Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 48
Fourier series and its Convergence – II

Welcome to the lecture series on Mathematical Methods and their Applications. So, in the last lecture, we have seen that what Fourier series are, and how we can expand the function f x in the form of cos and sine terms. Now, we will see that when the Fourier series is convergent. So, let us see first convergence of Fourier series for continuous functions.

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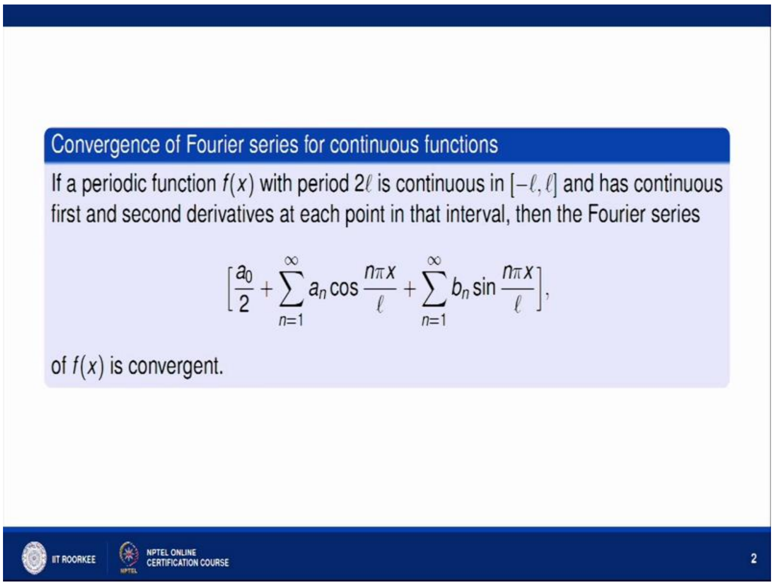

If a periodic function f x with period 2 l is continuous in interval minus l to plus l and has continuous first and second derivatives at each point in that interval then the Fourier series which we have already discussed a naught by 2 plus summation a n cos n pi x by l plus summation b n sin n pi x by l of f x is convergent. So, what is the proof? So, proof we can see.

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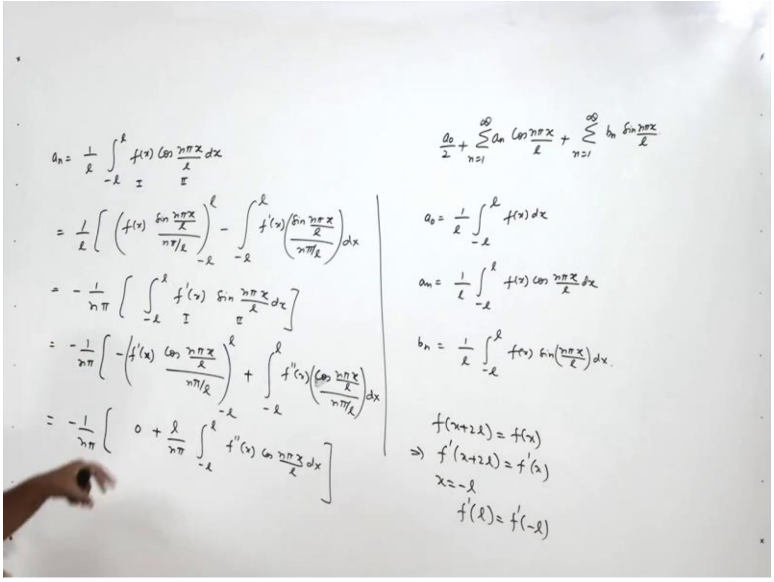

Now, in this series a naught by 2 plus summation n from 1 to infinity a n cos n pi x by l plus summation n from 1 to infinity b n sin n pi x by l. In this series, a naught, we already know is given by 1 upon l minus 1 minus l to l integral minus l to l f x dx, a n is given by 1 by l integral minus l to plus l f x cos n pi x by l into dx. And b n is given by one by l integral minus l to plus l f x sin n pi x by l into dx. So, this we already said in the last lecture that these are given by Euler's formula a naught, a n and b n.

