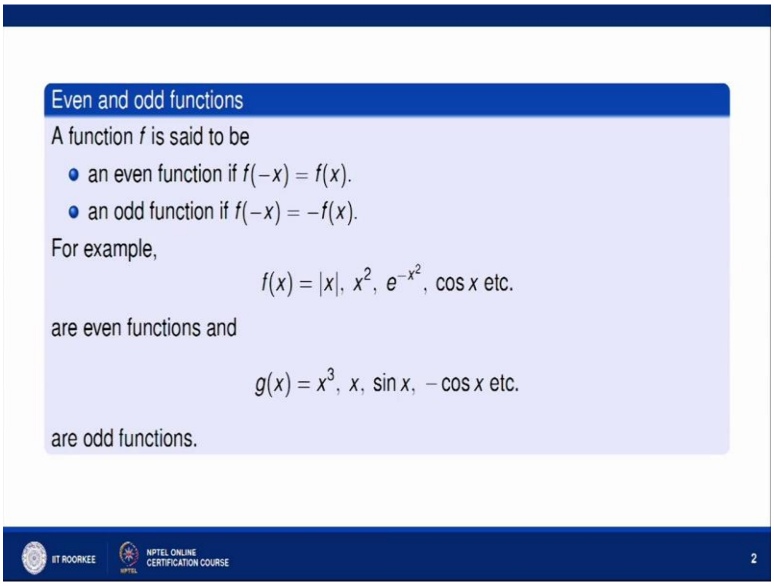
Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 49
Fourier Series of Even and Odd functions

Welcome to the lecture series on the Mathematical Methods and it is Applications. So, we have discussed in the last 2 lectures basically, we have discussed Fourier series and it is convergence. That is how we can expand a function f x periodic function f x. In fact, in the series form and the series of trigonometric functions of sine and cosine, that we have seen, and when a series when a x or function is continuous and having first and second derivatives are continuous, then function f x is convergent to that series of sine and cosine. Not even continuity, if we have piece wise continuous function, piece wise periodic continuous function. In fact, then also that f x will converge to the infinite series of sine and cosine, if left and right hand derivatives exist at each point that we have already seen.

Now, we will see Fourier series of even and odd functions. We already know what even function is, even function means f of minus x equals to f x that function is called an even function. And if f of minus x is minus of f x that function is called as odd function. We already know that mod x x square e k to the power minus x square cos x etcetera are all even function because, if we replace x by minus x in these functions then the functions remain as it is that is f of minus x is same as f x.

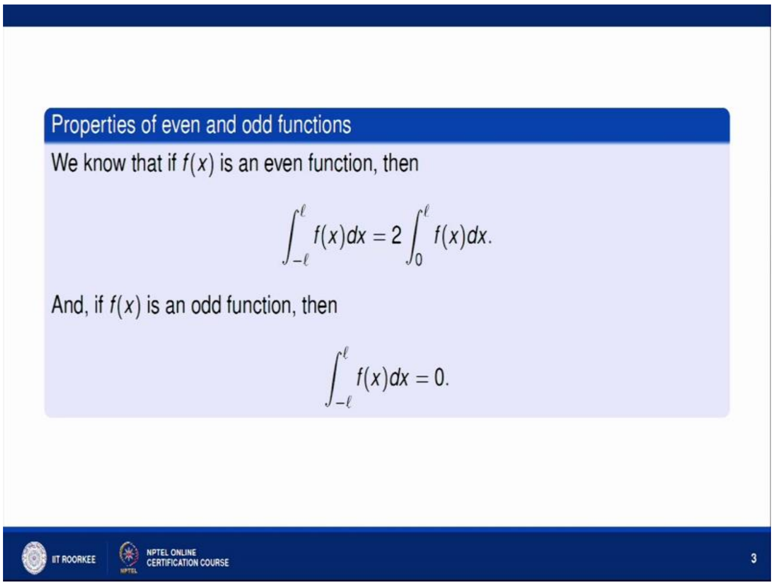
(Refer Slide Time: 01:56)



Therefore, these functions as even functions; however, if you see x cube x sin x or minus cos x then, these functions odd functions because if we replace x by minus x then the function will be equal to minus of f x; that means, f of minus x is minus of f x. Therefore, these functions are odd functions.

