Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 51
Parseval's Identity

Welcome on the lecture series on Mathematical Methods and its Applications. So, we were seeing Fourier series and what are various properties of Fourier series? That we have discussed in the last lectures. Now this is Parseval's identity. Now what Parseval identity is? And how it is important to solve few problems on Fourier series? Let us see.

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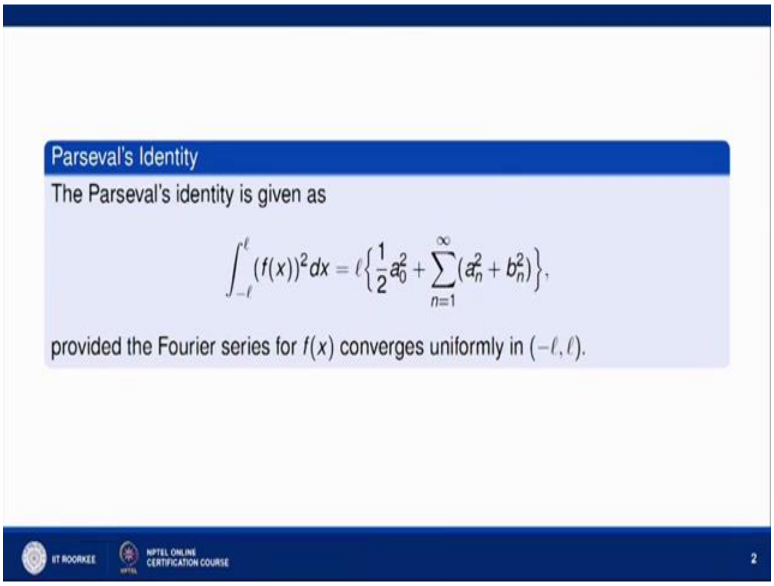

Now, it states that the Parseval's identity is given by integral minus l to plus l f x whole square d x is given by l times a naught square by 2 plus summation a n square b n square, provided Fourier series converges uniformly in minus l to l. Now what is the proof of this identity?

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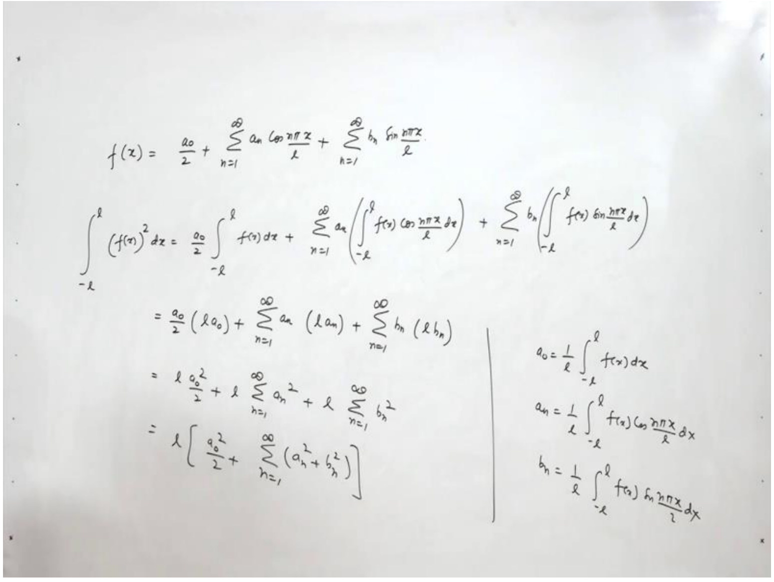

Now, what is the Fourier series? Expansion of f x, we know that Fourier series expansion of f x is nothing but it is a naught upon 2 plus summation n varying from 1 to infinity a n cos n pi x upon l plus summation n varying from 1 to infinity b n sin n pi x upon l. So, this is how we can expand the function in terms of sin and cosine terms. Now we want to find out in minus l to l integral, integral minus l to l f x whole square d x. So, you simply multiply both sides by f x and integrate from minus l to plus l.

So, what we will get? Integral minus l to l f x whole square d x from this side it is nothing a naught upon 2 integral minus l to plus l f x d x plus summation n varying from 1 to infinity a n because x is only here. So, it is integral minus l to plus l f x cos n pi x upon l into d x; this term plus summation n varying from 1 to infinity b n and integral from minus l to plus l f x sin n pi x upon l into d x.

