Mathematical methods and its applications
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Lecture - 52
Complex form of Fourier series

Welcome to the lecture series on Mathematical Methods and its Applications. So, we were discussing Fourier series, I told you that periodic function f x if satisfies some properties like piecewise continuity and left and right hand derivative exist at each point. Then f x can be expressed in terms of sine and cosine series that is f x will be something a naught by 2 plus summation a n cos n pi x by l plus summation b n sin n pi x by l. That I already told you, and we have also solved some problems based on that.

Now, the next is complex form of Fourier series, what is that and how it is important let us see. Now Fourier series of a periodic function f x of period 2 l is given by this expression this we already know.

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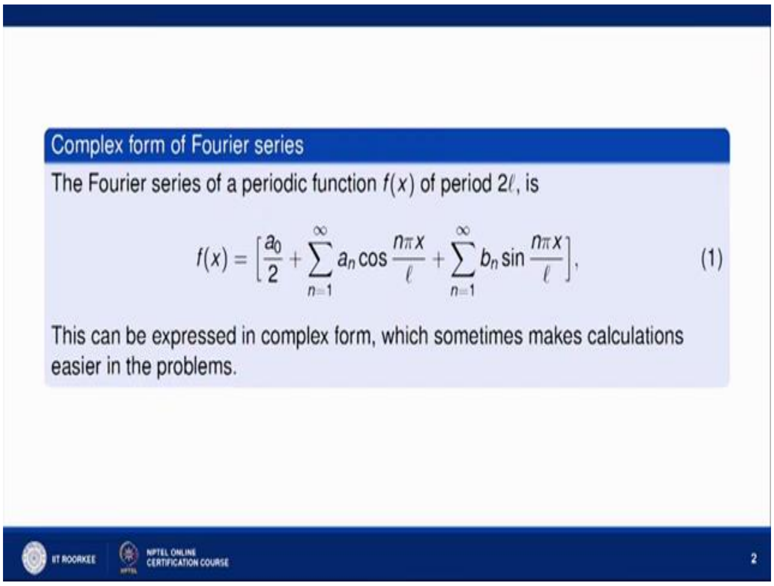

This can be expressed in complex form which sometimes makes calculation easier in the problems. So, why we need complex forms because sometimes it makes our calculation easier. Now how we can convert this form into a complex form let us see.

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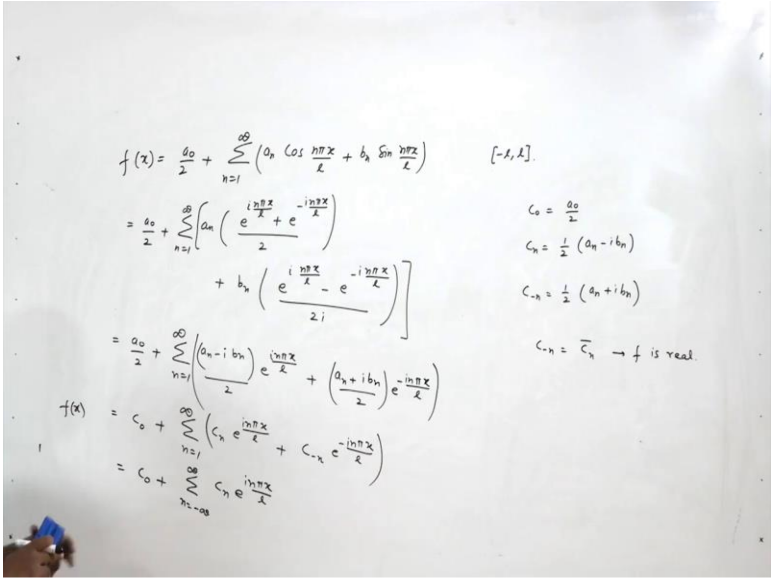

So, what f x is? F x is nothing but f x is equal to Fourier series expansion is a naught upon 2 plus summation n varying from 1 to infinity a n cos n pi x by l plus b n sin n pi x by l interval is we are taking interval from minus l to plus l. This interval we are taking and f x is a periodic function eta, these are Fourier series expansion of the function f x.

Now we know that e [FL] power iota theta is nothing but cos theta plus iota sin theta. This we already know and e [FL] power minus iota theta is nothing but cos theta minus iota sin theta. So, when we add these 2, we already know that cos theta is nothing but e [FL] power iota theta plus e [FL] power minus iota theta upon 2 and sin theta is nothing but e [FL] power iota theta minus e [FL] power minus iota theta upon 2 iota. So, these things, we already know by the complex numbers we already know that cos theta is nothing but e [FL] power iota theta plus e [FL] power minus iota theta upon 2 and sin theta is given by this expression.