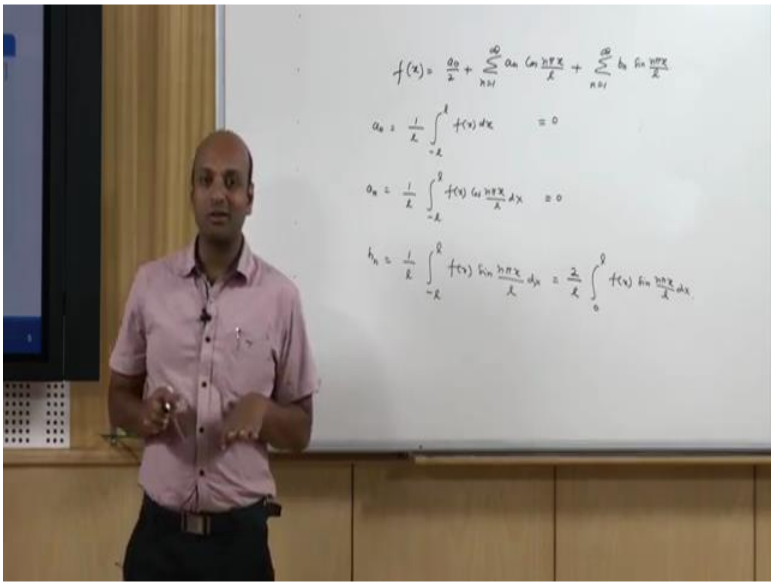
So, we have no functions which are neither even nor odd also like x minus x square which is neither even or odd.

(Refer Slide Time: 02:23)



Now, we already know this result that if f is an even function then integration minus l to plus l f x d x is nothing but 2 times 0 to l f x d x that we already know and if f is an odd function then minus l to l f x d x is simply 0. This is by the properties of definite integral we already know this. Now come to Fourier series representation.

(Refer Slide Time: 02:46)



Now, we already know that Fourier series representation of any f x is given by a naught upon 2 plus summation n from 1 to infinity a n cos n pi x by l plus summation n from 1 to infinity b n sin n pi x by l.

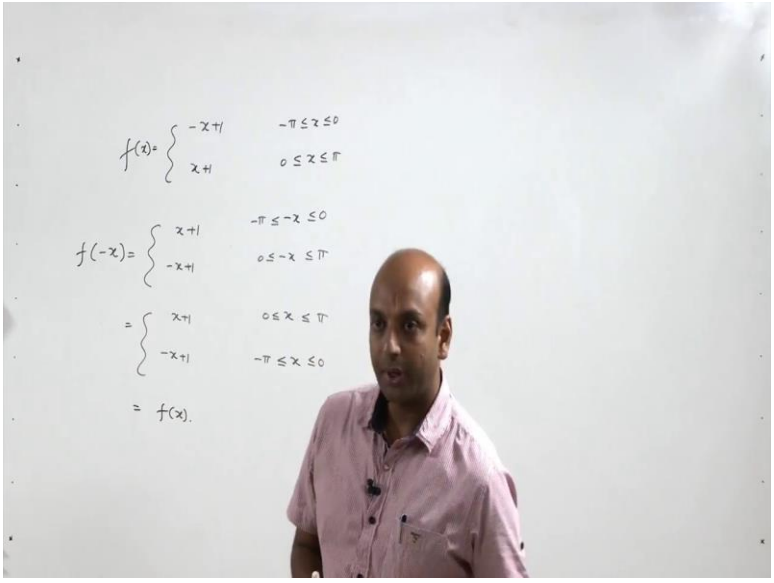
So, this is when interval is minus l to plus l, and the function is periodic with period 2 l. Now here a naught is given by that is 1 by l integration minus l to plus l f x d x, a n is given by 1 by l integral minus l to plus l f x cos n pi x by l into d x, and b n is given by 1 by l integral minus l to plus l f x sin n pi x by l into d x. Now suppose function which is given to us whose Fourier expansion is to find out is an even function. So, if it is even function, then minus l to plus l f x d x will be nothing but 2 times by the property of definite integral. And the cos is an even function it is cosine. Now cosine is an even function and f x is even function. So, even into even is an even function. So, minus l to plus l it will again give 2 upon l integral 0 to l f x into cos n pi x by l into d x.