Now if you recall the formulas of a naught a n and b n. Now what is a naught? We already know that a naught is nothing but 1 upon l integral minus l to plus l f x d x, a n is nothing but 1 upon l integral minus l to plus l f x cos n pi x upon l into d x and b n is nothing but 1 upon l integral minus l to plus l f x sin n pi x upon l into d x. This we already know. So, what this value is? Integral minus l to plus l f x d x, this is nothing but a naught into l. So, this value is nothing but a naught into l plus summation n from 1 to infinity a n.

Now, this value integral minus l to plus l f x cos n pi x by l d x is nothing but a naught a n into l. So, this is nothing but l into a n plus summation n varying from 1 to infinity, similarly this value is nothing but from here this value is nothing but b n into l. So, this is nothing but l into b n. So, hence this expression will be nothing but l into a naught square by 2 plus l into summation a n square n varying from 1 to infinity plus l times summation n from 1 to infinity b n square. So, this can be rewritten as, l we can take common a naught square upon 2 plus summation n varying from 1 to infinity a n square plus b n square. So, this identity is called Parseval's identity.

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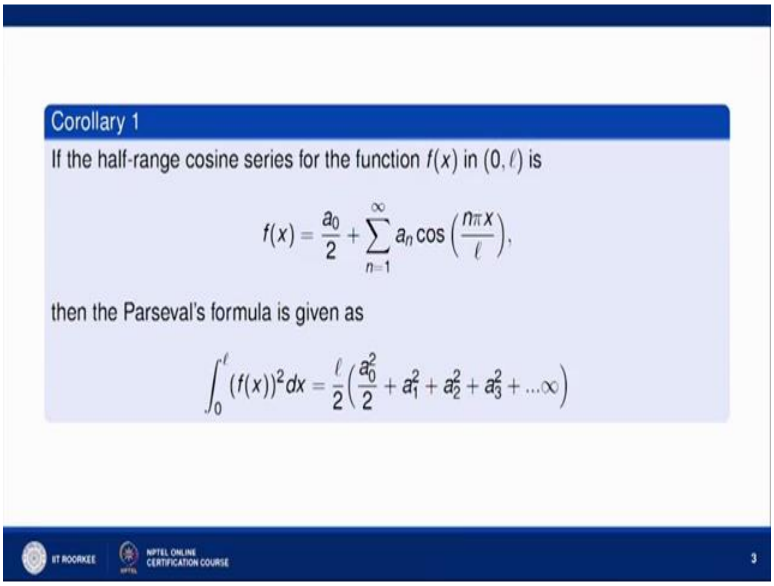

Now, so if we have half range series, which we have discussed in the last lecture; half range Fourier sin or cosine series. Half range cosine series of the function f x in the interval 0 to l. So, that can also be using Parseval formula that can also be derived basically, what was Fourier cosine series? If we have a half range, so and we want a even extension of the function, even periodic function that is in the terms of cosine series, cosine terms.

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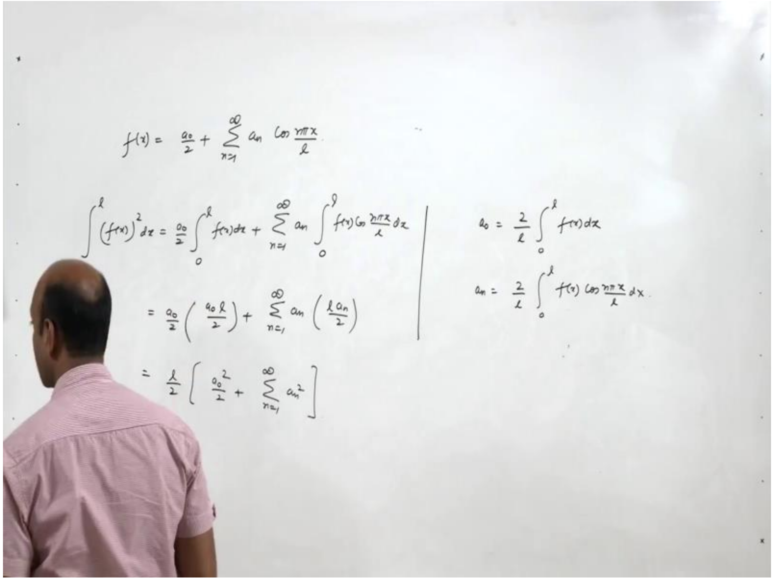

So, what is the expansion of that function? It is nothing but f x equals to a naught by 2 plus summation n varying from 1 to infinity a n cos n pi x by l. This is how we can write down the function f x in terms of cos. We take the even extension of the function. So, what is a naught here, a naught is nothing; upon 2 upon l integral 0 to l f x d x and what is a n? a n is again 2 upon l integral 0 to l f x cos n pi x upon l into d x.

