Mathematical methods and its applications
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Lecture - 50
Fourier half – range series

Welcome to the lecture series on Mathematical Methods and the Applications. Now, we will discuss Fourier half range series. We have already discussed in last lecture what how we can find out Fourier series expansion of even and odd functions. That we have already seen in the last lecture.

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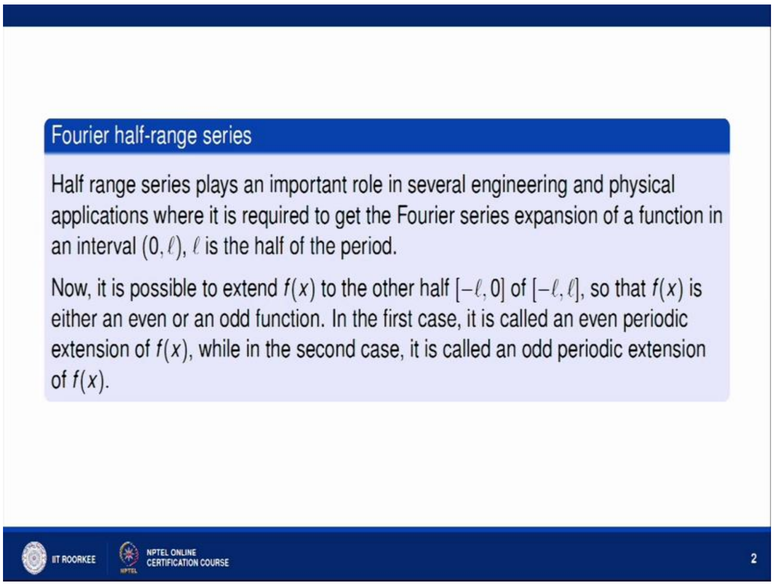

Now, Fourier half range series let us see half range series plays an important role in solving several engineering and physical applications, where it is required to get the Fourier series expansion of a function in an interval say 0 to l. So, it is a half range basically, in Fourier series basically in expansion of Fourier series, we need interval minus l to l and function must be periodic that is the condition for the Fourier series of the expansion f x.

Now, here the Fourier series expansion of a function is in the interval 0 to l, where l is an half of the period. Now how can we find out Fourier series expansion of such a function? And these problems are important in several engineering and physical applications. Now it is possible to extend the function f x to the other half, that is minus l to 0 of minus l to plus l. So, that f x is either an even or an odd function. Basically what we do, we know the nature of the function from 0 to l. And that is given to us. And depending upon the basically nature of the function, we find out we either extend the function, considering function as an even function or we either extend the function taking function as an odd function. Suppose function is given to you from 0 to l. Suppose, this function f function is given to us from 0 to l, this function is given to you from 0 to l. It is a half period.

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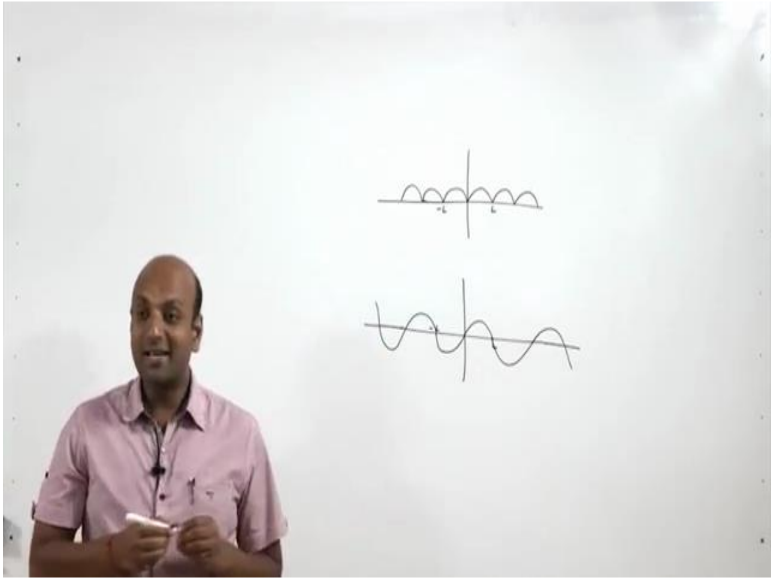

Now, you can find out the even extension of this function, you can extend this function as an even function from minus l to plus l. And then find out the Fourier series expansion of this function. So, you can take this function as an even function, whatever function is given to you from 0 to l. You can find out the even extension of this function, taking function as an even function. On the other way you can also find the; suppose this is given from 0 to l, you can find an odd extension of the function.