Now, if we see this function f x is an even function. And sine is an odd function. Even into odd is odd. So, this value will be 0. Hence if we have an even function, so we have only cosine terms in the Fourier series expansion because, b n is 0, b n is 0 means no sine terms. Only cosine terms will be there. So, if function is an even function, we have the Fourier series representation of that function will contain only cosine terms.

Now suppose we want to expand an odd periodic function. Suppose it is odd now suppose function is odd, minus l to plus l this will be 0. And f x is odd cos is even odd into even is again odd it will be 0. And this is odd sine is also odd, odd into odd is even this is an even function. So, when it is even, this will be nothing but 2 upon l times 0 to l f x sin n pi x by l; that means, that if we have an odd function then there will be no cosine term in that series expansion Fourier series expansion.

We will be having only sine terms, because there is only b n, a n a naught and n all are 0. So, that is how one can find even and odd extension odd I mean even and odd Fourier series expansion for even and odd functions. Now let us solve 2 examples on this. Suppose this function is given to us.

(Refer Slide Time: 06:35)

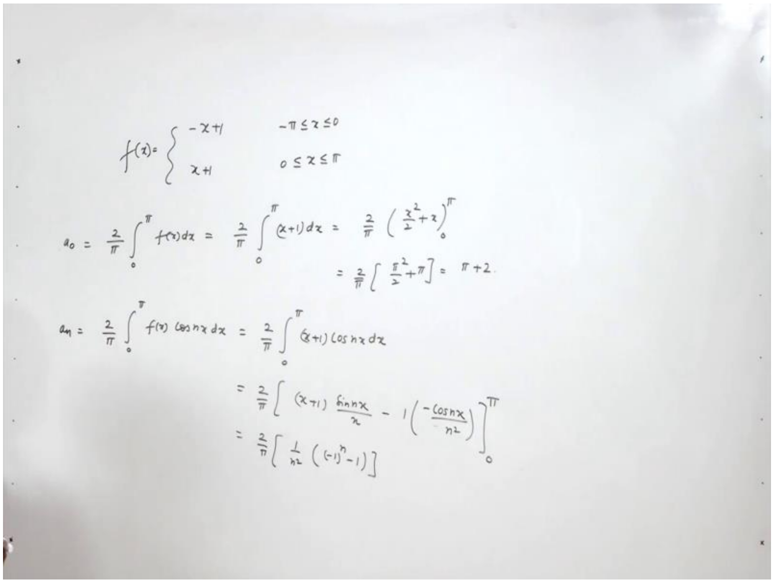


Now, what is this function? This function is f x equal to minus x plus 1, when minus pi less than equals to x less than equal to 0. And x plus 1 when 0 less than equals to x less than equals to pi. Of course, function is periodic with period 2 pi. So, we have to first check whether function is even and odd. So, to check whether function is even and odd we will find f of minus x, if it is equal to f x. So, it will be an even function and if it is equal to minus of f x then we say it is an odd function.

So, you find f of minus x. So, you replace x by minus x. So, it is x plus 1, when minus x less than equal to 0 greater than equal to minus pi. And minus x plus 1 when minus x less than equal to pi greater than equal to 0. So, this will be nothing but is equals to x plus 1 when x is less than greater than equal to 0 less than equals to pi. And it is minus x plus 1 when x is greater than equals to minus pi and less than equal to 0. So, from minus pi to 0, it is minus x plus 1 which is same as this function. And from 0 to pi it is x plus 1 which is same x plus 1 over here; that means, it is nothing but f of x. So, hence it is an even function. So, one can also see graphically, because if it is an even function then it is symmetrical about y axis. So, you can just plot the function and you can see the symmetry. If it is symmetrical about y axis, then you can simply say it is an even function.