Now, if we see a n, a n is nothing but what a n is given by 1 by l minus integral minus l to plus l f x cos n pi x by l into dx. Now, you integrate it by parts, you take it first function this second function this, integrate by parts. So, it is nothing but equals to 1 by l; now first as it is integration of second, it is nothing but sin n pi x by l upon n pi by l from minus l to plus l minus integral derivative of first and integral of second. Now, when x is l, so it is sin n pi which is 0 and when x is minus l which is again sin minus n pi which is again 0 this first term is 0, so it is nothing but minus it is l l cancels out it is minus 1 upon n pi. And it is integral minus l to plus l f dash sin n pi x by l into dx, so this we obtain.

Now again you apply by parts. So, it is first and second, so it is minus 1 by n pi now first as it is integration of second, integral second is nothing but minus of cos n pi x upon l divided by n pi by l. And it is from minus l to plus l minus integration derivative of first integration of second that is plus cos cos n pi x by l upon n pi by l into dx. Now, it is minus 1 upon n pi. Now, since f dash is continuous and f is periodic, so f dash l is same as f dash minus l. Since, f x is continuous and it is periodic. So, f dash l will be same as f dash minus l and at x equal to l and x equal to minus l, this value is same which is minus 1 k to the power n, so the first term will be 0.

The plus it is l upon n pi integral minus l to plus l f double dash x into cos n pi x by l into dx. This is 0 because function is periodic that is f of x plus 2 l is same as f of x and derivative is continuous. So, f dash of x plus 2 l is also equal to f dash x for all x. And when you substitute x as minus l, so f dash l will be same as f dash minus l. So, these two term are same f dash l is same as f dash minus l, and upper and lower limit the value of this is same this term is 0, and we left with this expression. Now, we have to show that anyhow that this is convergent.

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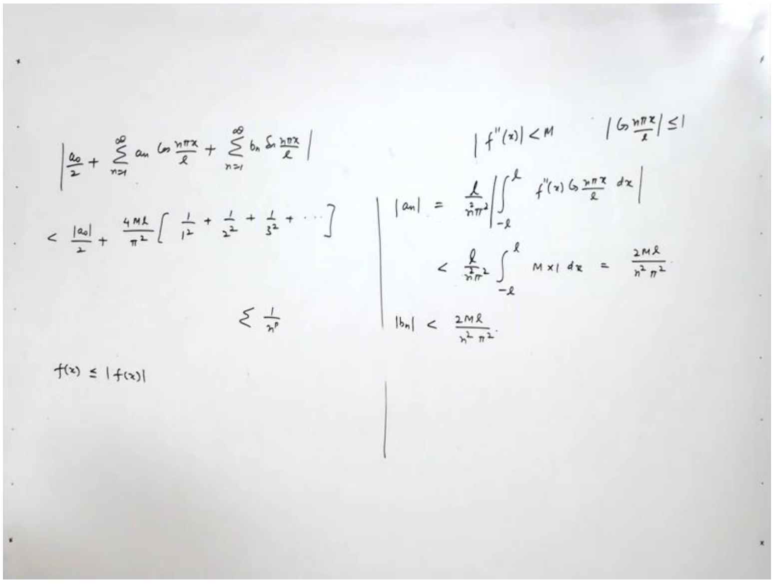

So, to show that this is convergent, again since f double dash x is continuous and in the interval minus l to l, so we can say that mod of f double dash x will be less than m for some m. Because it is continuous f double dash is continuous in the statement as given in the statement itself that f double dash is continuous; so there will exist some m, such that f double dash will be less than capital M. So, if we apply this over here eta mod of a n and mod of cos of n is always less than equal to 1 mod of cos of n x upon l is always less than equals to 1. So, mod of a n will be less than strictly less than 1 upon n pi integral minus l to plus l, we can take mod inside mod of f double dash x into cos n pi x by l into dx, which is equal to 1 upon n pi it is less than or it is less than mod of this not the entire expression.

