Mathematical methods and its applications

Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 05
Methods of undetermined coefficients

Hello friends. Welcome to my lecture on Method of Undetermined Coefficients. This method we will be used to solve second order linear differential equations, which are non-homogenous, that they are of the form y double dash plus a y dash plus by equal to r x. Now we have already solved the homogenous linear second order differential equations.

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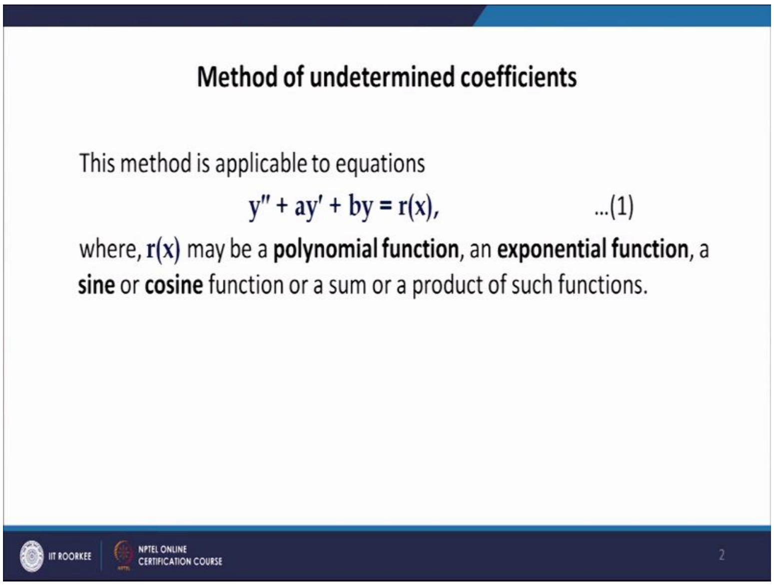

So, let us see how we shall solve this second order non homogenous linear differential equation.

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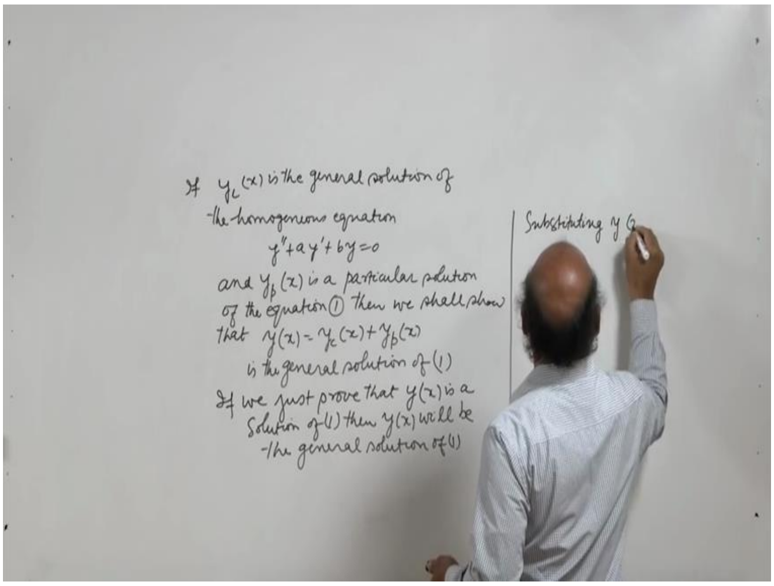

First of all, we shall show that if y c x is the general solution of the homogenous equation y double dash plus a y dash plus by equal to 0 and y p x is a particular solution of the non-homogenous equation of the equation 1, then we shall see that show that y x equal to y c x plus y p x is the general solution of equation 1. So, y c x is the general solution of the homogenous equation y double dash plus a by dash plus by equal to 0 means y C will contain 2 arbitrary constants.

And we know how to get the solution of this homogenous equation, y p x is a particular solution of the equation 1 means y p x is a solution of the equation 1, without any arbitrary constants. So, if we can show that y x equal to y c x plus y p x is a solution of the equation 1, then y x because it contains 2 arbitrary constants, will be the general solution of equation 1, y x contains 2 arbitrary constants, because y c x contains 2 arbitrary constants. And y p x is a solution of equation 1 without any arbitrary constants.

So, if we just prove that y x is a solution of equation 1, then y x will be the general solution of 1. So, let us see when you substitute y x equal to y c x plus y p x, if it satisfies the equation 1, it will be a solution of equation 1.

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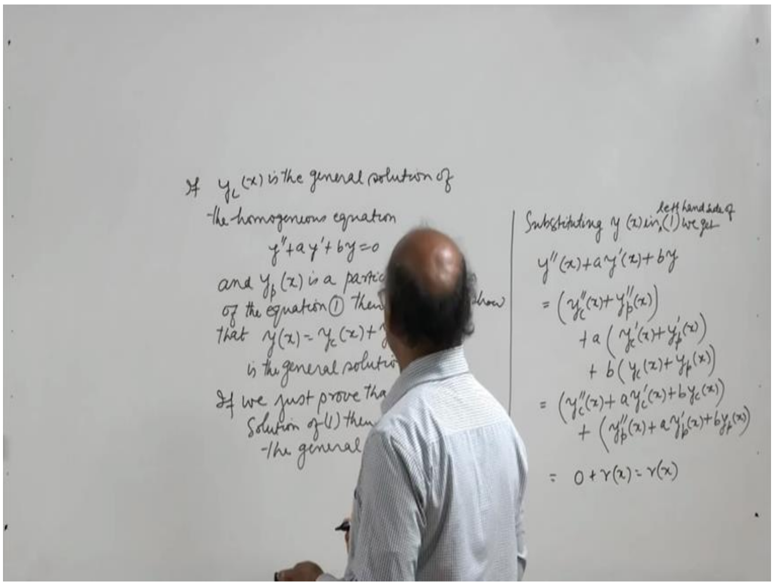

So, substituting in the left hand side of one, we get y p double dash x, plus a y p dash y double dash y double dash x, plus y dash x plus by dash by, equal to y C double dash x plus y p double dash x, plus a times y dash x, means y C dash x plus, y p dash x, plus B times y c x plus y p x. We can write it as y C double dash x plus a y C dash x plus by C x. Now since y c x is a general solution of y double dash plus a y dash plus by equal to 0. So, this first expression is 0. Then we have now y p x is a particular solution of the equation 1, therefore, y p double dash plus a y p dash plus by p is equal to r x. And thus we get y x is a solution of equation 1. Again I repeat y x will therefore, be the general solution of equation 1.

So, if we are having a non-homogenous second order linear differential equation with constant coefficients, then we will now discuss a method by which we can determine this particular solution y p x, y c x we know how to determine. So, our now method will determine the value of y p x. So, the method that we are going to discuss is called as the method of undetermined coefficients. In this method the by we make an educated guess for y p x the particular solution y p x of the equation 1 which we are looking for.