You can extend this function like this from 0 to minus l, minus l to plus l and in the same way over here. So, basically we need a Fourier series expansion of a function which is given in the half range from 0 to l. So, we can either to find the Fourier series expansion of such a function, we can either extend the function taking function as either as an odd function or taking function either as an even function.

So, in the first case it is called as even periodic extension of f x, while in the second case it is called an odd periodic extension of the function f x. If we do an even periodic extension of f x which is the first one, which is the first case, it is the even extension of the function f x. Then f x is an even function in minus l to plus l. Of course, it is an even function.

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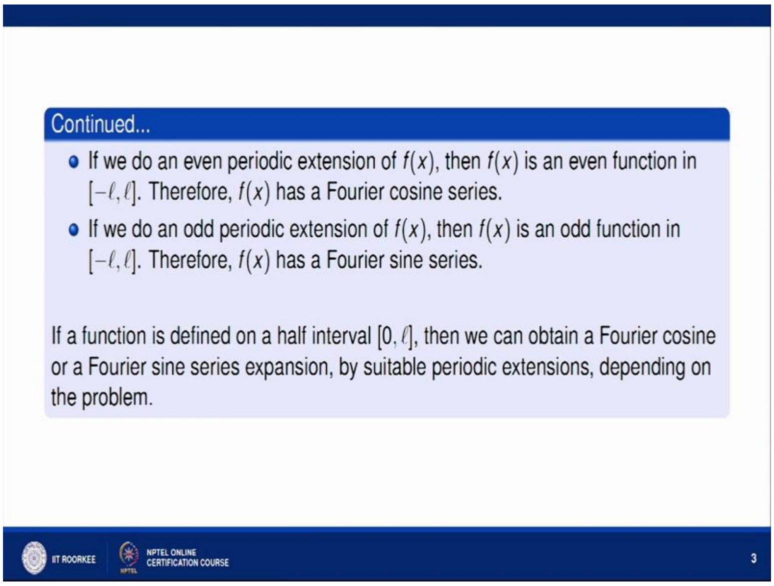

Then f x has Fourier cosine series that we already know, that if function is an even function. We extend the function as an even periodic function, then f x will be an even function. And in even function we only have a cosine series, cosine terms in the Fourier series expansion.

Now, if we do an odd periodic extension of the f x, then f x is an odd function in minus l to plus l. And therefore, f x has Fourier sine series. Suppose we extend the function as an odd periodic function. So, this f x will be an odd function and we know that if function is an odd function then, it contains only sine terms. The Fourier series expansion of that function will contain only sine terms.

If a function is defined on a half interval from 0 to l, then we can obtain a Fourier cosine or Fourier sine series expansion by suitable periodic extensions depending on the problem. If you want a Fourier cosine series expansion; that means, we take an even periodic function or we take function as an even function. And if we want Fourier sine series; that means, we will take as a odd periodic extension of the function that depends on the problem.

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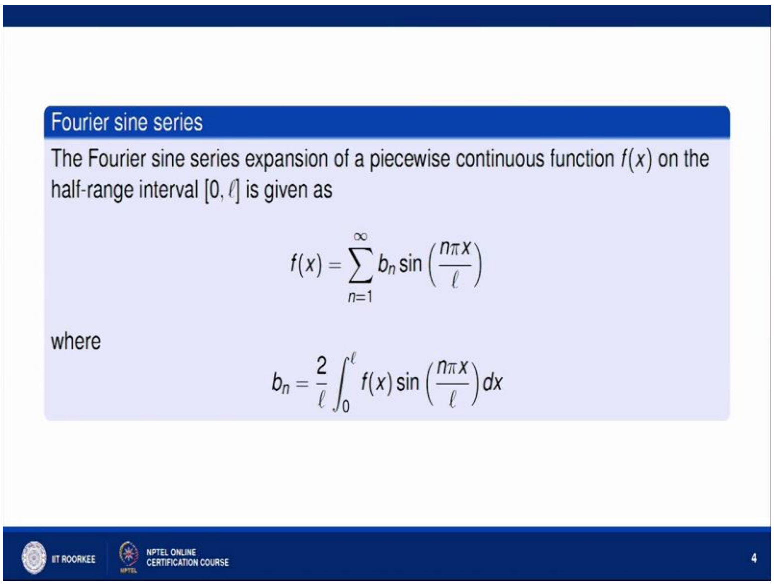

So, suppose you want Fourier sine series. So, we already know this result that if you want Fourier sine series. Now function is given in 0 to l. And you want an odd extension of the function. Odd extension means only sine terms. So, what will be the Fourier series of that that function?