Now, when you take this expression inside, when you take this inside and applying these inequalities. So, this is nothing but less than minus l to plus l m into 1 dx, which is nothing but now this into this is also there. So, it is l upon sorry it is l upon n square pi square, l upon n square pi square. So, it is nothing but 2 m l upon n square pi square. So, a n will be less than this. Now, similarly we can obtain that mod of b n will be less than 2 similarly it will be less than this, this we can similarly obtain.

So, what about this series? What about this series a naught by 2 plus summation n from 1 to infinity a n cos n pi x by l plus summation n from 1 to infinity b n sin n pi x by l, this mod will be less than mod a naught upon 2 plus. Now, this mod, mod of a n in each for each n mod of cos n x is less than equal to 1; mod of b n which is less than this; and mod of sin n x by l which is less than equal to 1. So, what we will obtain finally, this is 4 m by l upon pi square will come outside, two from here, two from here; this is 4 m l upon pi square. And it is summation 1 by n square that is 1 by 1 square plus 1 by 2 square plus 1 by 3 square and so on.

Because when you take the mod, and when mod come inside in each term then using these inequalities, we will obtain this expression because it is mod of a n, a naught by 2 plus; mod of a n is less than 2 m l by n square pi square and mod of this is less than equal to 1. Again this is less than equal to 1, we know that summation 1 by n to power p is convergent when p is greater than 1. So, here p is 2, it is convergent as this series is convergent. And this series is convergent means inside the series also convergent.

This series is convergent means because f x is always less than equal to mod of f x. And if f x is convergent then this f x is also convergent. So, this series will be convergent. So, hence we can say that if function is a periodic and first and second order derivatives are continuous then this series is convergent this series of f x is convergent. So, here is this is the same proof which I have discussed here.

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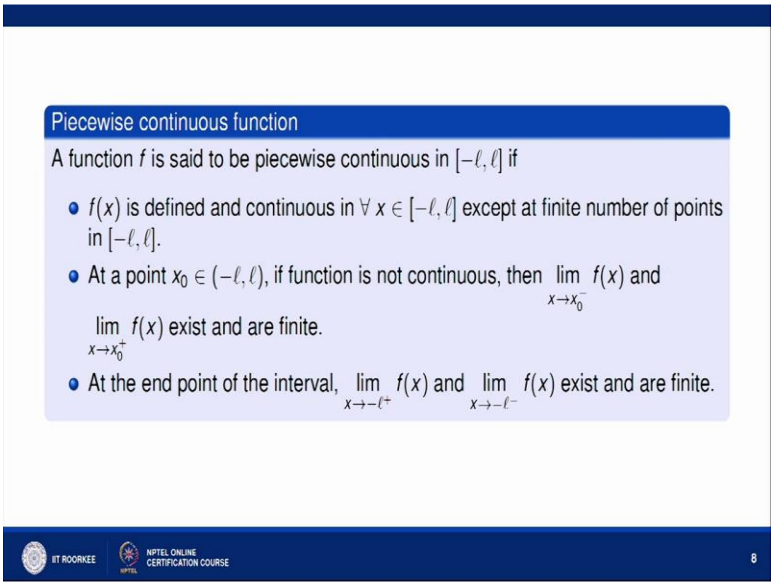

Now, piecewise continuous function, we have already discussed this thing in Laplace transforms also that piecewise continuous means that you divide interval a comma b into finite number of subintervals. And if in each subinterval function is continuous we say the function is piecewise continuous. The same thing we will define here. Suppose the function is said to be piecewise continuous in minus l to plus l, if function is defined and continuous for all l, except at finite number of points. If a function at a point say x naught is not continuous then left hand limit and right hand limit at that point exists and finite that means, discontinuity are of jump type.