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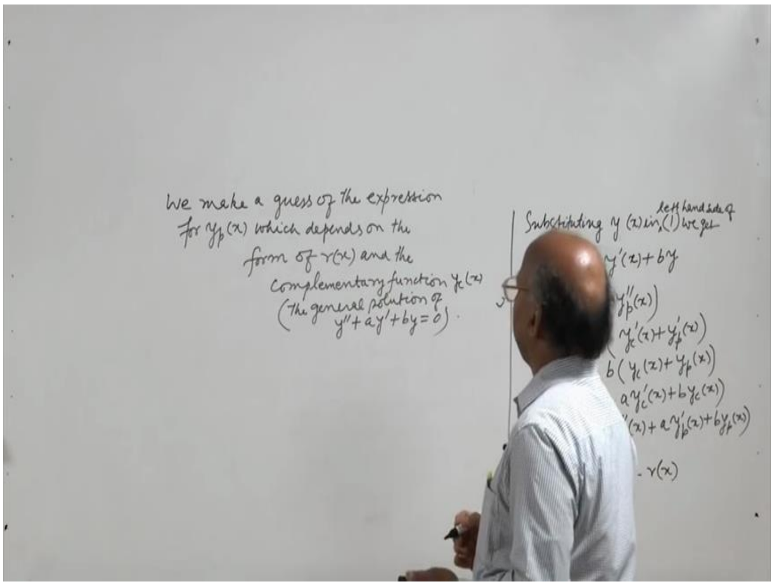

So, we make a guess of the expression for y p x, which depends on the form of r x, and the complimentary function. This complimentary function is nothing but the general solution of the associated homogenous equation. Complementary function is nothing but the general solution of. So, I can say y C x. So, this is not an arbitrary guess. It is a it is called an educated guess, because it depends on the form of r x, and also sometimes it depends on the form of the complimentary function y C x.

Now, this expression for y p x will contain constants which will be determined by substituting y p x in the equation 1. So, that is why we call this method as method of undetermined coefficients. Now here we have certain detritions. This method can be applied to the second order linear differential equation which are non-homogenous, when r x is of the following type. It can be a polynomial function, it can be an exponential function, it can be a sin or cosine function or a sum or products of such functions.

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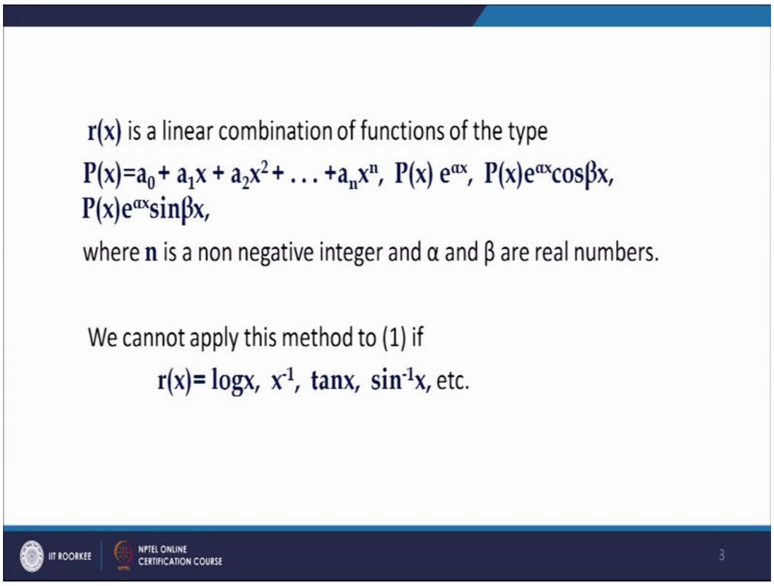

Now, or you can take linear combination of the functions which are polynomials or exponential functions or trigonometric functions sin x cosine x. So, p with p you can take r x to be function of the form p x, or you can take p x into e to the power alpha x, r p x e to the power x cos beta x, r p x e to the power alpha x and beta x. So, r x is a function of this type, where n is non negative integer and alpha and beta are real numbers, then we can apply this method of undetermined coefficients.

Now, if r x is of the type log x r x to the power minus 1 or 10 x r sin inverse x, then this method cannot be applied. So, for second order non homogenous linear differential equations, where r x is of those types log x 6 to power minus 1 10 x sin inverse x etcetera, then we will apply the general method which we shall discuss in the next lecture now. So, in the general method, we will be taking care of all the functions r x which cannot be handled by the method of undetermined coefficients.

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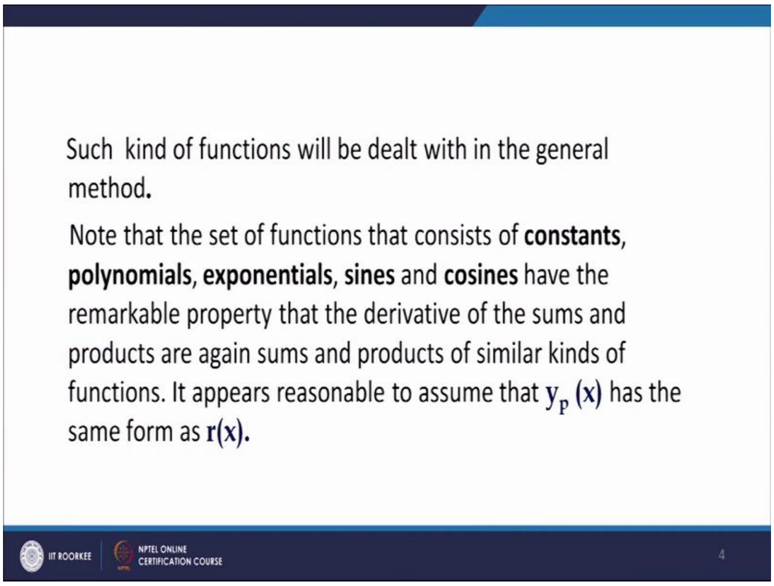

Now, let us see if you take the constants, polynomials, exponentials, sines or cosines, the remarkable property of those functions is that, the derivatives of the sums and products of such functions are again sums and products of similar kind of functions. So, in order to assume the form of y p x, it is reasonable to assume the same form as that of r x.

Now, let us take an example say y double dash plus 4 y dash, minus 2 y equal to 2 x square minus 3 x plus 6. So, here we can see r x is a polynomial in x of degree 2.