So, let us apply these things over here it is cos theta, theta is nothing but n pi n pi x by l and sin theta. So, let us apply these expressions over here. So, it is equal to a naught upon 2 plus summation n varying from 1 to infinity a n. So, what is cos n pi x by l; by this expression? It is nothing but e [FL] power iota n pi x by l plus e [FL] power minus iota n pi x upon l upon 2 and plus b n into summation is over entire bracket. So, b n is nothing but b n into sin theta sin theta is e [FL] power iota theta is n pi x upon x upon l minus e [FL] power minus iota n pi x upon l upon 2 iota. So, this is this we can do.

Now, this can be further written as a naught upon 2 plus now e [FL] power iota n pi x by l is here and here also. Let us combine these 2 terms and again combine these 2 terms e [FL] power minus iota n pi x by l here and here combine these 2 terms. So, what we obtain? We obtain summation n varying from 1 to infinity it is a n plus b n upon iota whole divided by 2 and e [FL] power iota n pi x upon l. So, we also know that we also know that 1 upon iota is nothing but minus iota because iota square is minus 1. So, this 1 upon iota can be written as minus iota eta. So, this is nothing but minus iota this we can easily write because of this property plus again it is a n a n and again 1 upon iota is minus iota minus minus plus, it is plus iota b n upon 2 time e [FL] power minus iota n x by n pi x by n pi x upon l. So, these are the expressions which we obtained over here.

Now, let us simplify it. So, this is suppose this a naught upon 2 is c naught plus summation n varying from 1 to infinity this suppose this term is nothing but suppose c n e [FL] power iota n pi x upon l plus and suppose this term is c minus n it is simply a notation c minus n e [FL] power minus iota pi x upon l. So, what is c n c naught is nothing but a naught upon 2, c n is nothing but half of a n minus iota b n and c minus n is nothing but 1 by 2 a n plus iota b n. So, 1 can easily see that if this f x is a real function f x is a real function then these coefficients will be real then this c n minus is nothing but c n bar one can easily this thing because its bar is this these are the conjugate of each other if f is real; this is if f is real.

Now, this can be further written as c naught plus summation it is c n e [FL] power iota n pi x upon l and we can write because all the plus values are covered from here and all the negative values are covered from here because instead of n we have minus n. So, we can easily write this as n from minus infinity to plus infinity because n from 1 to infinity is here n from minus infinity to minus 1 is here and c 0 will also disappear because when n is 0 n is 0 it is nothing but c 0 c 0 will also comes here. So, when we combine all these terms it is nothing but n varying from minus infinity to plus infinity c n e [FL] power n pi x by l. So, this is the complex form of the Fourier series.

So, Fourier series can be represented in this way or in this way, both ways both are equivalent. So this is a complex form of Fourier series, now, what are c n? How you define c n?

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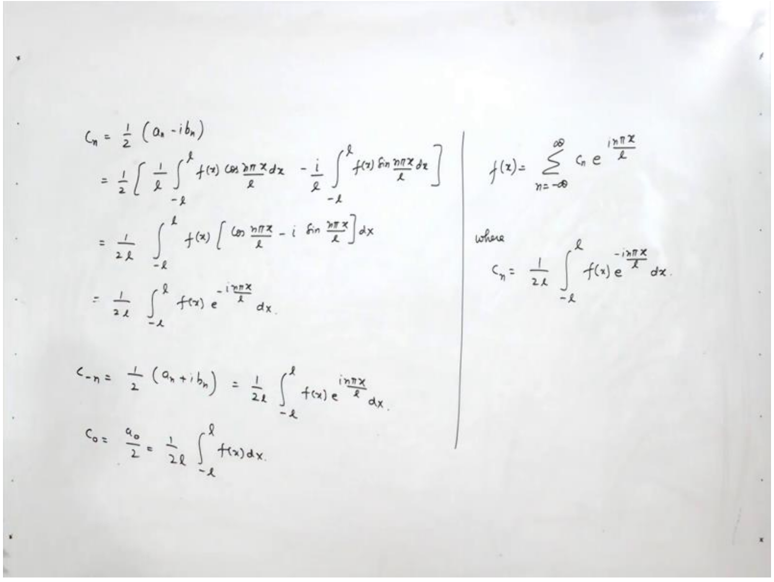

C n is given by c n is what? C n is nothing but 1 by 2 a n minus iota b n. It is equal to 1 by 2, what is a n? It is 1 by l integral minus l to plus l f x cos n pi x by l into d x. This is by definition of a n and minus iota times what is b n? B n is 1 upon l integral minus l to l f x sin n pi x upon l into d x. So, this entire expression can be rewrite as 1 upon 2 l integral minus l to plus l you can take f x common. So, this is nothing but cos n pi x upon l minus iota sin n pi x upon l and whole into d x. So, this can be further written as 1 upon 2 l integral minus l to plus l f x e [FL] power minus iota n pi x by l into d x.

