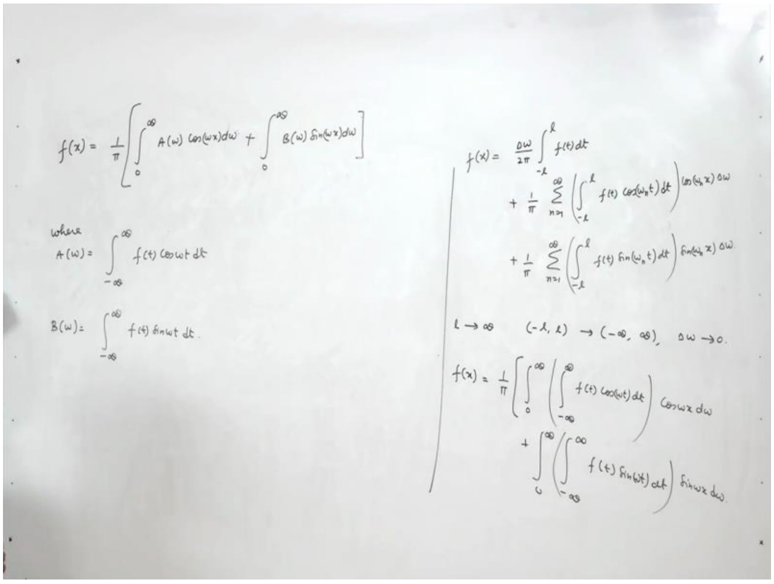

So, what will be f x, f x will be nothing but 1 by pi times integral 0 to infinity, a omega cos omega x t omega, plus integral 0 to infinity, it is b omega sin omega x, into d omega. So, that will be this expression, where a omega is nothing but minus infinity to plus infinity, f t cos omega t d t, and b omega is nothing but minus will plus infinity, f t sin omega t into d t. So, this is what the Fourier integral is, I mean integral representation of a function f x, for a integral representation of a function f x given by this, where a omega is a this and b omega is this. So, we expend the function over the entire real line, by assuming that l is tend to infinity. So, this is how we can obtain this thing. Now this is called Fourier integrals of f x. Now suppose f x satisfy the properties p 1 to p 3, which we have discussed.

Then the Fourier integral of f x converges to f x the point of continuity the same as in Fourier series. This we have also discussed Fourier series also. The same follows in Fourier integrals also. That if satisfy the some properties p 1 to p 3, then the Fourier integral of f x converges f x at point of continuity, and for discontinuity it converges to the average of left hand limit and the right hand limit at that point. Now let us try this problem now.

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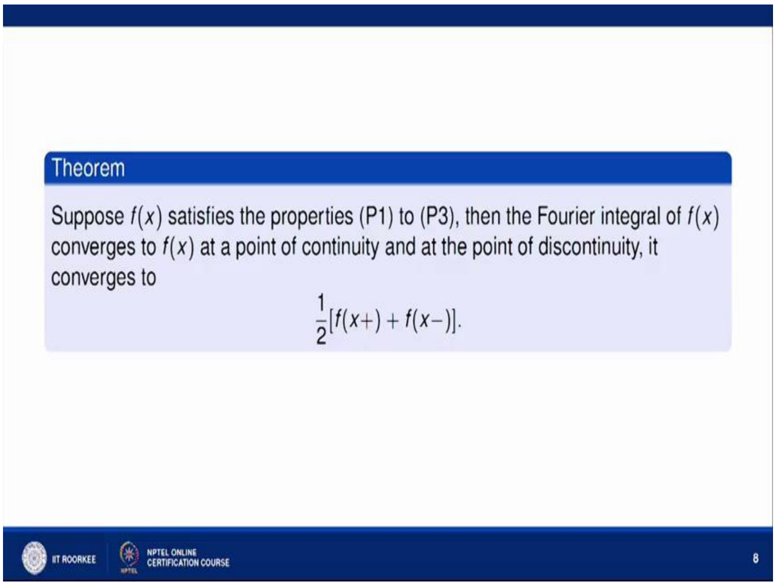