Now, at end point of interval that is at minus l and plus l, if it is minus l the right hand limit; if it is plus l then the left hand limit exist and they are finite. So, then we say the function is if these three properties hold, then we say that function is piecewise continuous in minus l to plus l. Or in other words, we can revise the same definition that to divide interval into finite number of subintervals and in each subinterval function is continuous and if there is a discontinuity that must be of jump type.

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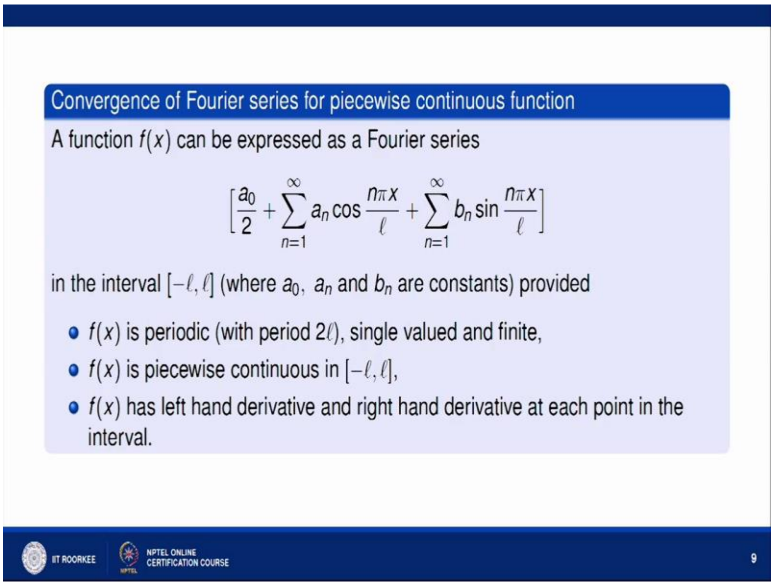

Now, if a function is piecewise continuous what is the convergence of condition for Fourier series. A function f x can be expressed as Fourier series this in the interval minus l to plus l, where a naught, a n, b n are constants provided function is periodic with period 2 l, single valued and finite. Function is piecewise continuous in minus l to plus l, piecewise continuous we already discussed what it is. And f x has left hand derivative and right hand derivative at each point in that interval. So, this is if function is piecewise continuous, if these properties holds then it is converges to this expression, then function f x will converge to this expression can be expressed in this way.

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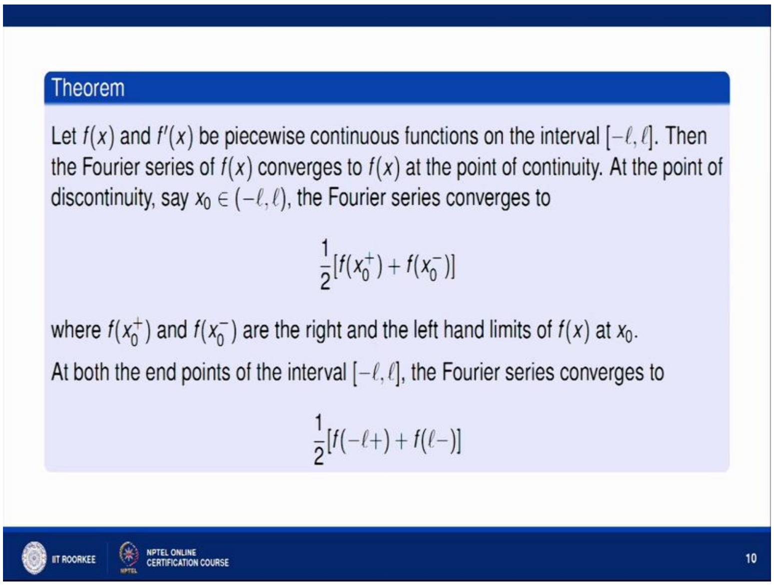

Now, we have a theorem that if f x and f dash x that is the first derivative be piecewise continuous function on the interval minus l to plus l then the Fourier series converges Fourier series f x converges to f x at the point of continuity. If it is a continuous point then f x will converge to suppose x naught is the point in minus l to plus l where it is continuous then f x will converge to f x naught. But if it is a point of discontinuity say x naught belongs to minus l to plus l, then Fourier series will converge to the average of left hand limit and right hand limit at that point will converge to the half of f x naught plus f x naught minus.