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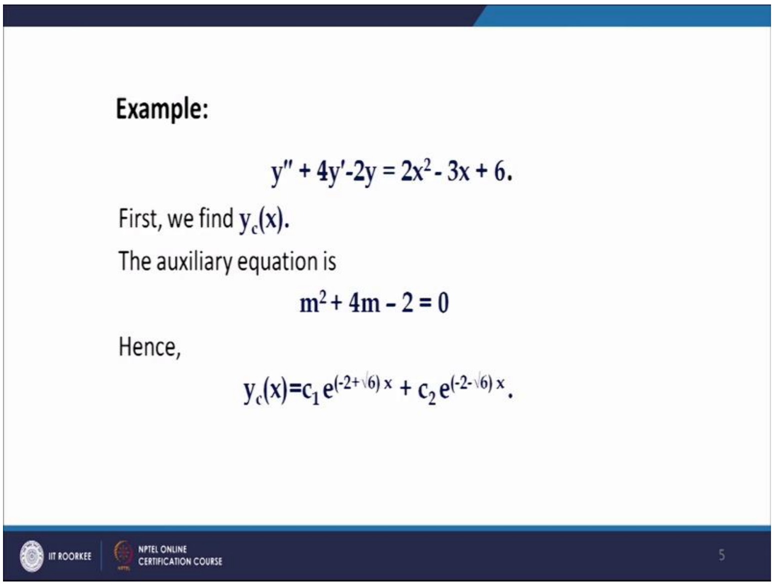

Now, first we shall find y C x. So, to find y c x we have the write the auxiliary equation m square plus 4 m minus 2 equal to 0, we can find the value of m for this.

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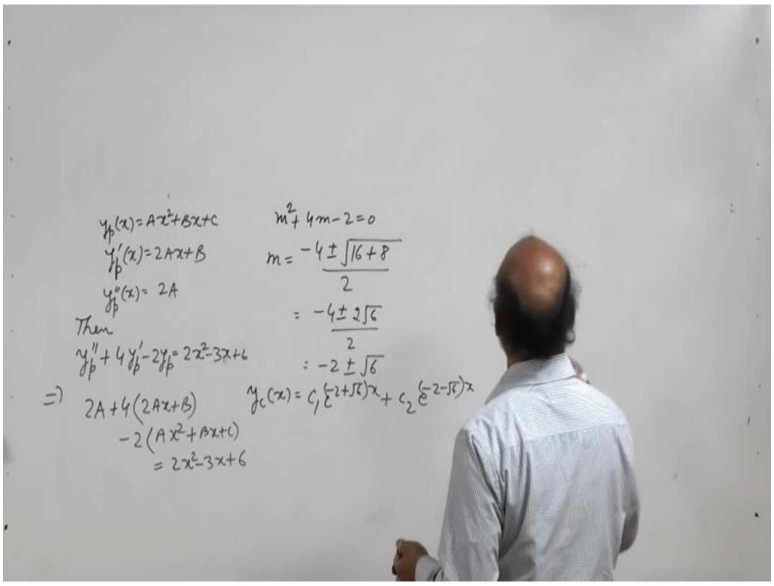

So, this will come out to be minus 4 plus minus. So, m has 2 distinct values minus 2 plus root 6 and minus 2 minus root 6 and therefore, y c x is equal to c 1 e to the power minus 2 plus root 6 into x plus c 2 e to the power minus 2 minus root 6 into x.

Now, since r x is equal to 2 x square minus 3 x plus 6, which is the polynomial of degree 2.

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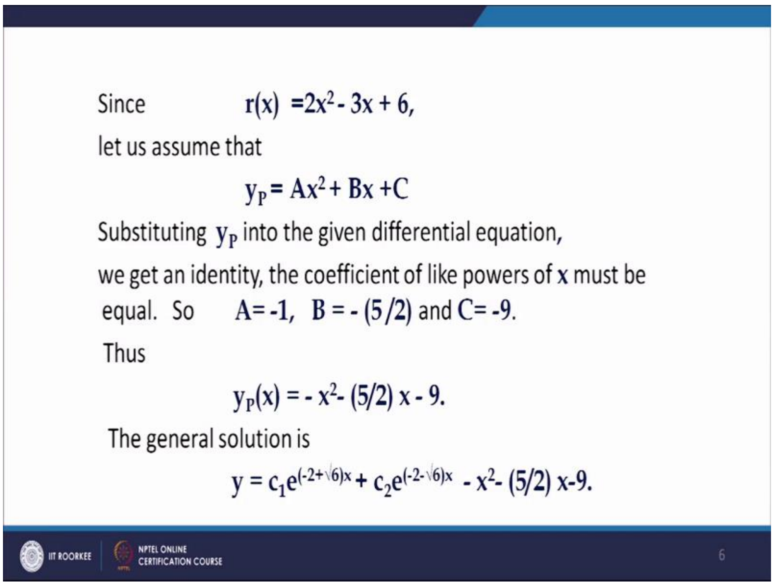

Let us assume y p x to be equal to A x square plus B x plus c; a polynomial in x of degree 2 where A B C are undetermined coefficients. We shall determine the values of these undetermined coefficients A B C by substituting this y p in to the in the given differential equation, because y p is a particular solution of the given differential equation. So, it should satisfy that equation. So, let us put y p equal to y p x equal to A x square p plus B x plus C. Let us put this value y p x in the given equation, y double dash. So, let us find y p dash x equal to 2 A x plus by p, double dash is equal to 2 A.

So, then y p double dash plus, 4 y p dash, minus 2 y p equal to, 2 x square minus 3 x plus 6 gives us 2 A. Now this equation is valid for all values of x. So, it must be an identity. And therefore, the coefficient of like powers of x must be same. So, if we equate the coefficients of like powers of x, we get the values of the unknown coefficients A B C. You can see that here x square coefficient x square is minus 2 A.

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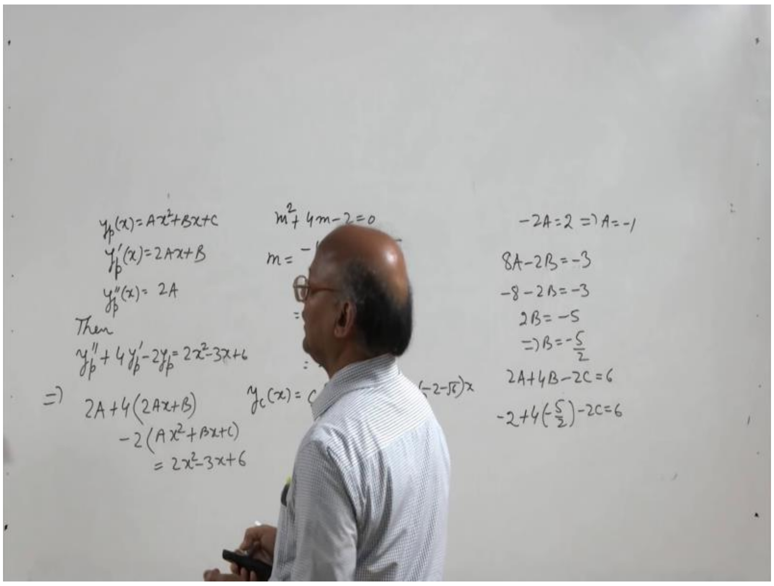

So, minus 2 A equal to 2, which gives us A equal to minus 1. When we equate the coefficient of x on both sides we get 8 A minus 2 B equal to minus 3. Putting the value of A equal to minus 1 we get minus 8 minus 2 B equal to minus 3. So, this will give you 2 B equal to minus 5, which give you B equal to minus 5 by 2. And when you equate the coefficients on both sides we get 2 A plus 4 B minus 2 C equal to 6. So, substituting the values of a and B minus 1 and minus 5 by 2, we shall have we shall have C equal to minus 9 and thus y p x will be equal to x square minus 5 by 2 x minus 9.