Now, you have to find the Fourier series expansion of this function. Now Fourier series representation can be find out, because it is an even function which we have already seen. So, it will contain only cosine terms no sine terms that is b n will be 0 that we have already seen.

(Refer Slide Time: 08:44)

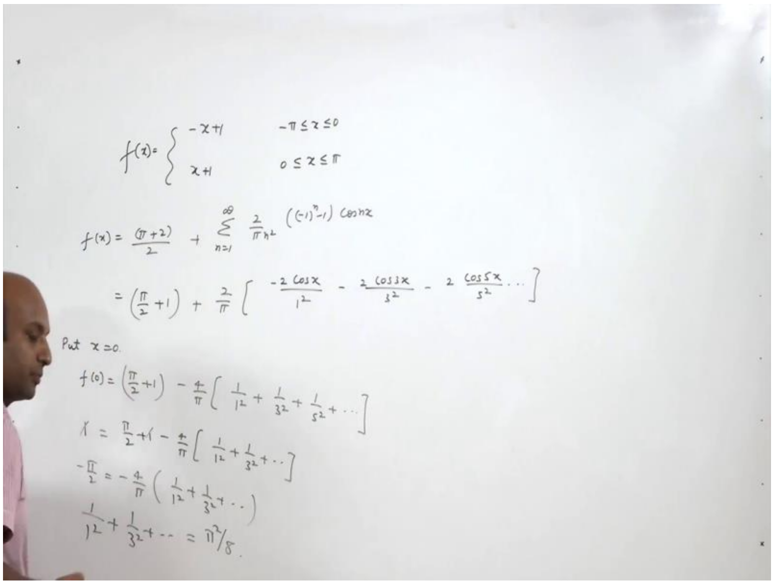


So, what will be a naught, this we have already seen it is 2 upon pi 0 to pi f x d x. So, that will be nothing but 2 upon pi, 2 upon l that is 0 to pi, l is pi here. So, it is 2 upon pi x plus 1 d x. And this is nothing but 2 upon pi it is x square by 2 plus x from 0 to pi. And which is nothing but 0 to pi, it is pi square by 2, plus pi which is equals to pi plus 2. Because 2 2 cancels out it is pi and it is 2 yeah it is pi plus 2.

Now, if you compute a n, a n is nothing but 2 upon pi integral 0 to pi, f x cos n x d x. It is again equal to 2 upon pi integral 0 to pi, what is f x, f x is x plus 1, into cos n x d x. And when you integrate it is 2 upon pi, it is first as it is integral of second is sin n x, upon n. We will apply integration by parts and derivative of this is 1, integration of this is minus cos n x upon n square, from 0 to pi, and this is nothing but 2 upon pi. Now it is 0 when x is pi or 0.

Now, here it is minus minus plus 1 by n square. It is minus 1 k power n minus 1. So, this will be a n, and this will be a naught b n is of course 0, because it is an even function. So, what will be the Fourier sine, Fourier representation of this function? So, a naught is pi plus 2.

(Refer Slide Time: 10:42)