Now, at both point, both the end points of the interval minus l to plus l, if you take minus what is the value of the function at minus l, what is the value of function at l then the Fourier series will converge to half of f of minus l plus that is the right hand limit at x equal to minus l and f l minus that is the left hand limit at x equal to l. The average of these two will be the value of the function at the endpoints of the interval minus l to plus l. I am not discussing the proofs here, proof of this theorem. So, the proof can be obtained, so these results we will used while solving some problems.

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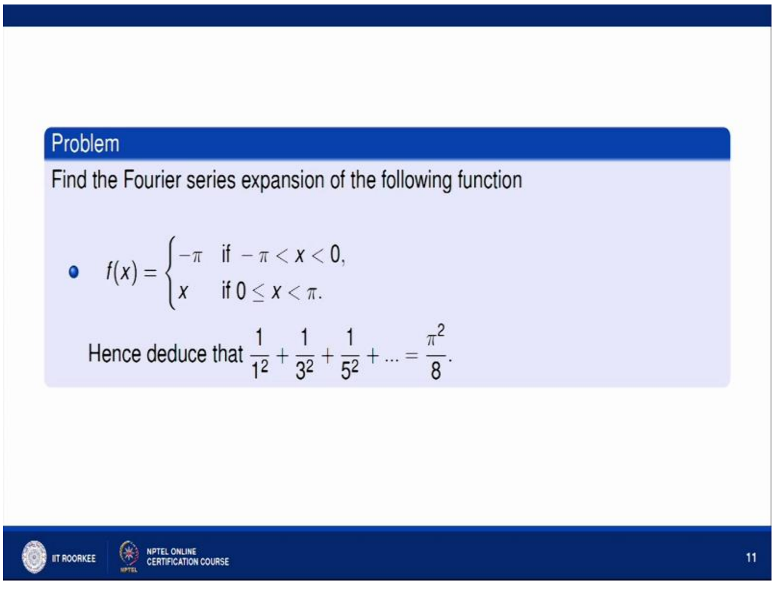

Now, let us try this problem. Find the Fourier series expansion of the function this and deduce this expression. So, we will solve the problem in the usual way as we did in earlier to express to find out the Fourier series expansion of this function. So, function is periodic, we assume here function is periodic of period 2 pi; function is defined from minus pi to plus pi, but function is periodic that is period is 2 pi. So, let us find a naught first, so what will be a naught a naught will be 1 by pi minus l to minus pi to plus pi f x d x by the definition.

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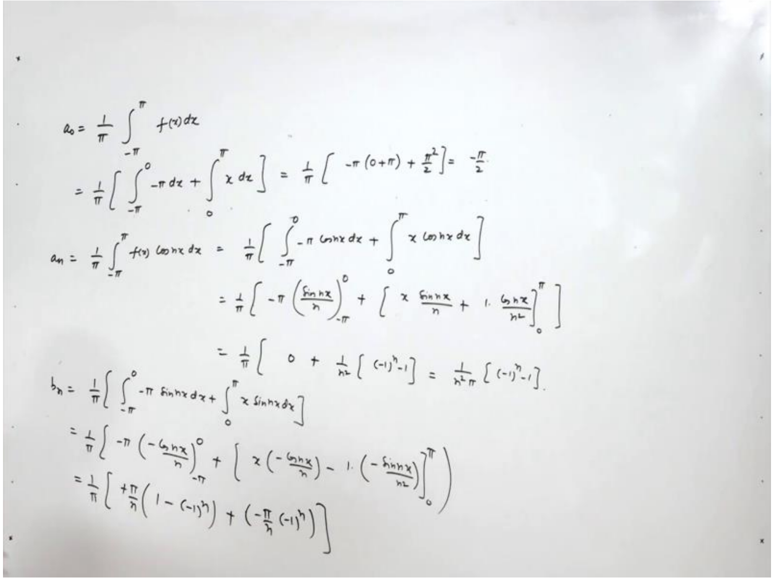