So, again we multiply both sides by f x integrate from minus l to plus l, so what we will obtain? Integral minus l to l f x d x, f x whole square d x is nothing but a naught upon 2 integral minus l to l f x d x plus summation n from 1 to infinity a n integral minus l to plus l f x cos n pi x upon l into d x. Now this is a naught upon 2, now this value it is from 0 to l; it is from 0 to l. It is not from minus l to plus l because it is half range. So, it is from 0 to l. So, integrate from 0 to l. Now 0 to l this integral is nothing but a naught l upon 2. So, it is a naught l upon 2 plus summation n from 1 to infinity; it is a n. It is again l into a n upon 2.

So, this is nothing but you can take l by 2 common and it is nothing but a naught square upon 2 plus summation n from 1 to infinity a n square. So, which is same expression l by 2, a raised to l square by 2 plus summation n square n from 1 to infinity. So, this is how the Parseval identity can be used, when we have a half range cosine series. Similarly, if we have a Fourier, if we have a half range sin series, that is we extend the function considering function as an odd function.

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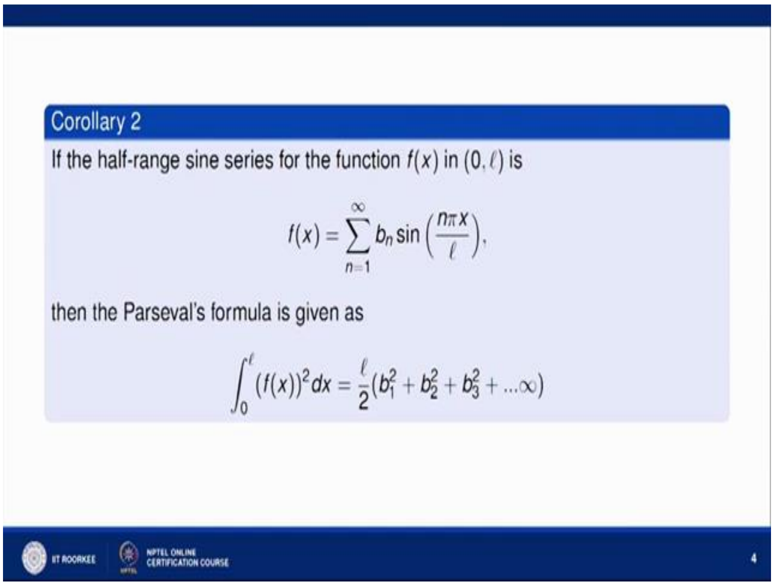

Then Fourier expansion is given by f x equals summation b n sin n pi x by l and using Parseval identity or the formula we will obtain 0 to l f x whole square d x will be nothing but l by 2 summation b n square. So, that can also be derived using the same lines, this formula can also be derived.

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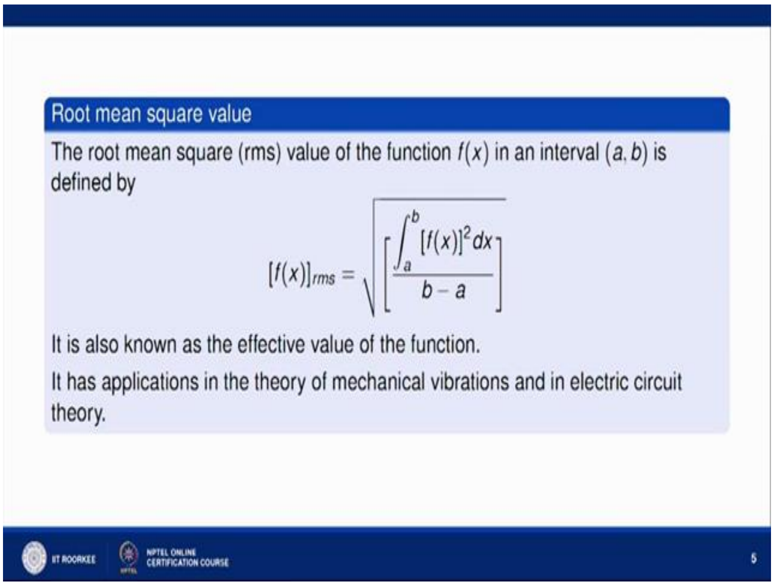