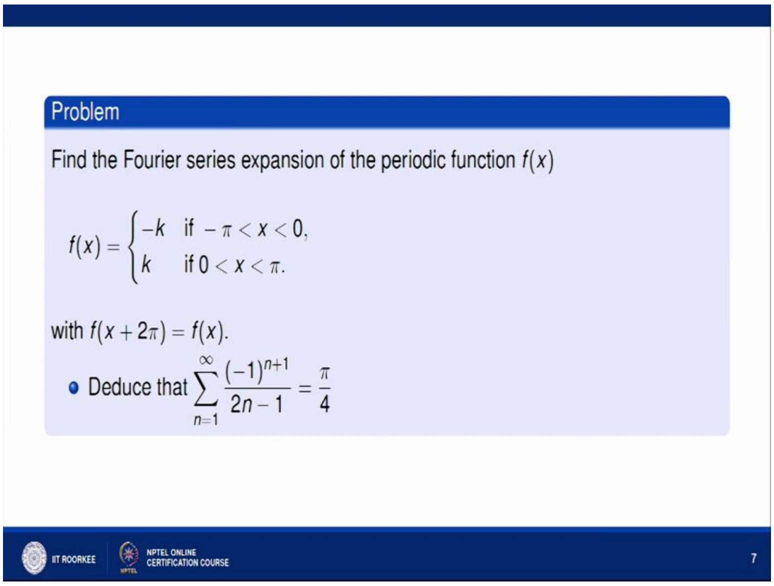


So, f x will be nothing but pi plus 2 a naught by 2, plus summation n varying from 1 to infinity a n, a n is 2 upon pi n square minus 1 k power n minus 1, into cos n x. This term come here now. So, this will be Fourier representation of this function. Now we have determined the value of this series, 1 upon, 1 square plus 1 upon 3 square plus 1 upon 5 square and so on. So, what the series is basically, it is pi by 2 plus 1 plus 2 upon pi.

Now, when you open this summation, when n is 1, it is minus 2. It is minus 2, cos x upon 1 square. Now when n is 2 minus 1 k to the power 2 is 1, 1 minus 1 is 0. Now when n is 3, it is again minus 2, minus 2 cos 3 x upon 3 square. Again when n is 4 it is 0, minus 2 cos 5 x upon 5 square and so on. So, it only contains odd terms, I mean 1 3 5 like this. So, and we have we have this series. Now we want value of this series. So, we want all these terms to be 1. So, substitute x equal to 0, so put x equal to 0. So, it is f 0 will be equal to pi plus 2 plus 1, minus 4 upon pi times 1 upon 1 square, plus 1 upon 3 square plus 1 upon 5 square and so on.

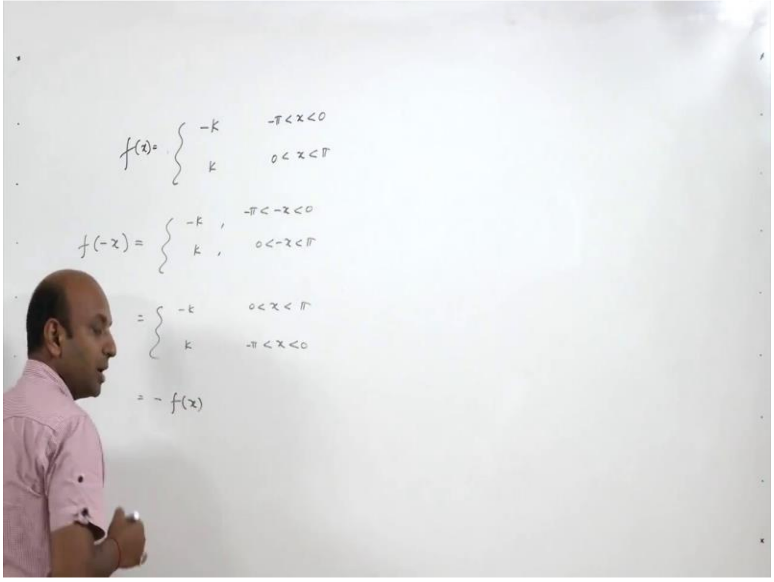
Now, what is f 0, is the function continuous at 0. Yes, it is continuous at 0 because from both the ends value is 1, we can easily check. So, the value of this will be one only. If it is not continuous then we will find in the same way like half of f 0, plus plus f 0 minus if it is not continuous at 0. Here it is continuous at 0. So, it will be one only. It is pi by 2 plus 1 minus 4 upon pi times this expression. So, what this value is 1, 1 cancels out. This is minus pi by 2 is equals to minus 4 by pi times this expression. So, hence this value is nothing but pi square by 8. This value is from here we can say that this value is nothing but pi square by 8. So, hence we got this value. Because it is an even function calculation become easy, otherwise we have to split the integral from minus pi to 0 and 0 to pi.

(Refer Slide Time: 14:09)



Now, see this problem. Now f x is minus k when between x lying between minus pi to 0, and k when it is lying between 0 to pi.