So, the general solution of this differential equation is the sum of the complimentary function C 1, e to the power minus 2 plus root 6 into x c 2 e to the power minus 2 minus root 6 into x, and then minus x square minus 5 by 2 x minus 9. Now let us take another example, y double dash minus y dash plus y equal to sin 3 x.

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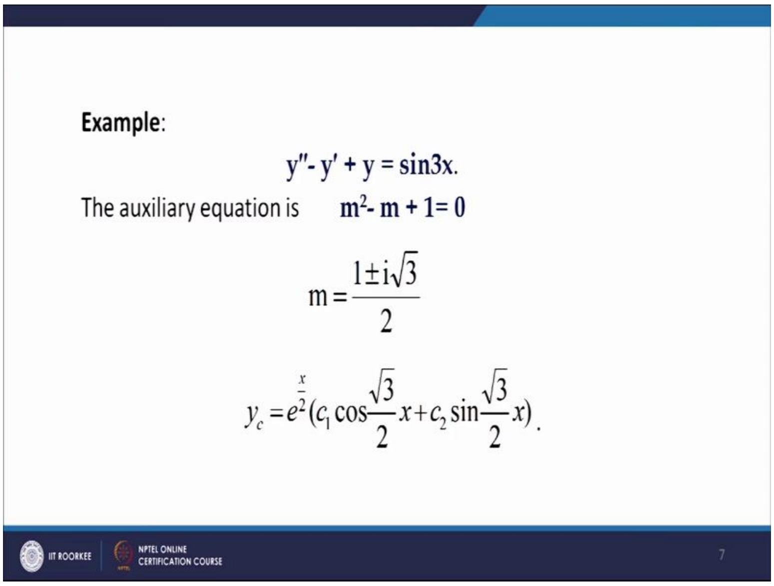

Here the auxiliary equation is m square minus m plus 1 equal to 0. You can find the values of m they are 1 plus minus i root 3 by 2. So, we can write the complimentary function, because these are complex roots. So, e to the power alpha x that is e to the power half x into c 1 cos theta x beta is root 3 by 2. So, c 1 cos root 3 by 2 x plus c 2 sin root 3 by 2 x.

Now, let us make an guess for y p x. So, the natural guess when you look at the form of r x r x is equal to sin 3 x. So, natural guess for the y p x would be that we write a times sin 3 x, but then when you differentiate sin 3 x it produces 3 times cos 3 x.

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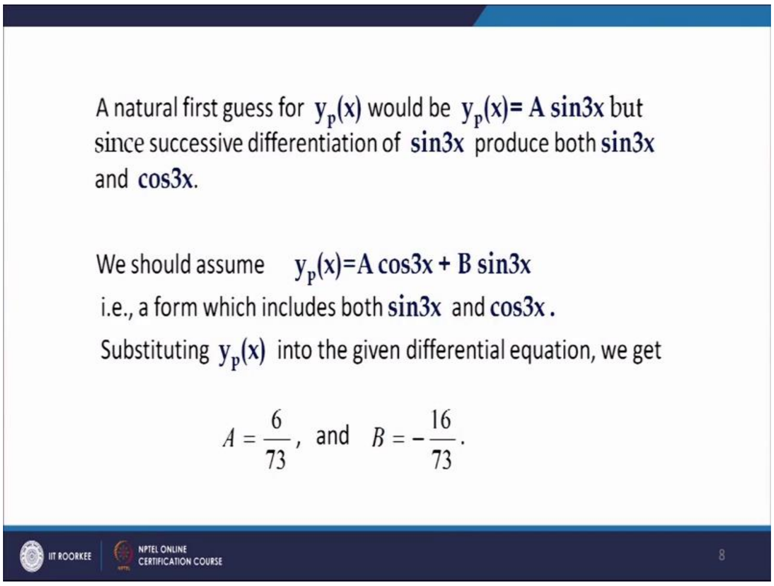

And then when you differentiated again you get sin 3 x. So, it differentiation of sin 3 x produces both sin 3 x and cos 3 x, and therefore, what we should do is we should assume y p x to be a linear combination of sin 3 x and cos 3 x functions. So, let us assume y p x to be having both the functions sin 3 x and cos 3 x that is A cos 3 x plus B sin 3 x.

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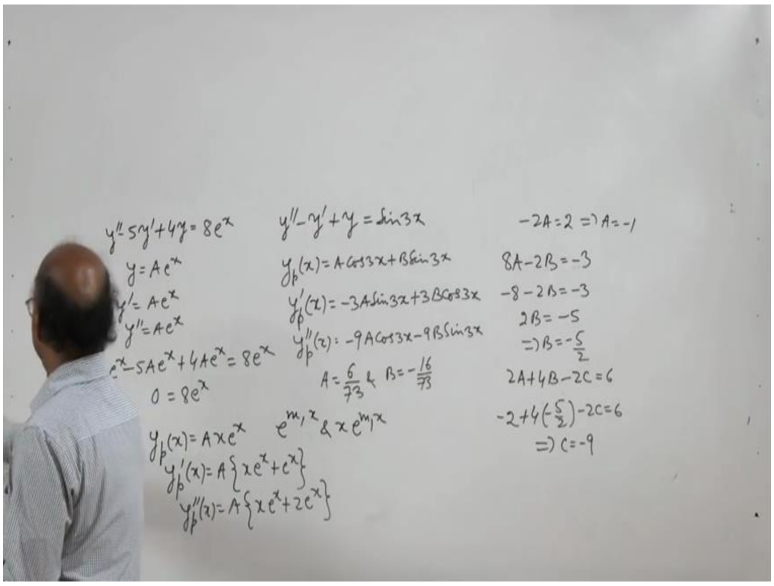

Now, then you substitute this y p x in the given differential equation, the given differential equation is y double dash minus y dash plus y equal to sin 3 x. So, if you take y p x equal to A cos 3 x plus B sin 3 x. And then put it in the given differential equation, what you will get is unequating the coefficients of sin 3 x and cos 3 x both sides, we shall get a to be 6 by 73 and B to be minus 16 by 73.