Now, next is root mean square value. The root mean square value of the function f x in the interval a comma b is given by it is under root of integral a to b f x whole square d x divided by b minus a. Under root of the entire expression, it is also known as the effective value of the function and it has application in theory of mechanical vibrations and in electrical circuits' theory, this root mean square value. Now let us solve few problems based on Parseval identity.

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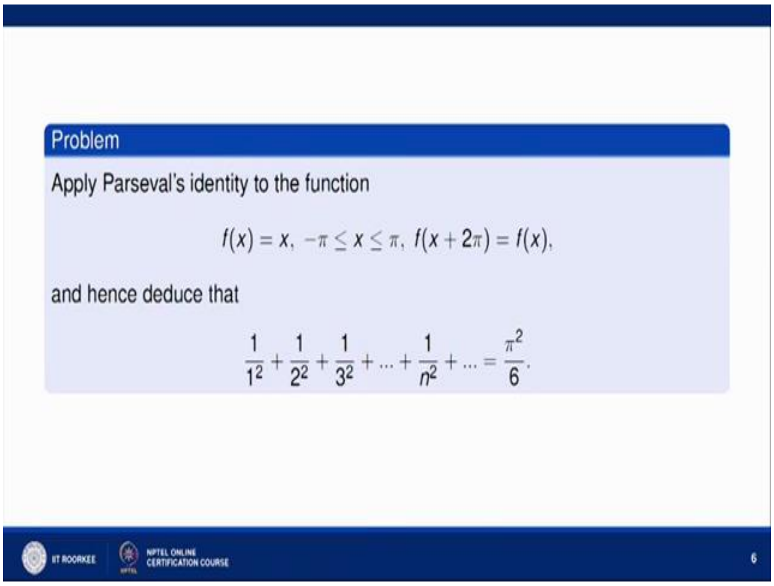

Now, apply Parseval identity to the function this and deduce that this equal to this. So, this series we have to find out using Parseval identity.

So, now f x equal to x is an odd function, f x equal to x from minus pi to plus pi is an odd function. It will contain only sin terms, sin series a naught and a n are 0.

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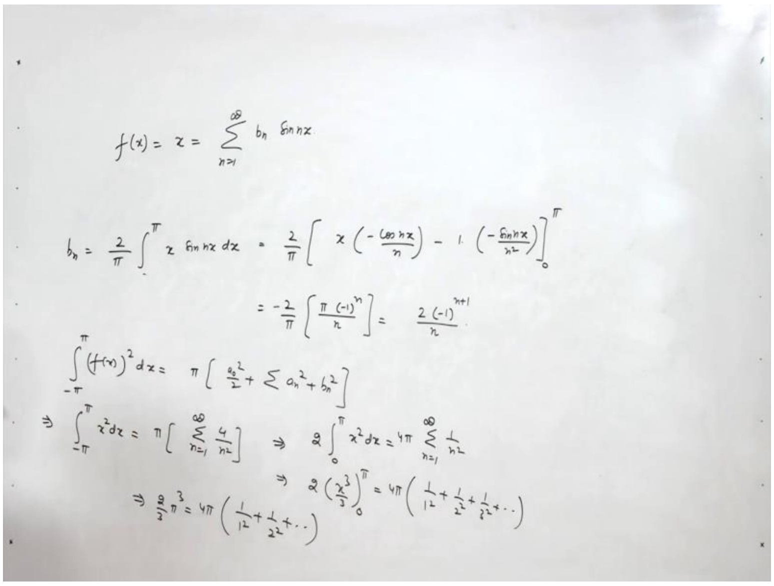

So, the f x which is equal to x will be nothing but summation n from 1 to infinity b n sin n x because l is pi. So, what will be b n? b n will be given by 2 upon pi integral 0 to pi x sin n x d x, f x sin n pi x by l into d x. Now you integrate this, it is 2 upon pi, first as it is integration of second which is minus cos n x upon n minus derivative of first into integration of second and 0 to pi.