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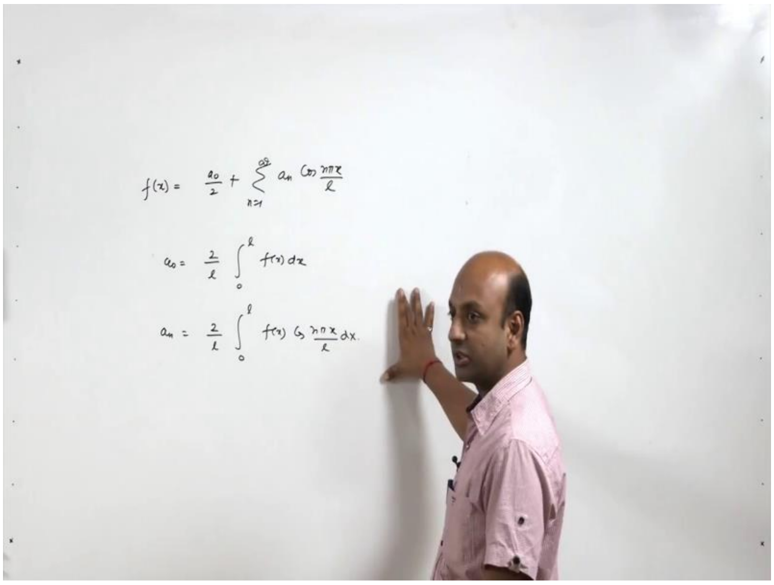

The Fourier series of that function will be nothing but it will be given by summation n from 1 to infinity, b n, sin n pi x by l, and where b n is nothing but 2 upon l integral 0 to l, f x sin n pi x by l into d x. So, this is an odd extension of the function. Now if we want

Fourier cosine series expansion or the function defined on a half range from 0 to l. So, that will contain only cosine terms. So, that will be given by, that will be given by a naught by 2 plus summation a n cos n pi x by l, n varying from 1 to infinity. And here a naught will be nothing but 2 upon l integral 0 to l f x d x and a n will be given by 2 upon l integral 0 to l f x cos n pi x by l into d x. So, this will be the even extension of the function. Function is given from 0 to l and we defined an even extension of a function as a cosine terms in the form of cosine terms.

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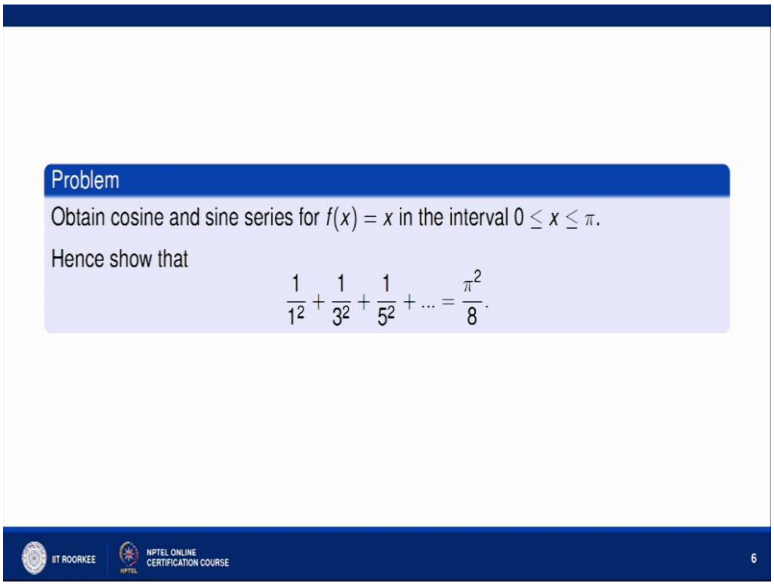

Now, let us try to solve this problem. Obtain cosine and sine series for f x equal to x in the interval from 0 to pi and hence show this result.