So, this will be c n where c n is nothing but this value and what is c minus n c minus n if you evaluate eta that will be nothing but c minus n will be nothing but 1 by 2 a n plus iota b n using the same steps follow the same steps. So, this is nothing but when we simplify it; it is nothing but 1 by 2 l integral minus l to plus l f x e [FL] power iota n pi x by l into d x because in this we have only the positive sign here instead of this negative this is positive this is positive this is positive this would be positive. So, c minus n will be nothing but simply replace n by minus n in this expression in this expression simply replace n by minus n and what will be c naught. C naught is nothing but a naught upon 2 and a naught is nothing but 1 by l integral minus l to plus l f x d x.

So, it means when you replace n by 0 in this expression you get c naught. So, if you find c n. So, it contains c minus n also when you replace n by minus n and it contains c naught also when you replace n by 0 so; that means, when we write the Fourier complex form of Fourier series which is given by this expression f x is equals to summation n varying from minus infinity to plus infinity c n e [FL] power iota n pi x upon l and here c n will be nothing but 1 by 2, l integral minus l to plus l f x iota n pi x by l into d x. So, that will be the complex form of Fourier series it contain negative c n also which is also covered here it contains c naught also which is also covered here. So, we can easily write that the Fourier series complex form of Fourier series is given by this expression where c n is given by this expression. So, this is a complex form of Fourier series

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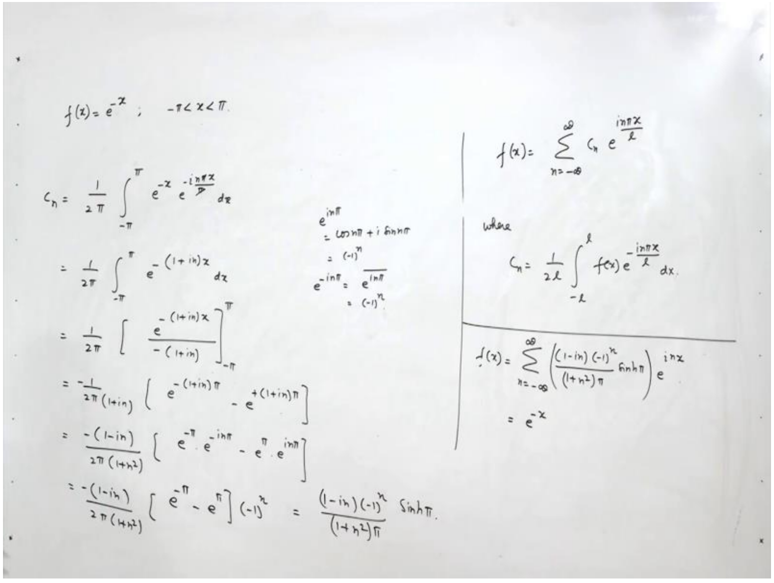

Now, let us solve few problems based on this. Now let us try these 2 problems based on this. Now in the first problem, f x is what? F x is e [FL] power minus x and the function is defined from minus pi to plus pi of course, function is periodic. Now we have to express this function as a as the complex form of Fourier series. So, how can we do that we first find c n which is given by this expression and when we substitute c n over here. So, that will be the complex form of the Fourier series expression of this function f x.

So, what is c n? C n will be given by 1 by 2 l here l is pi 1 by 2 pi integral minus pi to plus pi f x is e [FL] power minus x into e [FL] power minus iota n pi x by pi into d x. So, pi pi cancels out. So, this is nothing but 1 upon 2 pi integral minus pi to plus pi e [FL] power minus will be outside and 1 plus iota n into x into d x. So, this expression we will be having. So, this is further can be written as 1 upon 2 pi. Now when we integrate this, it is nothing but e [FL] power minus 1 plus iota n x upon minus 1 plus iota n and x is varying from minus pi to plus pi.