So, applying integration by parts, we obtain this. Now we take the limits, it is 2 upon pi negative will come outside. It is 0, when x is pi or 0 and when x is pi it is nothing but pi into minus 1 to power n upon n and when x is 0, it is 0. So, it is nothing but 2 into minus one to power n plus 1 upon n. So, this will be b n. Now what is Parseval's identity? It is integration from minus pi to plus pi, f x whole square d x will be equal to l times, l is pi here pi times a naught square upon 2 plus summation a n square plus b n square. So, this is basically Parseval's identity. So, this is equal to, now this implies minus pi to plus pi this is x d x this is equal to x square because f x is x and pi a naught is 0, a n is 0 only b n is here and b n value is this when you square this it is summation n from 1 to infinity. It is 4 upon n square, when you take b n square.

So, this implies, now it is an even function it is nothing but 2 times 0 to pi x square d x which is equal to pi into 4 will come out, summation 1 by n square n from 1 to infinity. So, this further implies 2 times x cube by 3 from 0 to pi and it is 4 pi. It is 1 by 1 square plus 1 by 2 square plus 1 by 3 square and so on. So, this implies 2 upon 3 pi cube is equal to 4 pi into this expression. So, therefore, this value is nothing but when you simplify this it is nothing but pi square by 6 when you simplify this. So, this expression is nothing but pi square by 6. So, hence using Parseval identity, we can easily find out the value of the series.

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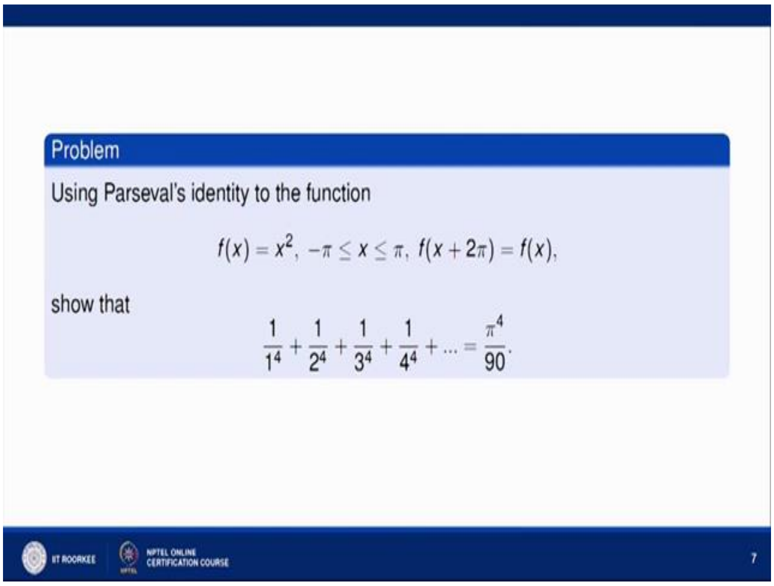

Now, the next problem is f x equal to x square, it is a even function and when we apply Parseval identity for this problem, it is an even function; that means, it will contain only cosine terms. So, b n will be 0.

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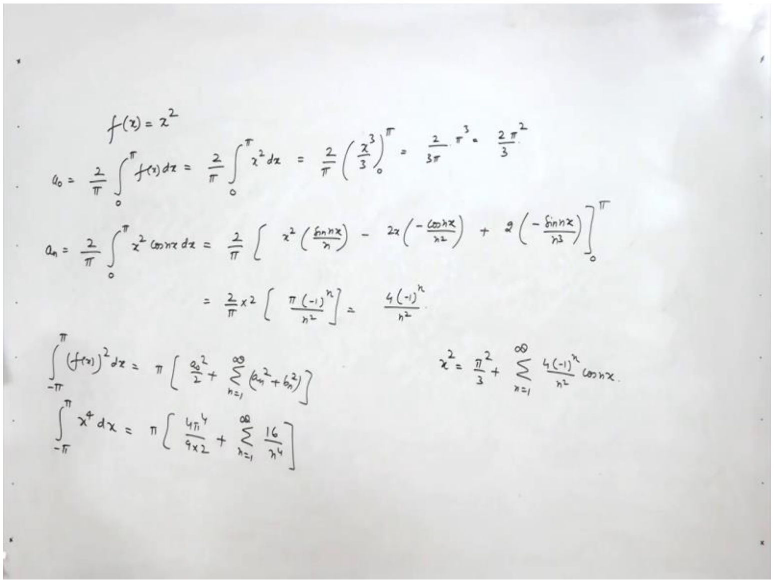