So, we can find y p dash x here this is minus 3 A sin 3 x plus 3 B cos 3 x. We differentiate it again should will give you minus 9 A cos 3 x plus 9 B minus 9 B sin 3 x. Now substitute the values of y p, y p dash, y p double dash in this differential equation, and equate the coefficient of sin 3 x and cos 3 x both sides, we shall get a equal to 6 by 73 and B equal to minus 16 by 73.

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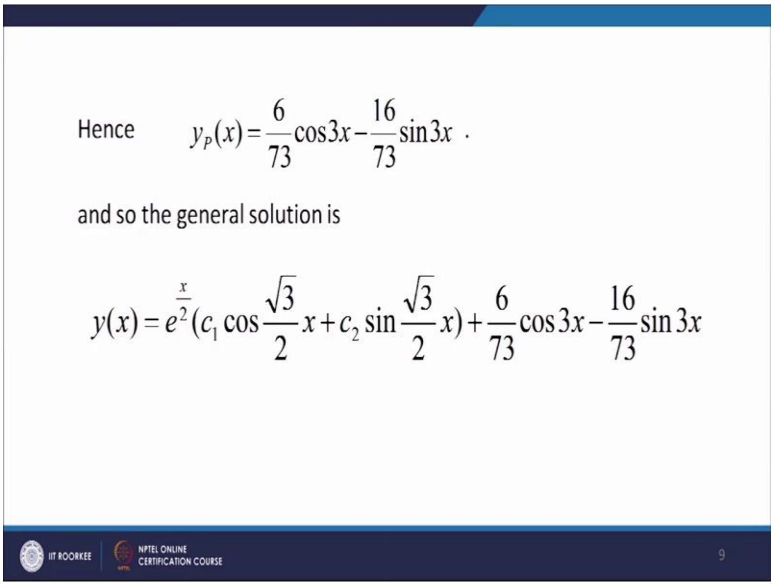

So, we will get y p x to be equal to minus 6 by 73 cos x, minus 16 by 73 sin 3 x. And thus the general solution y x is the sum of the complementary function. This is the complimentary function plus the particular integral. The particular solution of the equation 1 is also called as particular integral; so 6 by 73 cos 3 x minus 16 by 73s sin 3 x.

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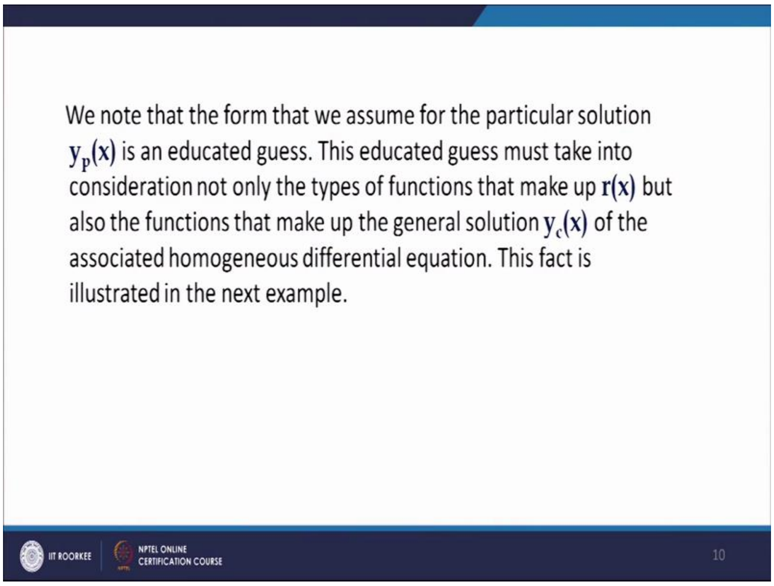

Now, now let us look at this comment. Here the as I said in the beginning y p x is not an arbitrary. It is not guessed arbitrary, it is an educated guess, because it depends on the form of r x, the functions that make up r x, as well as the functions that make up the complimentary function, which is the solution of the associated homogenous linear differential equation, that is y double dash plus a y dash plus by equal to 0 let us see how it depends on the complimentary function. We have seen that the function y p x depends on the form of r x, in the next example we shall see how y p x depends on the form of the complimentary function.

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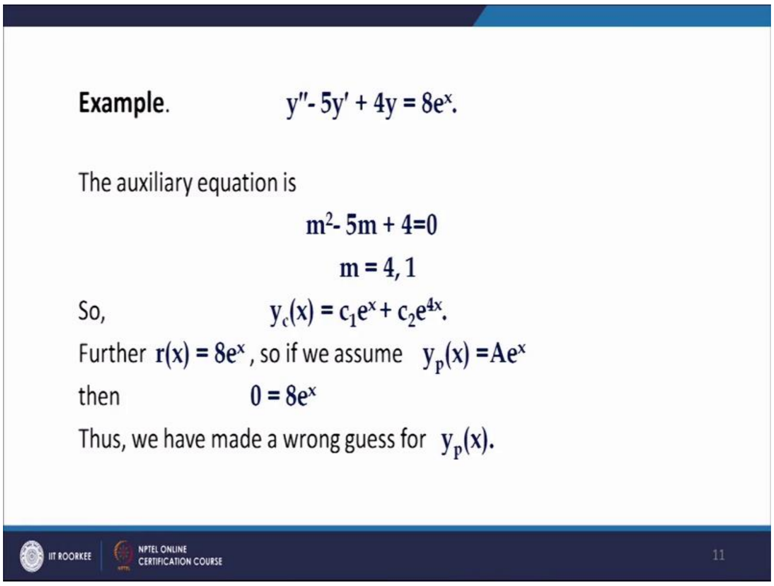

So, let us take up this this problem, y double dash minus 5 y dash plus 4 y equal to a t to the power x. Here we can write the auxiliary equation as m square minus 4 5 m plus 4 equal to 0. So, we get 2 values of m which are 4 and one they are distinct values. So, the complementary function we can write easily, y c x equal to c 1 e to the power x plus c 2 e to the power 4 x.

Now, when we look at the form of r x, r x is 8 e to the power x, and successive differentiation of e to the power x produce e to the power x. So, we can assume y p x to be equal to A times e to the power x, but when you put y p x equal to 8 e to the power x, in this equation in the given differential equation, y double dash minus 5 y dash plus 4 y equal to 8 e to the power x. Let us see what happens, y we are taking as A e to the power x. So, let us find y dash it is A e to the power x. Let us find y double dash it is again A to the power x. So, A e to the power x minus 5 A e to the power x plus 4 A e to the power x equal to 8 e to the power x now this is 5 A e to the power x minus 5 A to the power x. So, this is 0 equal to 8 e to the power x, which is not correct of course, because e to the power x is never 0. So, thus we have made a wrong guess for y p x. So, what should be the form of y p x.