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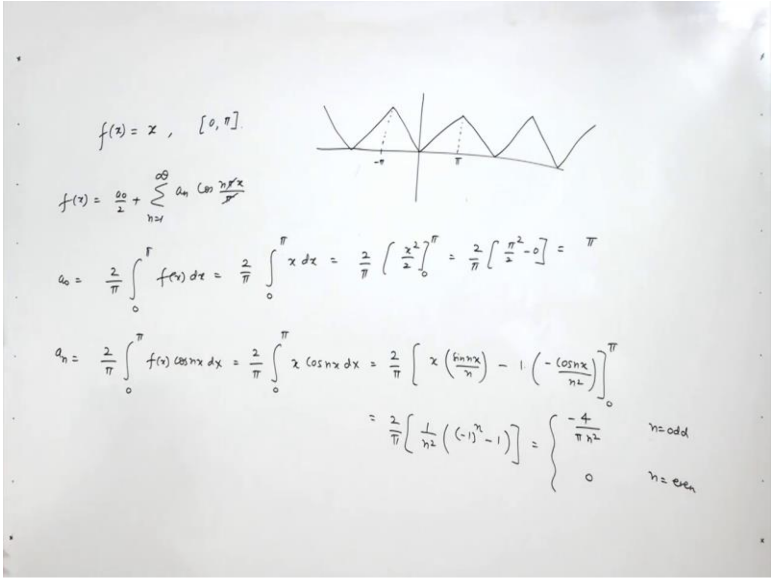

So, first of all function is given function is x, f x equal to x. So, a very simple function f x equal to x interval is 0 to 0 to pi. Now suppose you want cosine series expansion of this function. Cosine series expansion means we want to extend this function as an even function. We want to extend this function as an even function. So, this function is like this it is from 0 to pi it is it is x, f x equal to x from 0 to pi. And we want to extend the function as an even periodic function. So, from minus pi also if it is an even function it must be symmetrical about x axis. So, we will draw this function like this. We extend this function like this.

And similarly, here also like this and similarly here also like this and similarly here also. This will be an even extension of this function or now we can write this function in the cosine series form. How f x will be nothing but a naught by 2 plus summation a n, cos n pi x by pi, n varying from 1 to infinity. And what will be a naught now it is 2 upon pi integral 0 to pi. So, here pi pi cancels out. 0 to pi f x d x f x is nothing but f x is nothing but x d x. So, it is nothing but 2 upon pi. It is x square upon 2 from 0 to pi. So, it is nothing but 2 upon pi, pi square by 2 minus 0. So, it is nothing but pi. So, a naught is nothing but pi. And a n what will be a n, a n will be nothing but 2 upon pi integral 0 to pi f x cos n x d x. So, it is equal to 2 upon pi integral 0 to pi. Now what is f x, f x is x; so x cos n x d x.

So, now we will apply integration by parts to simplify this expression. So, this will be nothing but 2 upon pi. So, first as it is integral of second, sin n x upon n, minus derivative of this and integration of this. So, this is nothing but minus cos n x upon n square 0 to pi. So, that is further equal to 2 upon pi. Now the first term is 0 when x is pi or x is 0. Now second term is 1 upon n square times when x is pi it is minus 1 k to the power n and when x is 0 it is 1. So, this would be this term. So, this a n is equal to now when n is odd when n is odd. So, this is nothing but when n is odd it is minus 1, minus 1, minus 2, minus 2, minus into 2 minus 4 upon pi into n square. So, this will be nothing but this term when n is odd. And when n is even it is nothing but 0. Because minus k to the power even is 1 one minus 1 is 0. So, n is defined like this.

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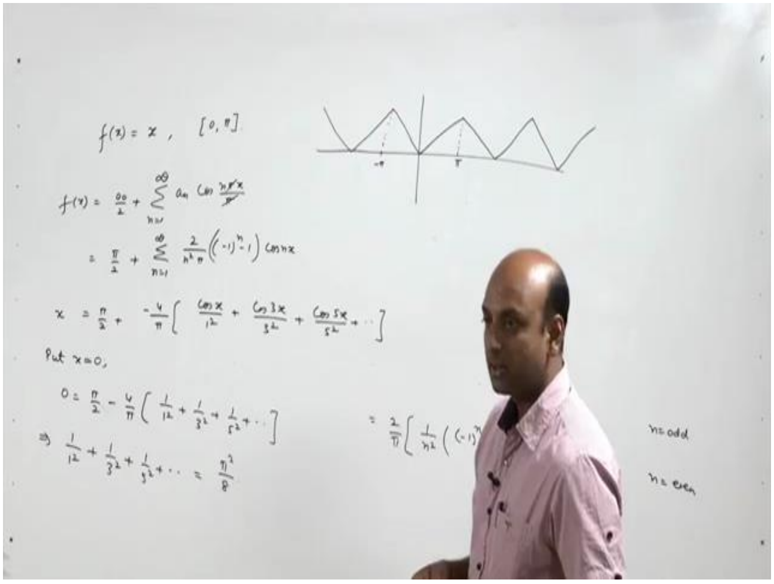