So, what will be function f x? f x is x square. So, first let us compute a naught, a naught will be nothing but 2 upon pi integral 0 to pi f x d x which is nothing but 2 upon pi integral 0 to pi x square d x which is equals to 2 upon pi, integration is x cube by 3 from 0 to pi. So, it is 2 upon 3 pi into pi cube, which is 2 pi square upon 3. So, this is a naught.

Now, a n is 2 upon pi integral 0 to pi, f x cos n x d x. Now you integrate by parts, it is 2 upon pi, it is x square sin n x upon n minus 2 x minus cos n x upon n square minus minus plus 2 into minus sin n x upon n cube from 0 to pi. Now sin terms will be 0, when x is 0 or x is pi. So, only this term is remaining and it is also 0, when x is 0. So, it is nothing but 2 upon pi, 2 this will also come out and it is nothing but pi into minus 1 to power n upon n square. So, this is nothing but 4 into minus 1 to power n upon n square. So, this will be a n.

So, we can easily find out the Fourier series of this f x, which is nothing but we already know it is nothing but x square is nothing but a naught upon 2, that is pi square upon 3 plus summation n from 1 to infinity; a n is this value which is 4 minus 1 to power n upon n square into cos n x. So, this will be the Fourier series of this function x square. Now if we apply Parseval identity for this problem. So, what will be that? It will be minus pi to pi f x whole square d x will be equals to l times a naught square upon 2 plus summation a n square plus b n square, n varying from 1 to infinity.

Now, this b n is 0 because it is an even function. So, this is nothing but minus pi to plus pi x to power 4 d x and it is pi times a naught square upon 2, a naught square is 4 pi to power 4 upon 9 into 2 plus n square, n square is 16 upon n to power 4; n is varying from 1 to infinity. Now when you simplify this, what we will get?

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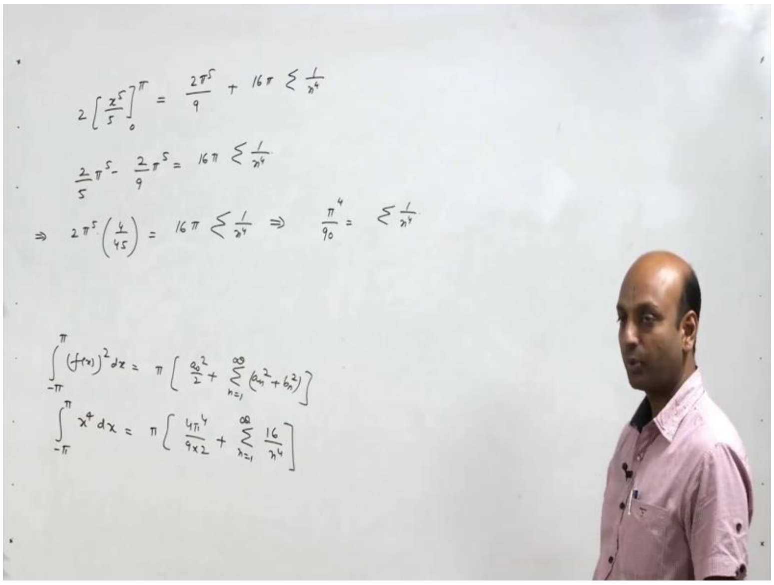

So now using Parseval identity in this expression we obtain this, now if you simplify this, this is an even function; it is 2 times x to power 5 upon 5 from 0 to pi, which is equal to, it is 2 pi to power 5 upon 9 plus 16 pi into summation 1 by n to power 4.

Now, this is nothing but 2 upon 5 pi to power 5 minus 2 upon 9 pi to power 5, which is equals to 16 pi into summation 1 by n to power 4. So, this implies, it is 2 upon, 2 this is common. So, it is nothing but 4 upon 45, which is equals to 16 pi into summation 1 by n to power 4. So, this implies it is nothing but pi to power 4 upon 90 will be equals to summation 1 by n to power 4. So, hence we have proved this, that summation one by n to power 4 is nothing but pi k to power 4 upon 90. So, this is how we can apply Parseval identity in this problem, to solve this equation.