So, it is 1 by pi. Now, from minus pi to 0, it is nothing but minus pi; and from 0 to pi, it is nothing but x. So, using this, it is nothing but 1 by pi; and it is minus pi 0, minus minus plus pi and it is plus pi square by 2 so that will be nothing but minus pi by 2, so that is a naught. Now, if you find a n, a n is nothing but 1 by pi integral minus pi to plus pi f x cos n x dx, because here l is pi, here l is pi, so instead of cos n pi x by l, if you take l equal to pi, so it will be nothing but cos n x dx. Now, it can be expressed as written as 1 upon pi integral minus pi to pi, it is minus pi to 0, it is nothing but minus pi, it is nothing but minus pi cos n x d x plus 0 to pi it is nothing but x cos n x dx.

Now, it is 1 by pi minus pi will come out it is sin n x upon n from minus pi to 0 plus it is x sin n x upon n minus 1 into integral of this that is minus, minus - plus cos n x upon n square from 0 to pi. So, it is nothing but 1 by pi, it is minus pi. Now, it is 0, when x is 0; it is 0 again when x is minus pi this term is 0 plus. Again this term when x is pi and 0, it is 0, so it is 0. Now, here it is 1 by n square times, it is minus 1 k to the power n minus 1, this is nothing but 1 upon pi square n minus 1 k to the power n minus 1. So, this will be a n.

Now, what is b n, b n will be nothing but again 1 by pi. I break it here minus pi to 0, it is minus pi sin n x d x plus 0 to pi it is nothing but x sin n x dx. Now, when we take it, it is 1 by pi it is minus pi it is minus cos n x upon n, you integrate it, from minus pi to pi minus pi to 0 plus you apply by parts here. So, we will obtain it is x minus cos n x upon n minus 1 into minus sin n x upon n square, so it is 0 to pi. So, when you simplify this, it is 1 by pi minus pi minus, minus - plus cos 0 is 1 minus it is minus 1 k to the power n one by n will come out. Now, here plus this term is 0, when x is pi and 0 because of sin n x; and this term is 0 when x is 0; only one term remaining that is minus of pi by n into minus 1 k to the power n. Now, this is minus, this is minus.

And now what we will obtain from here it is nothing but it is minus, and here it is also minus. It is nothing but let us see it once again, it is minus pi it is minus cos n x upon n minus pi to 0, so when 0 it is 1, when minus pi it is minus 1 k to the power n it is plus. Now, x is the integration of this is minus cos n x upon n minus derivative of this and integral of this so that is minus sin n x upon n square from 0 to pi. Now, when it is 0, so it is one it is this and when it is pi it is minus pi upon n minus 1 k to the power n. So, it is nothing but something like 1 upon n will come out pi will cancel, and it is 1 minus it is 1, it is minus of this and minus of this will add up, so it is two times, so it will be minus 2 times minus 1 k to the power n. So, this will come here. So, now we have a naught, a n and b n in our hand. So, what will be the Fourier series now, a naught is minus pi by 2.