So, what will be the cosine series of this function f x. So, a naught is pi, a naught is pi. So, it will be nothing but pi by 2 plus summation n from 1 to infinity. It is 2 upon n square pi minus 1 k to the power n minus 1 into cos n x. Which is further equal to pi by 2 plus 2 upon pi can come out and when n is odd it is minus 4 upon it is I mean minus 2 will come out. So, when n is odd it is minus 4 upon pi into 1 by n square that is cos 1 upon 1m square plus cos 3 x upon 3 square plus cos 5 x into 5 upon 5 square and so on. And when n is even it is 0. So, this will be the expansion of x, f x is x. So, this is an even extension of the function x.

So, x is defined in only 0 to pi. We extend this function take assuming the function as an even function. And that will contain only cosine terms these are the cosine series expansion of function f x equal to x. Now to reduce this expression 1 is 1 upon 1 square, 1 upon 3 square, 1 upon 5 square and so on. With all plus signs, we can simply substitute x equal to 0, so put x equal to 0 both sides. So, when you put x equal to 0, it is pi by 2 plus or minus, minus 4 upon pi, it is 1 upon 1 square, plus 1 upon 3 square, plus 1 upon 5 square and so on. So, that implies 1 upon 1 square plus 1 upon 3 square plus 1 upon 5 square is simply equal to pi square by 8. So, that we have proved this result also.

Now, to obtain Fourier sine series of this function f x equal to x; so how you will obtain that the first part is over of this problem. Now we want to obtain Fourier sine series of this function; that means, we have to extend the function f x equal to x, as an odd function. Then only we can obtain Fourier sine series that is the series containing sine terms only. So, how can we do that we have the function f x equal to x here?

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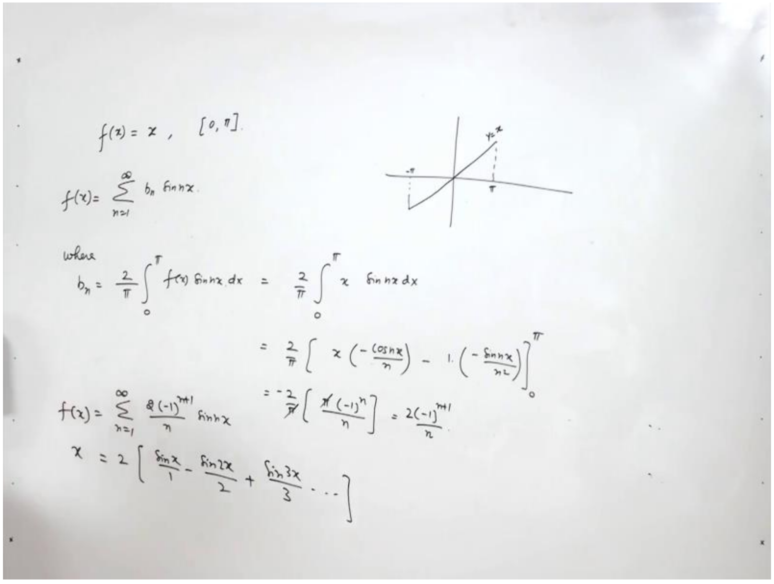

We have the function f x equal to x here is y equal to x and we have to extend this function as an odd function, to obtain the Fourier sine series of this function f x equal to x. So, it must be like this. And assuming function as periodic function we can extend this function as an odd function. So, what will be the Fourier sine series of this function? It will be summation n from 1 to infinity b n, sin n x, where b n is nothing but 2 upon pi integral 0 to pi, f x, sin n x, d x. So, this is nothing but is equal to 2 upon pi integral 0 to pi f x is x, sin n x, d x. So, this is nothing but 2 upon pi integral of this is x into minus cos n x upon n, minus derivative of this and integration of this will be minus sin n x upon n square from 0 to pi.