We want Fourier integral representation of this function. So, we know that the Fourier integral representation of a function is given by this expression, f x is given by this expression a omega is given by this b omega is given by this expression. Now their function is what? Function is defined function is 1 when mod x is less than 1 and 0 when mod x is greater than 1. So, this is the function now we want Fourier integral representation of this function. So, this function is defined over the entire real line. So, we want the Fourier integral representation of this function how you can find it. So, we first find a omega and b omega, we substitute the values of a omega b omega over here.

So, that will give the Fourier integral representation of this function. So, what is a omega, let us find a omega first what is a omega from here a omega is minus plus infinity f t cos omega t d t.

Now, it is defined from minus 1 to plus 1 is 1. So, minus 1 to plus 1 is one otherwise it is 0. So, it is cos omega t d t, and cos omega t is an even function. So, it is nothing but 2 times 0 to 1, cos omega t d t. And it is equals to 2 times sin omega t upon omega from 0 to 1, which is nothing but 2 sin omega upon omega, because sin 0 is 0. Now what is b omega? B omega will be nothing but again minus infinity plus infinity, by this definition f t sin omega t into d t, which is equals to minus 1 to plus 1 sin omega t into d t, because from minus 1 to plus 1 it is 1. And it is an odd function it is 0. So, we have computed both a omega and b omega, and the Fourier integral representation of a function f x is given by this expression as you already know.

So, this will be nothing but 1 upon pi integral 0 to infinity.

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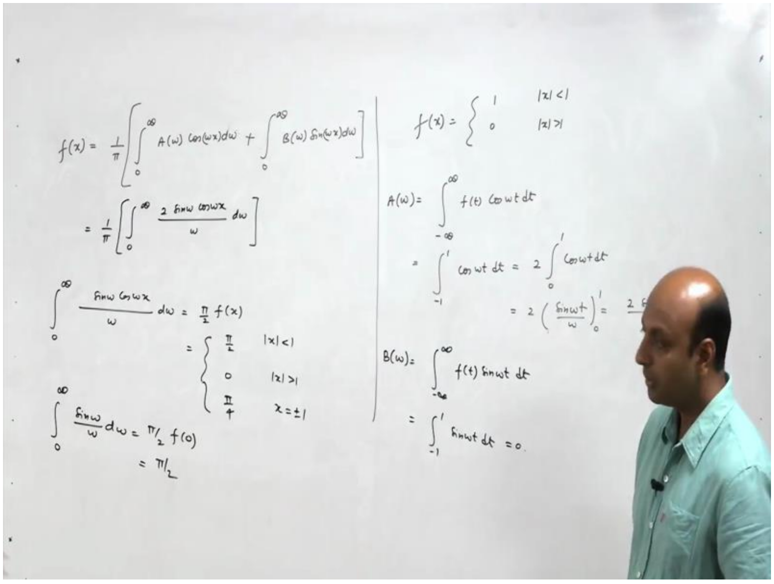

Now, what is a omega is 2 sin omega cos omega x, upon omega into d omega, d omega is 0. So, that will be the Fourier integral representation of this function f x. Now we can also write this function as 0 to infinity, 2 sin omega cos omega x upon omega, d omega which will be equals to pi times f x, and which is equals to pi you multiply simply that by pi, when mod x less than 1, that is 0 when mod x greater than 1. This 2 you can also bring here pi by 2. So, it is pi by 2, and what happens when x is 1 when x is 1. So, it is it is discontinues. So, it converges to average of left hand limit and right hand limit. Which is nothing but what will be f 1? F 1 will nothing but half of f 1 minus plus f 1 plus.