So, this will be further equal to 1 upon 2 pi, you can take this term outside. So, this is negative of 1 plus iota n upper limit minus lower limit e [FL] power minus 1 plus iota n into pi minus e [FL] power minus minus plus 1 plus iota n into pi this you can simplify very easily multiply this by its conjugate 1 minus iota n multiply and divide by 1 minus iota n. So, we will get minus 1 minus iota n upon 2 pi into 1 plus n square and here when you simplify. So, this is nothing but e [FL] power minus pi into e [FL] power minus iota n pi minus e [FL] power pi into e [FL] power iota n pi.

Now, what is e [FL] power iota n pi? E [FL] power iota n pi will be nothing but cos n pi plus iota sin n pi and it is minus 1 [FL] power n and it is sin n pi is 0 and similarly when you find e [FL] power minus iota n pi that is nothing but the conjugate of e [FL] power iota n pi. So, that will be remaining minus 1 [FL] power n. So, from here what we obtained it is equal to minus 1 minus iota n upon 2 pi 1 plus n square when you simplify this. So, this is also minus 1 [FL] power n this is also minus 1 [FL] power n both will come out and what we will be having e [FL] power minus pi minus e [FL] power pi and whole multiplied by minus 1 [FL] power n. So, this negative you can get inside and upon 2. So, this will be nothing but when you simplify further. So, it is nothing but 1 minus iota n into minus 1 [FL] power n upon 1 plus n square which we are obtaining from here into pi and this nothing but it is sine hyperbolic pi because sine hyperbolic pi is e [FL] power pi minus e [FL] power minus pi upon 2. So, that will be c n.

So, what will be the Fourier series expression of this function? The Fourier series expression of this function f x now will be nothing but f x will be equal to summation n varying from minus infinity to plus infinity, c n is what? C n is 1 minus iota n minus 1 [FL] power n upon 1 plus n square into pi sin hyperbolic pi into e [FL] power iota n x. So, that will be the Fourier series it is equal to e [FL] power x because we find the Fourier series complex form of Fourier series this function. So, function is e [FL] power minus x and that e [FL] power minus x the complex form of this is nothing but this expression. So, that is how we can find out the complex form of Fourier series of a function f x.

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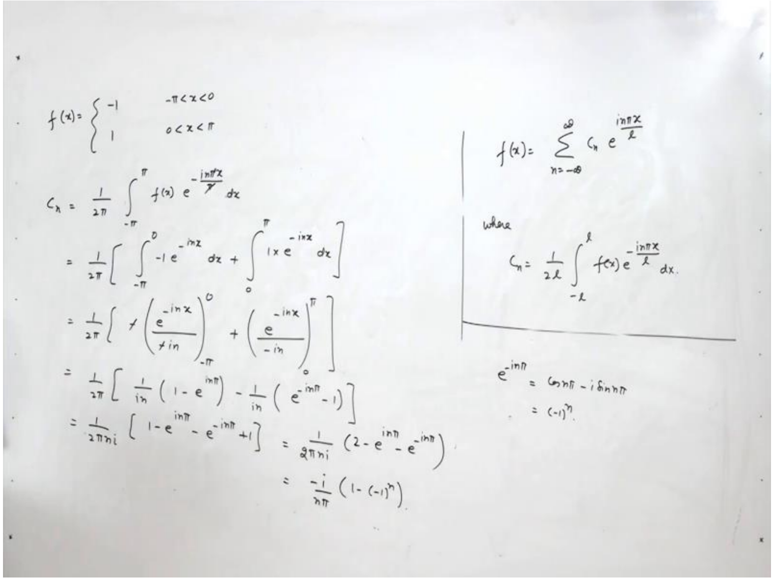

Now, let us solve second next problem. Again we have to write down the complex form of Fourier series of this function now. So, what is the; how the function is defined? Now function is f x is equals to it is minus 1 when x is varying from minus pi to 0 and it is 1 when x is varying from 0 to pi and function is periodic with period 2 pi now again to find out the complex form of this f x first we will find c n which is given by this expression and after finding c n we will substitute it here. So, that will be the complex form of the Fourier series of this function f x.

So, what is c n? Now c n we can find out it is 1 upon 2 pi because l is pi minus pi to plus pi f x e [FL] power minus iota n pi x upon l into d x that is nothing but 1 upon 2 pi. Now you can split this function because function from minus pi to 0 is minus 1 and from 0 to pi it is 1 and it is periodic 1 branch in the negative side of x axis 1 branch in the positive side of x axis. So, this is minus pi to 0 minus 1 e [FL] power minus iota n pi x upon l d x plus 0 to pi it is 1 into e [FL] power minus iota n pi x upon l into d x.