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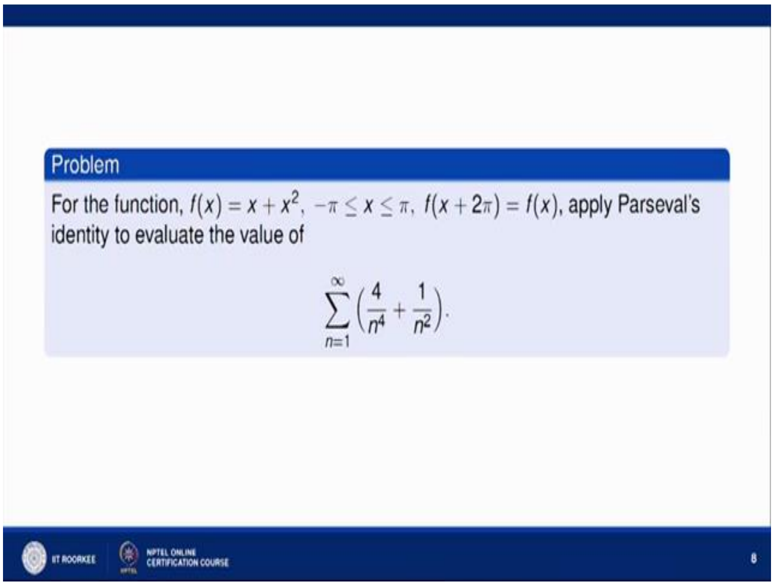

Now, let us try one more problem, which is neither even nor odd. It is x plus x square and we have to apply Parseval identity to find out the value of this expression. So, again we can apply Parseval identity in this problem. Let us see.

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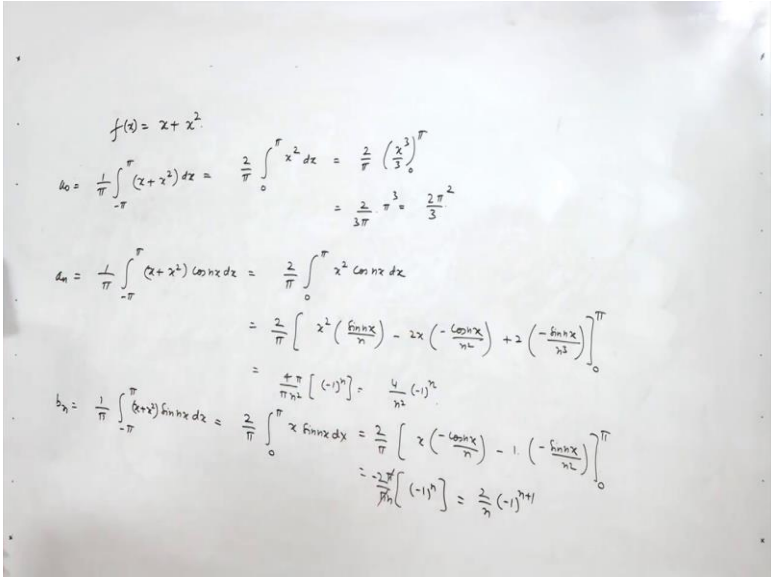

So, what will be f x? What is f x? f x is x plus x square. So, it will contain all terms sin terms, cosine terms it will contain all the terms. So, we have to find a naught a n and b n. So, what will be a naught? a naught will be nothing but 1 upon pi integral minus pi to plus pi x plus x square d x. So, this x is an odd function. So, this integral will be 0, only this will be left. So, it is 2 upon pi integral 0 to pi x square d x.

So, this is nothing but x square d x. So, it is nothing but 2 upon pi, it is x cube upon 3; 0 to pi, which is equals to 2 upon 3 pi into pi cube, which is 2 pi square upon 3. So, this is a naught. Now what will be a n? a n is 1 upon pi integral minus pi to pi, x plus x square cos n x d x. Now x into cos n x is an odd function with respect to x it is 0 and x square into cos n x is an even function. So it will be 2 times pi from 0 to pi, it is x square cos n x.