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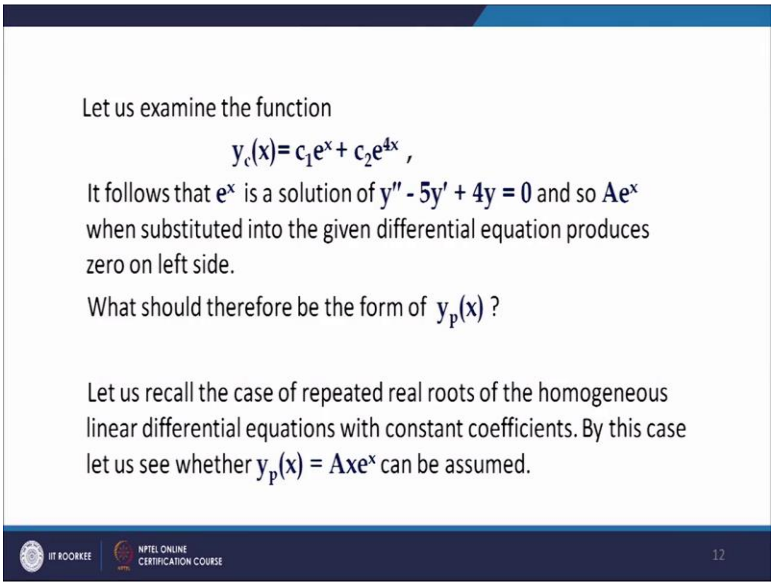

Now, let us look at the complimentary function. The complimentary function is c 1 e to the power x plus c 2 e to the power 4 x. And therefore, e to the power x and e to the power 4 x are both solutions of the associated homogenous linear differential equation of second order, that is y double dash minus 5 y dash plus 4 y equal to 0. And So, A e to the power x one substituted in the given differential equation produces 0 on the left hand side. Now what should therefore, be a form of y c x this is the question.

So, let us recall the case of repeated v l rules of the homogenous linear differential equation, with constant coefficients. There we see that when the 2 roots are same, if value of m is equal to m 1 which is repeated then the 2 solutions independent solutions of the homogenous equations are e to the power m 1 x, and x times e to the power m 1 x. So, let us see whether if we assume y p x equal to a times x e to the power x, do we get the solution of the given differential equation.

So, let us see. If we do this then y p dash x will be equal to A times this, y p double dash will be equal to A e x times x e to the power x plus 2 e to the power x.

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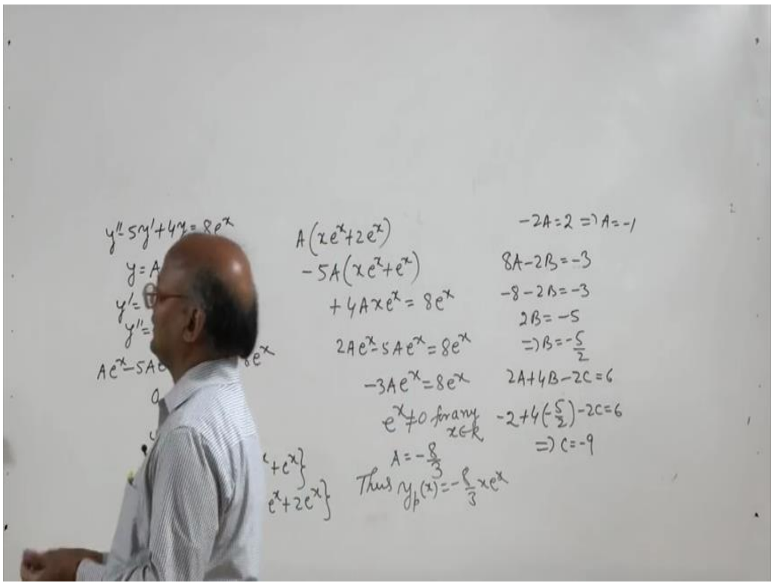

Now, when you put this in the given differential equation let us see what do we get. So, A times x e to the power x, plus 2 e to the power x, that is y double y p double dash minus 5, y p dash, y p dash means A into x e to the power x plus e to the power x, and then we have 4 times y which is A x e to the power x, equal to 8 e to the power x. Now let us see what do we get.

So, A times x e to the power x 4 A times x e to the power x minus 5 A times x e to the power x they cancel, and what we get is 2 A e to the power x 2 A e to the power x, minus

5 A e to the power x equal to 8 e to the power x. So, we get minus 3 A e to the power x is never 0, for any x. So, we can divide by e to the power x, and we get the value of A x minus 8 by 3, and thus y p x we get as minus 8 by 3 x, e to the power x. So, the general solution therefore, is given by c 1 y equal to c 1 e to the power e x plus c 2 e to the power 4 x minus 8 by 3 x e to the power x.

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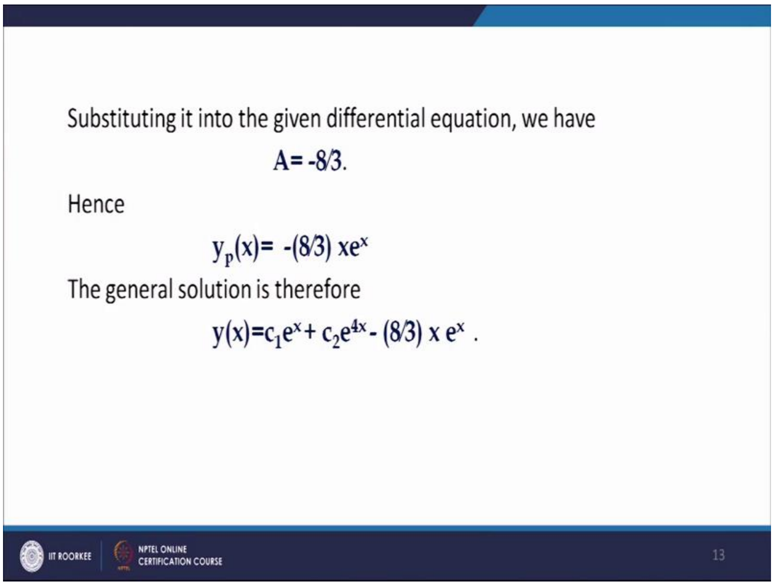