So, that will be nothing but 1 by 2, 1 plus 0 that is 1 by 2 and you have to multiply pi by 2 also. So, that will be pi by four. So, in this way you can find out the value, the same you can find out when x equals to minus 1 also. When x equal to minus 1, we are the same value it is x equal to x equal plus or minus 1. So, so this integral is equal to basically this integral. Now we have to find out integral 0 to infinity sin omega upon omega d omega. So, that will be nothing but you substitute x as 0, it is a function of omega. So, you can arbitrary choose any value of x. So, when x is 0. So, when x is 0. So, this is nothing.

But, you can substitute both side as 0. So, it is 0 to infinity, sin omega upon omega d omega is equals to pi by 2 f 0. And where it is 0, when f is 0 f is 1, it is pi by 2. You can directly see from here when x is 0 value is pi by 2, value of this expression is pi by 2. So, you can easily see that this value when x is 0. In fact, when you put x equal to say half; when you put x equal to half to when x equal to half satisfy this expression, when x equal to half, the value of that will also be pi by 2. You can easily find out under different values of x the value of this expression nothing but this expression now let us try to prove this that this is equal to this expression.

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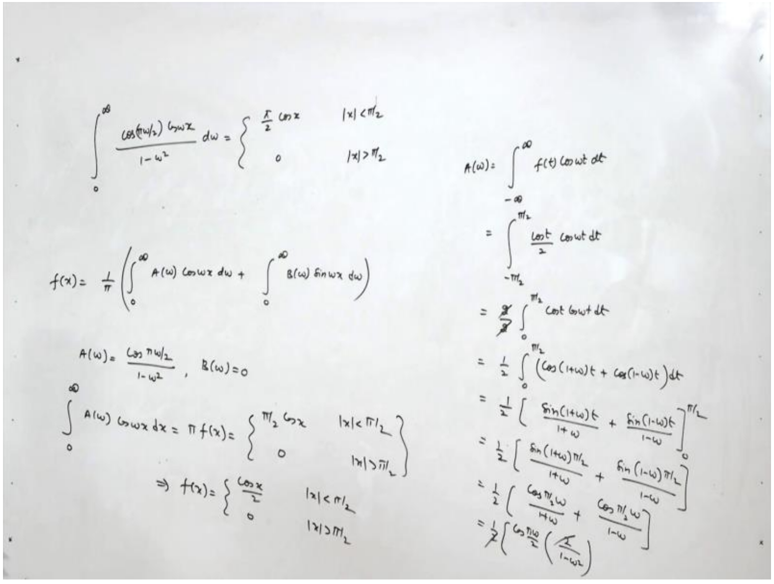

So, how we can prove this is nothing but Fourier integral representation of function f x. So, what is the Fourier integral representation of function? So, what is the expression it is 0 to I mean infinity, it is cos pi omega by 2 into cos omega x, upon 1 minus omega square d omega is equals to it is given to you pi by 2 cos x, when mod x is less than pi by 2, and 0 when mod x is. So, we have to prove this. We have to prove this. So, what is the Fourier integral representation of function f x is nothing but 1 upon pi times integral 0 to infinity, a omega cos omega x t x cos omega x, d omega plus integral 0 to infinity, d omega sin omega x into d omega is it.

So, Fourier integral representation is into d omega, let us same thing and here it is d t. Now what is a omega? A omega is minus minus will plus infinity it is f t cos omega t d t, and d omega is given by minus will plus infinity, f t sin omega t d t. That you already know. Now if you compare this with this. So, what will be a omega. So, it is pi f x which is equals to if you compare this by this. So, cos omega x coefficient here is simply a omega.

So, we can simply say that a omega is nothing but cos pi omega by 2, upon 1 minus omega square, and d omega is 0, because there is no term of sin omega x. And this integral 0 to infinity a omega cos omega x d x is nothing but pi times f x, which is equals to which is equals to pi by 2 cos x, when mod x is less than pi by 2, and 0 when x is mod x is greater than pi by 2. So, this implies f x is cos x by 2, when mod x is less than pi by 2, and 0 when mod x is greater than pi by 2. So, you take this f x you take this f x, and try to obtain the value of a omega. If you obtain a omega as this; that means, we have proved. Anyhow any how we have assumed that this is true to assuming this to be true, we have find out the coefficients a omega and d omega and we have find f x. For this f x if a omega comes out to be this; that means, we have proved. So, what will be a omega for this f x? And b omega a omega should be this b omega should be this. So, what will be a omega? A omega will be given by minus infinity plus infinity f t, cos omega t d t.