Now, this can be written as 1 upon 2 pi now you can integrate it minus e [FL] power minus iota n pi x upon l here l is pi here l is pi. So, this is pi eta pi pi cancels out. So, you can eliminate this. So, this is this and this is this minus iota n x because l is pi. So, this is this function now this is n x and divided by minus iota n from minus pi to 0 plus it is e [FL] power minus iota n x upon minus iota n from 0 to pi it is further equal to; now negative, negative cancels out it is 1 upon 2 pi.

Now, you apply upper limit minus lower limit the upper limit is 1 upon iota n will be outside upper limit is 1 minus e [FL] power e [FL] power iota n pi and it is minus 1 upon iota n will be outside and it is nothing but when it is pi it is e [FL] power minus iota n pi minus 1 now it is further equal to 1 upon 2 pi n into iota it is 1 minus e [FL] power e [FL] power iota n pi minus e [FL] power minus iota n pi minus minus plus. So, this is nothing but when we simplify this. So, it is 2 it is 2 times of this 2 2 canceled out. So, it is nothing but 1 upon pi n iota and 1 minus e [FL] power iota n pi ok it is not 2 times. So, it is it is 2 into minus e [FL] power minus iota n pi. So, 2 is remaining here.

Now, e [FL] power iota n pi is cos n pi plus iota sin n pi which is minus 1 [FL] power n and this value is again cos n pi this value is what e [FL] power minus iota n pi will be nothing but cos n pi minus iota sin n pi which is nothing but minus 1 [FL] power n. So, this is minus 1 [FL] power n this is minus 1 [FL] power n. So now, it is 2 times 2 2 cancels out and 1 upon iota is nothing but minus iota. So, it is minus iota upon n pi and it is 1 minus minus 1 [FL] power n. So, this will be the final expression for final expression for c n you can see when it is when n is even it is 0 and when n is odd it is 2 it is inside bracket expression the expression is at the bracket when n is even minus 1 [FL] power even is 1. So, 1 minus 1 is 0 when n is odd it is 2.

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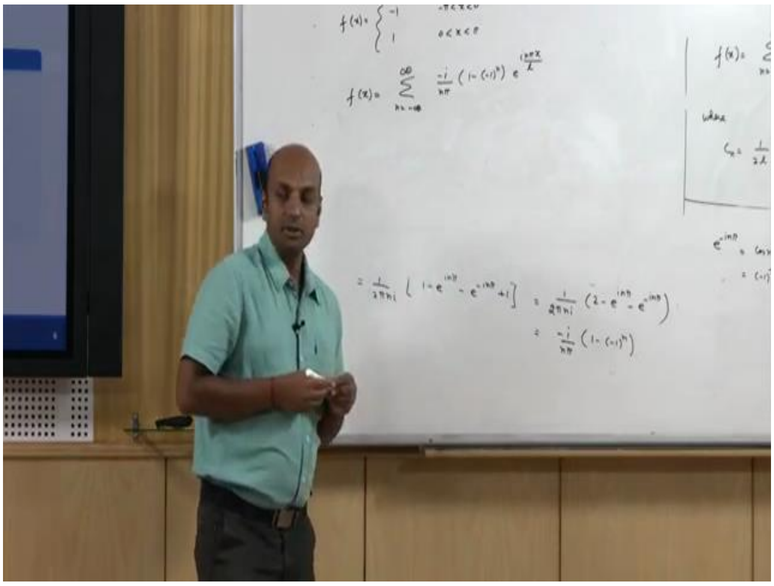

So, what will be the complex form of Fourier series for this function? So, for this function, the complex form is given by f x will be equal to its summation n from minus infinity to plus infinity, c n is what? C n is minus iota upon n pi 1 minus minus 1 [FL] power n into e [FL] power iota n pi x upon l. So, this will be the complex form of this Fourier series. So, hence whether the function is continuous like the first example is continuous, the second example is discontinuous at x equal to 0. So, if we have functions that are continuous or discontinuous or I mean piecewise continuous the second example is piecewise continuous.

So, if we have such problems then problems can be either converted into the sine or cosine terms as we have done before or in the complex form of Fourier series which is given by this expression where c n is given by this expression. Here the benefit is we have to find only c n and there we have to find a naught a n and b n the 3 coefficients. So, that that is how we can find out the complex form of Fourier series of any function f x.

Thank you.