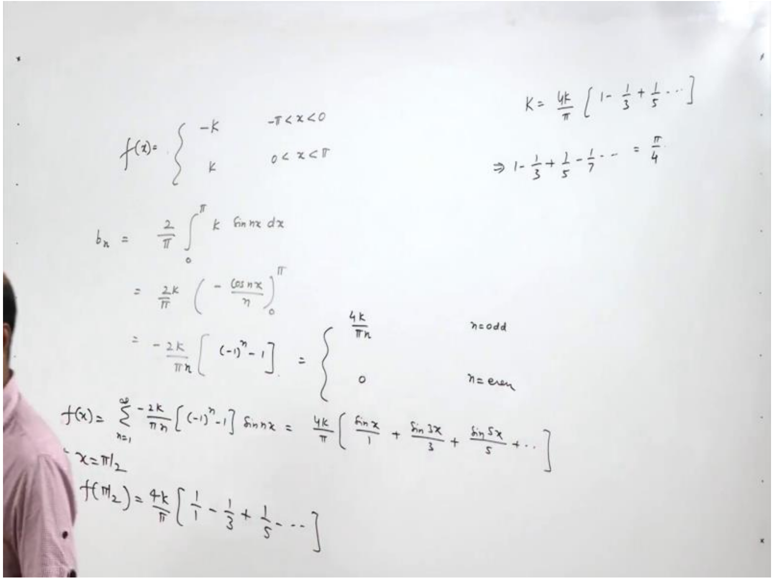
(Refer Slide Time: 14:22)



First let us check whether it is even function odd function or neither. So, we have to verify this first. So, let us see it is minus k when x lying between minus pi to 0. And it is k when x lying between 0 to pi. And it is a periodic function with period 2 pi. So, first we will see whether it is even function odd function. So, again we will find f of minus x. If it is equal to f x, it is an even function. If it is equal to f of minus of f x it means it is an odd function. So, you replace x by minus x. So, it will be minus k, when minus x lying between 0 and minus pi and it is k when minus x lying pi and 0. So, it is nothing but minus k when x is greater than 0 less than pi. And it is k when x is less than minus pi greater than minus pi and less than 0.

So, if we see this and this function, from minus pi to 0 it is k, from minus pi to 0 it is minus k from 0 to pi it is minus k, and from 0 to pi it is k that is this is nothing but minus of f x. So, hence this function is an odd function. Now in odd function. So, it will contain only sine terms, that is a naught and a n are 0. We already know this. So, this will only contain Fourier series expansion of this function, it will only contain sine terms.

(Refer Slide Time: 15:56)



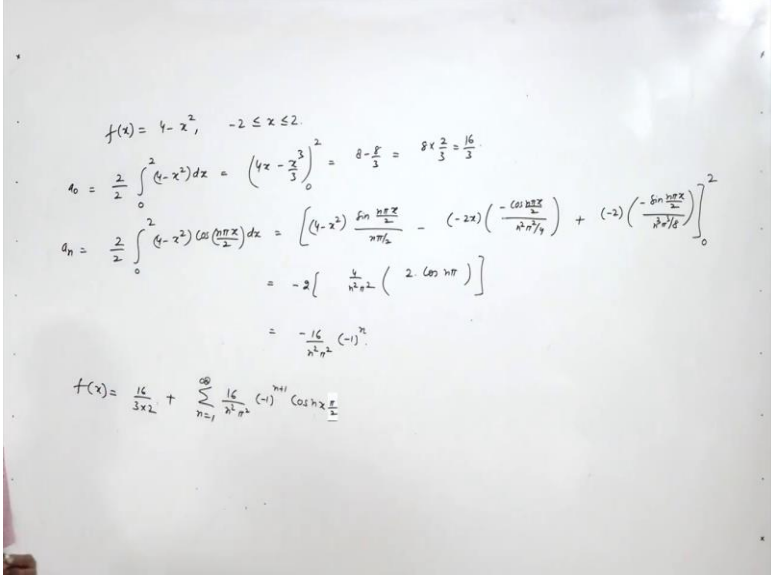
So, how you find b n; only b n will exist. So, b n will be nothing but 2 upon pi integral 0 to pi, k sin x, sin n x d x, f x is k. And it is nothing but 2 upon pi k is outside, integral of sine is minus cos n x upon n, from 0 to pi, minus will come outside 2 k upon pi and it is nothing but when you integrate this what you will get find the apply the limit is minus 1 power n minus 1. So, which is equals to when n is odd, it is minus 1 minus, 1 minus, 2 minus, 2 into minus 2 is 4, that is 4 k upon pi. So, n is also there, n is it will come here.