Again we will apply integration by parts to solve this. It is 2 upon pi x square integration of this is sin n x upon n minus derivative of this, integration of this plus 2 integration of this from 0 to pi. So, only one term is left which is nothing but 4 upon pi n square and it is nothing but when x is pi, so it is pi times minus 1 to power n. It is 4 upon pi n pi, pi cancels out; it is 4 n square into minus 1 to power n. So, this will be a n.

Now, let us compute b n, b n is nothing but 1 upon pi integral minus pi to plus pi x sin n x d x, x plus x square sorry it is f x plus x square, so it is x plus x square. Now x into sin n x is an even function and x square sin n x is an odd function. So x square sin n x will be 0, it will be 2 upon pi integral 0 to pi x sin n x. So, again we will apply integration by parts here. So it is nothing but 2 upon pi x into minus cos n x upon n minus derivative of first integration of second minus sin n x upon n square and it is from 0 to pi. So, it is nothing but 1 upon pi minus will come out, n will come out and it is nothing but pi times minus 1 to power n. So, pi, pi cancel out it is nothing but 2 upon n minus 1 to power n plus 1. So, this expression will be here.

So, we got the values of a naught a n and b n. We can directly apply Parseval identity to find out the value of this expression. So, what is Parseval identity now? So, let us note down the values of a naught, a naught we obtained 2 pi upon square upon 3, a n we obtained as 4 upon n square minus 1 to power n and b n we obtained as 2 upon n minus 1 to power n plus 1. So, these values we have obtained, now we can apply Parseval's identity.

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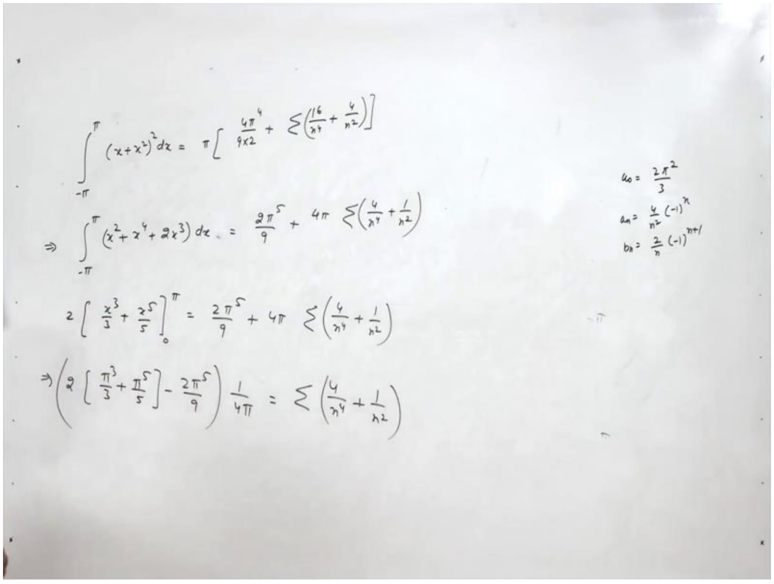

So, this is integral minus l to plus l, f x square; f x whole square d x which is equals to l into a naught square upon 2, that is 4 pi to power 4 upon 9 into 2 plus summation; summation a n square, that is 16 upon n to power 4 plus b n square 4 upon n square.

So, this implies. So, first we simplify this. So, this integral is nothing but minus pi to pi, it is x square plus x to power 4 plus 2 x cube whole into d x plus it is which is equal to it is 2 pi to power 5 upon 9 plus 4 pi and summation 4 upon n to power 4 plus 1 by n square. So, this is the required expression which you want to find out, required expression. So we can easily simplify now. It is a even function and odd function.

o, this will be 0 and it is 2 times it is x cube upon 3 plus x to power 5 upon 5 from 0 to pi and it is equals to 2 pi to power 5 plus upon 9 plus 4 pi into summation 4 upon n to power 4 plus 1 by n square. So, this implies 2 into it is pi cube by 3 plus pi to power 5 by 5 minus 2 pi to power 5 upon 9 and whole divided by 1 upon 4 pi I mean 4 pi is equal to summation 4 upon n square plus 1 upon here, 4 plus 1 upon n square. So, we can simplify this value, on simplify we will get the value of this expression.

So, this is how one can apply a Parseval identity to find out the value of some series expansion, using Fourier transforms, using Fourier series sorry.

Thank you very much.