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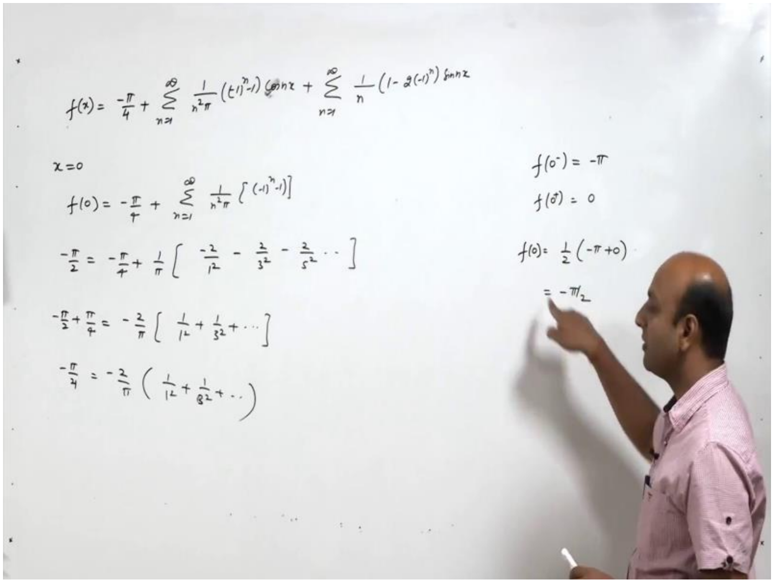

So, Fourier series of f x will be nothing but a naught is minus pi by 2, so a naught by 2 plus summation n from 1 to infinity it is 1 upon n square pi minus 1 k to the power n minus 1 plus into cos n x plus summation n from 1 to infinity 1 upon n into 1 minus 2 into minus 1 k to the power n into sin n x. So, this is the Fourier series of f x which we will obtain from here after simplifying. Now, this will converge to this because function is we have seen that the conditions of convergence of Fourier series is piecewise continuous holds here.

Now, we want to deduce this expression. So, it is 1 by 1 square. Now, n square is only here so; that means, this term is here, but we do not want this term. So, we can put x equal to 0, we put x equal to 0. So, it is nothing but f 0 is equals to minus pi by 4 plus summation n from 1 to infinity 1 upon n square pi and minus 1 to power n minus 1, and cos 0 is 1. Now, what will be f 0 because it is not continuous at x equal to 0 left hand limit is minus pi and right hand limit is 0 because what is f 0 minus, f 0 minus is nothing but minus pi, and f 0 plus is nothing but 0 by the definition of function.

So, what will be f 0 is the average of two 1 by 2 minus pi plus 0 which is nothing but minus pi by 2. So, this is minus pi by 2 is equals to minus pi by 4 plus 1 upon pi come outside, when n is 1, so it is minus 1 minus 1 minus 2 minus 2 upon 1 square. Now when n is 2, it is 0; when n is 3, it is again minus 2 upon 3 square minus 2 upon 5 square and so on. So, when you simplify this it is minus pi by 2 plus pi by 4 is equals to minus 2 upon pi 1 upon 1 square plus 1 upon 3 square plus 1 upon 5 square and so on. So, it is minus pi by 2 minus pi by 4 which is minus 2 by pi into this expression. So, hence the value of this expression is nothing but pi square by 8, pi square by 8. So, hence we obtain the required result.

So, in this way if the function is discontinuous at a point what I want to illustrate in this example, I want to illustrate that if you want to find out the value of function at a point where it is discontinuous then to find out the value of that point is nothing but the average of the left hand limit at that point and right hand limit at that point. You find out left hand limit and right hand limit and the average of these two will be the value of function at that point. So, all other things are as usual we did earlier, but the important point to note here is what how can you finds the value of function at the point of discontinuity, if function is discontinuous.