So, after integration by parts, we obtain this thing. So, this is nothing but it is 2 upon pi when you put x equal to pi here or 0 here, this term is 0. And here when you put x equal to 0 it is 0. So, only term is left here. So, negative will come here. So, it is pi into minus 1 k power n upon n. So, pi pi cancels out it is nothing but minus 1 k to the power n plus n plus 1 upon n. So, this will be 2 into 2 into minus 2 n k to the power n plus 1 upon n. So, this will be b n. So, what will be f x, f x will be nothing but summation n from 1 to infinity b n, b n is 2 into minus 1 k power n plus 1, upon n into sin n x. So, this is nothing but 2, when n is 1 it is sin x upon 1, minus sin 2 x upon 2, plus sin 3 x upon 3 and so on. F x is x. So, this will be the odd extension of the sine series expression of the function f x in terms of sine. So, this will be the final solution of this problem.

So, now let us try to solve one more problem of this function, which is we find as an, this function is also given from 0 to 1. And that is in half range. And we want function to expand as an Fourier sine series; that means, we want an odd extension of this function. So, how we can expand this as an odd function? Again for the odd expansion that is we want the Fourier sine series of this function eta f x will be given by this expression. So, what will be b n, b n will be nothing but 2 upon l.

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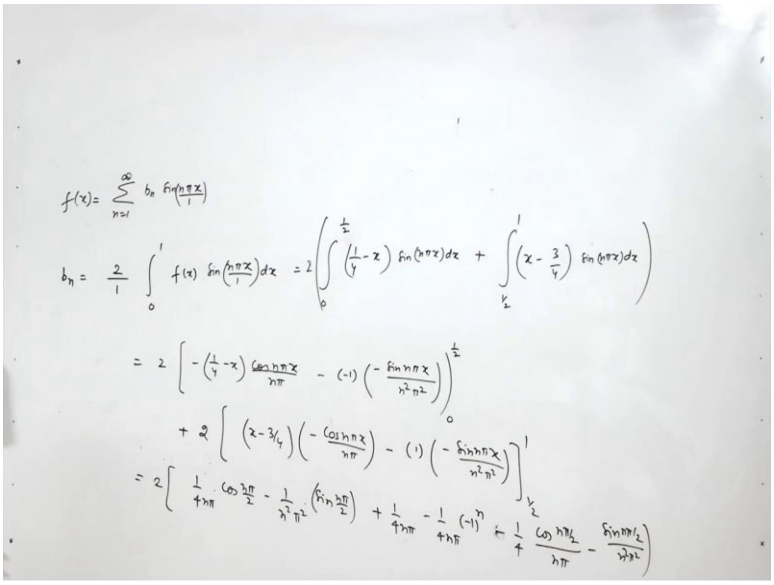

Now, here l is 1 in this problem l is 1, 0 to 1, f x and here also it is sin pi x upon 1, because l is 1, f x sin n pi x upon 1, into d x.

Now, we have to split the function. From 0 to half it is 1 by 4 minus x. From 0 to half, from 0 to half sorry 0 to half it is it is 1 by 4 minus x into sin n pi x d x plus half to 1, it is x minus 3 by 4, sin n pi x into d x, and whole 2 times. So, we can simplify this to find out the sin series expansion of this function. So, how we can simplify, simplification is simple we have to apply by parts only 2 times. It is nothing but 1 by 4 minus x, into minus of cos of n pi x upon n pi and minus derivative of this which is nothing but minus 1 and integration of this which is nothing but minus sin n pi x upon n square pi square.

And that whole from 0 to 1 by 2 I simply apply integration by parts first. As it is integration of second minus derivative of this and integration of this which is nothing but this again for the second part plus 2, times again 2 is here for second part. It is x minus 3 by 4 that you will easily solve by your own you can easily apply integration by parts and simplify to find the Fourier series expansion of this function. So, it is nothing but minus cos n pi x, upon n pi .and it is minus plus 1 into minus sin n pi x, upon n square pi square, and it is varying from half to 1. So, so after simplification what we will get 2 times. So, when x is half 1 by 4 minus half is minus 1 by 4, minus minus plus. So, it is 1 by 4, into cos n pi remains same into cos n pi by 2. When x is half, here it is nothing but minus of 1 upon n square pi square. And when n is half it is sin n pi by 2. When x is 0, it is 0, when x is 0, it is 1, 1 upon 1 by four. So, it is minus minus plus 1 by 4 into n pi.