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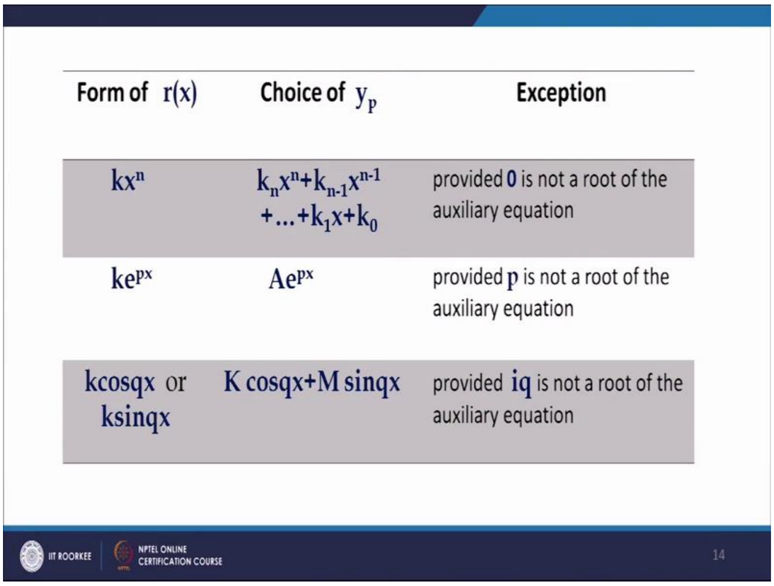

Form of $r(x)$	Choice of y_p	Exception
kx^n	$k_nx^n+k_{n-1}x^{n-1}+\ldots+k_1x+k_0$	provided **0** is not a root of the auxiliary equation
ke^{px}	Ae^{px}	provided p is not a root of the auxiliary equation
$k\cos qx$ or $k\sin qx$	$K\cos qx+M\sin qx$	provided iq is not a root of the auxiliary equation

In this slide we have listed the choice of y p x depending on the choice of r x. So, if in the first column we have given the form of r x in the second column we have given the choice of y p x, provided we have these exceptions. So, suppose r x is of the form k x to the power n, then where k is a constant. Then we shall make the choice of y p x to be a polynomial in x of degree n that is k n x to the power n plus kn minus 1, x to the n minus 1 plus 1 k 1 x plus k naught. This will be the choice of y p x, provided 0 is not a root of the auxiliary equation. If 0 is the root of the auxiliary equation what we have to do, we shall see next.

Now, in the next slide, when r x is k e to the power p x, then we shall choose y p x to be A e to the power p x provided p is not a root of the auxiliary equation, like in the previous example we have seen that, r x here the m equal to 1 m equal m equal to was a root of the auxiliary equation. So, the choice of a times e to the power x did not was not a correct choice. So, so this is an exceptional case. So, when p is not root of the auxiliary equation, we will make a choice of our pi p x as A e to the power p x. If r x is k cos k q x, or k sin q x, then the choice of y p x will be taken as k cos q x plus m sin q x, provided i q is not a root of the auxiliary equation.

Now these are the exceptional cases. When 0 is a root of the auxiliary equation r p is a root of the auxiliary equation or i q is the root of auxiliary equation, what choice of y p x we will have to make it is given in the next slide.

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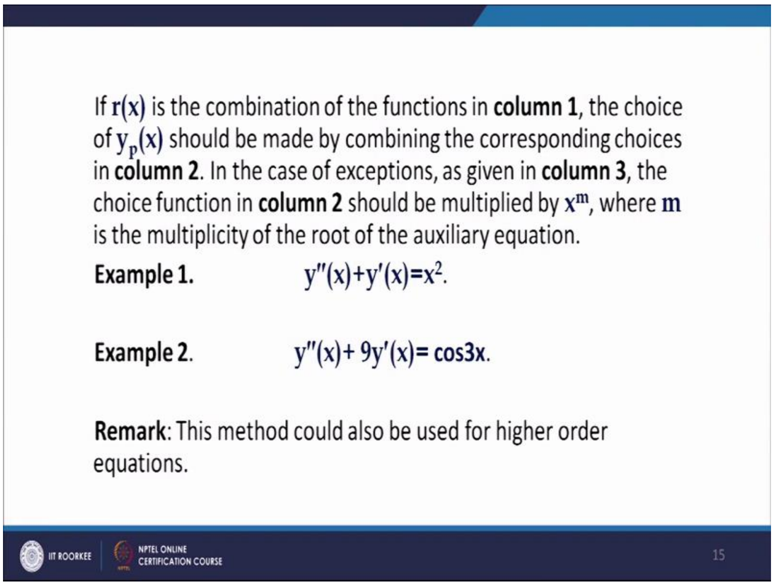

So, here let us see if r x is a combination of the functions in column 1, if r x is the combination of functions given in the column 1, then the choice of y p x should be made by combining the corresponding choices in column 2. In the case of exceptions as given in column 3, the choice function in column 2 should be multiplied by x to the power m. Let us see now we have seen that m equal to 1, p equal to 1 was a root of the auxiliary equation here, p equal to 1 was a root of the auxiliary equation here.

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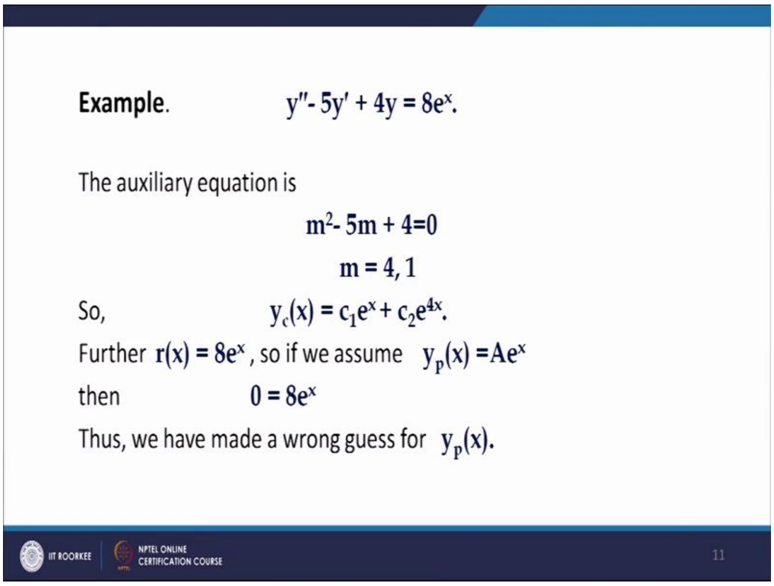

So, we had to consider and it occurred once it is multiplicity is 1. So, we had to multiply e to the power x by x. We had to assume y p x equal to A times x into e to the power x. So, in the case of exceptions as given in column 3 the choice function column 2 should be multiplied by x to the power m, where m is the multiplicity of the root of the auxiliary equation.