Now it is given by f x is defined like this, from minus pi by 2 to plus pi by 2, it is cos t upon 2, into cos omega t d t, which is nothing but it is even function it is 2 times by 2 0 to pi by 2, cos t into cos omega t d t, which is equals to now 2 2 cancels out multiply divided by 2 again 0 to pi by 2 it is 2 cos a cos b which is nothing but sin a plus b plus sin a minus b it is equals to 1 by 2 integral of sin is minus cos. So, put minus outside put minus on outside it is cos of 1 plus omega t upon 1 plus both will be cos sorry cos a to cos a cos b is nothing but cos a plus b plus cos a minus b. So, it will be cos it will be cos, and minus is not. So, it will be sin and plus sin 1 minus omega t upon 1 minus omega and t varying from 0 to pi by 2. So, this will be nothing but 1 by 2. It is sin 1 plus omega into pi by 2 upon 1 plus omega, plus sin 1 minus omega into pi by 2, upon 1 minus omega. Now we were simplifying this is nothing but 1 by 2.

Now, sin 90 plus theta is sin theta, to that is sin pi by 2 omega upon 1 plus omega. Again sin 90 minus theta no sin 90 plus theta is cos theta sorry it is cos theta. So, it will be cos theta again here it is cos theta. So, this will comes out, this cos pi omega by 2 will come out. And it will be nothing but 2 upon 1 minus omega square. So, 2 2 cancels out. So, this will comes equal to this cos pi omega by 2 into 1 upon 1 minus omega square.

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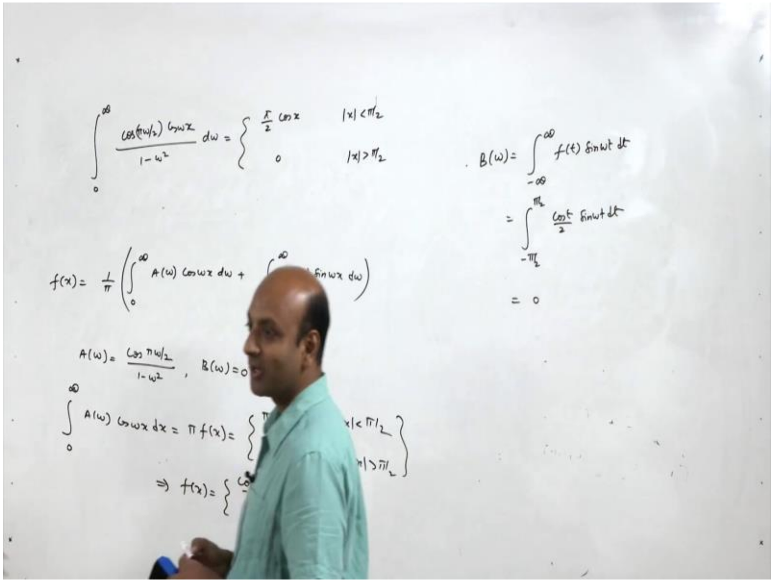

Now, when you compute b, omega b omega must be 0 when you compute b omega for the same problem. So, b omega will be nothing but what, minus plus infinity f t sin omega t d t. Now it is nothing but minus pi 2 plus pi by 2. It is cos t by 2 sin omega t d t. It is an odd function. So, it is 0. So, hence we have proved. Because we have obtained that a omega is this and b omega is 0, for this function; that means, this results hold. So, we have shown the result. So, this how we can find out Fourier integral representation of a function f x.

Thank you.