Now, for this term when x is 1 it is 1 minus 3 by 4, that is 1 by 4 negative is here, so minus 1 by 4, and pi into minus 1 k power n. And when x is 1, here it is 0. Now when x is half it is half minus 3 by 4, half minus 3 by 4 is minus, 1 by 4 it is minus 1 by 4 minus minus plus 1 by 4. And minus is again here. So, it is minus and then it is cos n pi by 2 upon n pi. And when n is half here it is nothing but minus minus plus and minus again. So, it is sin n pi by 2 upon n square pi square. So, you can simplify this entire expression, to find out the value of b n.

And that b n you can substitute here to find out the Fourier series expansion Fourier series expansion of this function f x.

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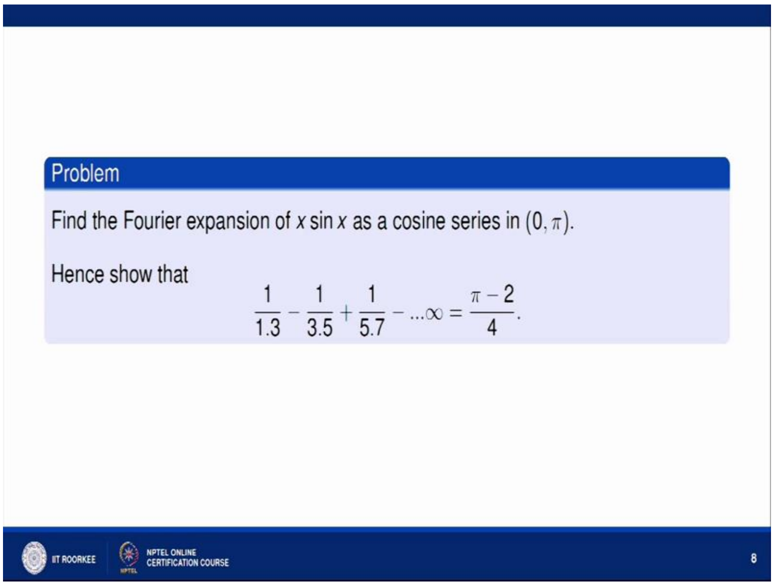

Now, on the in the same lines, we can also find out the Fourier series expansion of x sin x as a cosine series. Cosine series means you want an even extension of this function.

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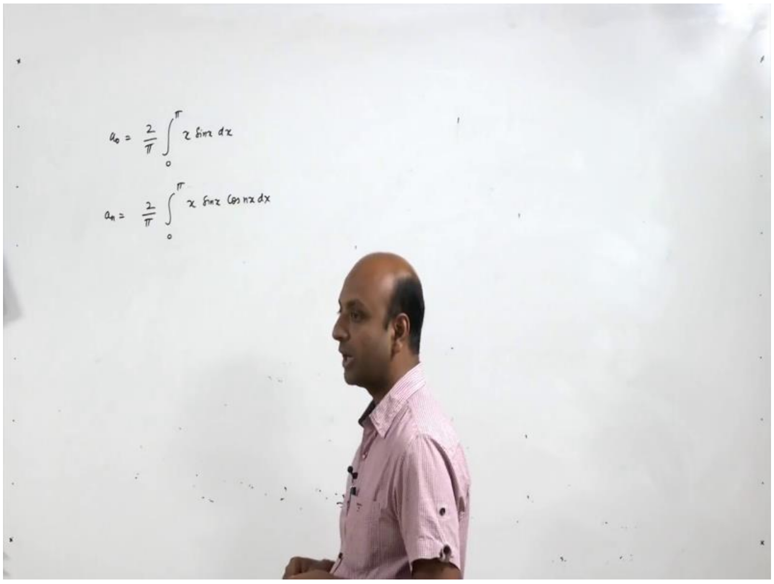

That means, for this function x, x sin x you will simply find a naught and a n. And a naught will be given by a naught will be given by. So, for this problem a naught will be given by 2 upon pi integral 0 to pi x sin x d x. And a n will be given by 2 upon pi, integral 0 to pi it will be x sin x into cos n x d x. So, this is how we can find a naught and a n in this expression. And finally, for some x we can also find out the series expansion of this function.

So, thank you for this lecture. In the next lecture we will study an identity and some problems based on that and also complex form of Fourier series expansion.

Thank you.