Because n is also there, now when n is odd when n is even it is minus plus 1 minus 1 0. So, when n is even it is 0. So, in this way we can define b n. Now what will be the Fourier series of this function then, Fourier series of this function will be nothing but summation b n, with b n is b n is this term, it is minus 2 k upon pi n minus minus 1 k power n minus 1 sin n x. And when we open this, when n is odd it is and when n is even it is 0. So, you can substitute values of n. When n is when n is 1. So, it is 4 k upon 4 k upon pi will come outside, it is sin 1 sin x upon 1 plus sin 3 x upon 3 plus sin 5 x upon 5 and so on. Because when n is even, it is 0. We have the existence of b n only when n is odd.

So, we have this series expansion. Now we want to compute this expression summation n from 1 to infinity minus, 1 k power n plus 1, 2 n minus 1 equal to pi by 4. This we have to show. So, because it is minus 1 k power n plus 1; that means, we have alternate negative positive sign in the series of this. So, to have alternate sign of plus minus sign over here, so put x equals to pi by 2. So, it will be f pi by 2, will be equals to 4 k upon pi, sin pi by 2 is 1, 1 by 1, sin 3 pi by 2 is minus 1. So, minus 1 by 3 sin 5, pi by 2 is plus 1 and so on. And what is sin pi by 2, f pi by 2 f pi by 2 comes from here it is k. So, this expression will be nothing but k is equals to 4 k upon pi, 1 minus 1 by 3, plus 1 by 5 and so on. So, hence this implies 1 minus 1 by 3, plus 1 by 5, minus 1 by 7 and so on will be nothing but pi by 4. So, this you have to derive yeah. So, it is pi by 4. It comes pi by 4. So, hence we have proved the result.

So, this is simple problem Fourier series expansion. 4 minus x square is an even function one can easily see and it has a period 4. So, we have only cosine terms in the series. So, how can we find the Fourier series expansion of this function? Using the same technique same expressions. So, f x is 4 minus x square, x lying between minus 2 to plus 2. And it is a periodic function with period 4.

(Refer Slide Time: 20:22)



So, again we know that it will contain only cosine terms because it is an even function. So, what will be a naught, a naught will be 2 upon 2 because period is 2 0 to 2 f x d x. So, this will be nothing but 4 x minus x cube by 3 from 0 to 2. So, this value we can compute, it is 8 minus 8 by 3 that is nothing but 8 into 2 by 3 that is 16 by 3.

Now, a n, a n will be nothing but 2 by 2, integral 0 to 2, 4 minus x square, cos n pi x by 2 into d x, because l is 2. So, we can easily integrate this. There is no problem in this it is 4 minus x square. Integration of this will be sin n pi x by 2, upon n pi by 2, minus derivative of this and integration of this, will be minus cos n pi x by 2, upon n square pi square by 4 plus derivative of this and integration of this will be nothing but minus sin n pi x by 2 upon n cube pi cube by 8, and whole multiplied by limit from 0 to 2.

Now, when you take limit is 2 here, it is 0 and when x is 0 it is 0. So, it is 0, now here also when x is 2, it is n pi, it is 0 when it is 0. It will be 0. So, only this term will there. So, it is minus, minus, minus, it is minus 2 comes outside when x is 2. So, it 4 into it is 4 upon n square pi square. It will also come outside 4 upon this. So, it is this will come outside 2 will also outside. So, it is 2 into cos n pi, and at 0 it is 0. So, it is nothing but minus 16, upon n square pi square into minus 1 k power n. It is minus 16 upon this. So Fourier series expansion of this function will be nothing but 16 upon 3, into 2 a naught by 2, plus summation n from 1 to infinity, 16 upon n square pi square minus 1 k to the power n plus 1 cos n x. So, this should be the Fourier series this expression.

Thank you.