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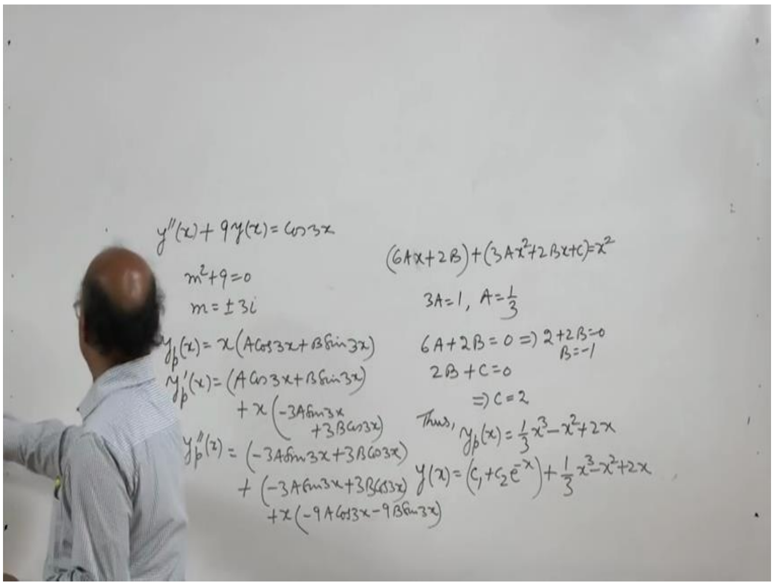

Now, let us take the example y double dash plus y dash x equal to r x square. Let us see what do we how do we, choose r x, y p x here; so y double dash, plus y dash x equal to x square. So, auxiliary equation here is m square plus m equal to 0. So, m equal to 0 and minus 1. And therefore, complimentary function y c x is equal to c 1 e to the power 0 x plus c 2 e 2 the power minus x. Or we can say c 1 plus c 2 e to the power minus x. Now let us look at the table 0 is a root of the auxiliary equation.

So, this is an exceptional case. So, in this case and 0 occurs once and therefore, we have to multiply y x. So, the y p x, because r x is equal to x square here, r x is given to be x square here. So, natural choice of y p x should have been A x square plus B x plus C, but in this exceptional case we will multiply A x square plus B x plus C y x. Because 0 is the root of the auxiliary equation and it occurs once. So, we have to multiply the choice A x square plus B x plus C by x with this choice of y p x when you when you substitute this y p x into the given differential equation, what you will have. So, these xs cube.

So, 3 A x square plus 2 B x plus C and then y p double dash x you will get as 6 a s plus 2 b. So, let us put these values of y p x and the derivatives in the given differential equation. So, 6 A x plus 2 B which is y 3 double dash plus y dash so; that means, y p dash. So, 3 A x square plus 2 B x plus C equal to x square. So, what do we notice is that. So, we notice that the coefficient of x be it is an identity it is true for all values of x. So, we should equate the corresponding powers of like powers of x. So, 3 A equal to 1. So, we get a equal to 1 by 3. Then we have to look at the coefficient of x here. So, 6 a plus 2 B equal to 0. Because there is no coefficient of x, there the it is 0 and then we have 2 B plus C equal to 0.

So, substituting a equal to 1 by 3 here, this will give you 6 a equal to 2. 2 plus 2 B equal to 0 which will give you B equal to minus 1. And B equal to minus 1, then gives you C equal to 2. So, thus y p x is equal to A x cube, A is equal to 1 by 3. So, 1 by 3 x cube and then B x square B x square means minus x square and then we have C x. So, 2 x. So, the general solution is y x equal to y c x, y c x is c 1 plus c 2, e to the power minus x this is y c x plus y p x. So, this is the general solution of the given differential equation.

We can have one more example y double dash x plus 9 y dash x equal to cos 3 x which will take care of the other exceptional case. Where i q this exceptional case i q is not a root of the auxiliary equation. So, let us discuss now another example y double dash x plus 9 y x equal to cos 3 x. In this case we have the auxiliary equation as m square plus 9 equal to 0. So, m equal to plus minus 3 i.

Now this is an exceptional case, because in the r x here is cos 3 x and we see that r x is equal to cos 3 x, and i 3 i is a root of the auxiliary equation. So, what we see that 3 i occurs once here therefore, y p x will be assumed as x times A cos 3 x plus B sin 3 x. So, if you assume y p x to be equal to this, then y p dash x will be equal to A cos 3 x plus B sin 3 x plus, x times minus 3 A sin 3 x, plus 3 B cos 3 x. And then y p double dash x will be equal to minus 3 A sin 3 x, plus 3 B cos 3 x, plus minus 3 A sin 3 x, plus 3 B cos 3 x, and then x times minus 9 A cos 3 x minus 9 B sin 3 x.

So, when you put these values in the given differential equation, let us see what we get. So, y p double dash.

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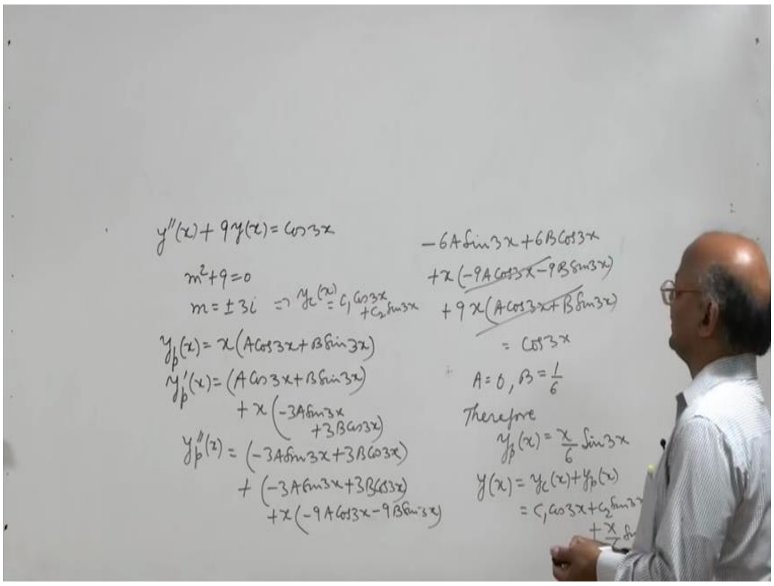

Now, this is minus 3 sin 3 x minus 3 A sin 3 x. So, minus 6 A sin 3 x and we get 3 B cos 3 B cos 3 x 6 B cos 3 x. And then x times minus 9 A cos 3 x minus 9 B sin 3 x. So, this is y p double dash plus 9 times y A y p x. So, y p x is x times A cos 3 x, plus B sin 3 x, equal to cos 3 x. Now 9 x into A cos 3 x will cancel with minus 9 A into x cos 3 x. 9 x B sin 3 x will cancel with minus 9 B sin 3 x.

So, this and this cancel now equating the coefficients of sin 3 x and cos 3 x both sides, we get a equal to 0 and B equal to 1 by 6. And therefore, y p x equal to x times a 0 B is 1 by 6. So, x by 6 sin 3 x. And thus, general solution y x is equal to y c x plus y p x, which is equal to now y c x here is some constant c 1 cos 3 x plus c 2 sin 3 x. So, this will be c 1 cos 3 x plus c 2 sin 3 x, plus x y 6, sin 3 x. So, this is how we shall solve this equation.

Now in my next lecture we shall discuss the general method for obtaining the general solution of second order no non homogenous linear differential equation.

Thank you.