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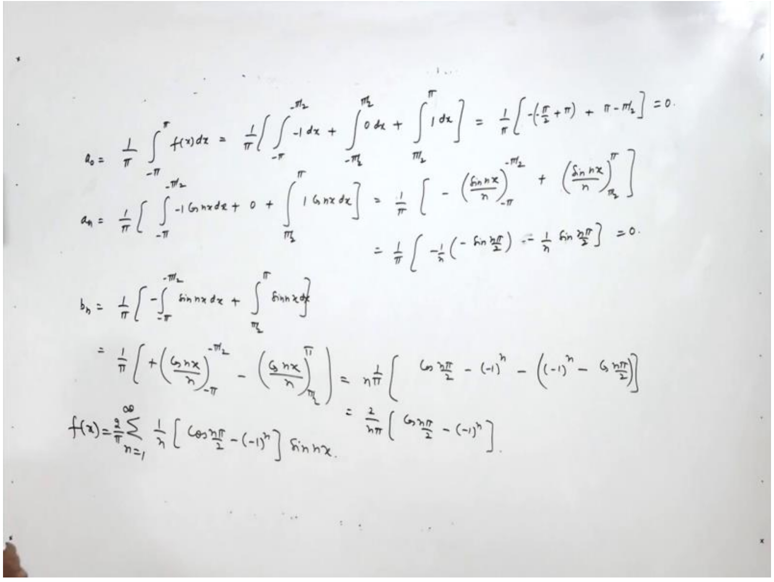

Now, it is the same type of example, so we can easily solve this example also again suppose you want to find out a naught. So, a naught is nothing but now it is from minus pi to plus pi, it is 1 by pi integral minus pi to plus pi f x dx. So, it is 1 by pi. Now, you can break it, from minus pi to minus pi by 2, it is minus 1; from minus pi by 2 to pi by 2 it is 0; and from pi by 2 to pi, it is 1. So, you can simplify this it is a very simple thing you can very easily simplify it is minus pi by 2 minus plus negative will come outside actually negative of minus pi by 2 minus minus plus pi, and it is 0, it is pi minus pi by 2, and which is nothing but when you simplify, so it is 0.

Now, what will be a n, again a n you can compute, a n will be 1 upon pi. Again you will break the interval into three from minus pi to minus pi by 2 it is, minus 1 into cos n x from minus pi by 2 to pi by 2 it is 0, no need to calculate that this is 0 plus integral pi by 2 to pi, it is 1 into cos n x dx. Now, it is nothing but 1 upon pi negative of sin n x upon n from minus pi to minus pi by 2 and plus sin n x upon n again from pi by 2 to pi. So, it is nothing but 1 upon pi minus of sin minus 1 by n comes out and it is sin, it is negative of negative of sin n pi by 2 and sin pi is 0. Again here sin n pi is 0 and it is nothing but minus of 1 by n sin n pi by 2 which is nothing but again it is 0. So, it is 0, so that you can easily simplify.

Similarly, when you find b n, what will be b n, it is 1 upon pi again minus pi to plus pi minus pi by 2 it is minus of sin n x dx and plus pi by 2 to pi it is sin n x dx. So, when you simplify, it is 1 upon pi it is minus minus plus cos n x upon n from minus pi to minus pi by 2 and it is minus of cos n x upon n from pi by 2 to pi. So, this value is nothing but 1 upon pi it is cos n pi by 2 upon n 1 by n 1 by n comes out and minus minus 1 k to the power n minus again minus 1 k to the power n and minus cos n pi by 2, is it ok? 1 upon n pi will come out and it is sin cos n pi by 2 minus of cos n pi minus 1 k to the power n negative is here. So, it is nothing but minus 1 k to the power minus minus 1 k to the power n and it is minus of cos n pi by 2, so it is ok, minus is outside. So, minus is outside, so minus will be here something like this minus minus is outside. So, it is nothing but 1 upon n pi. So, it is 2 times 2 will come outside 2 times cos n pi by 2 minus minus 1 k to the power n.

So, only we have in this expression a naught a n are 0. So, for this function f x, we have only the sine terms on the right hand side, because a naught and a n are 0, so we have on the right hand side only sine terms. So, what will be f x, f x will be nothing but summation n from 1 to infinity 2 upon pi can come outside, it is 1 upon n and it is cos n pi by 2 minus minus 1 k to the power n into sin n x. So, this will be the expression of this function.

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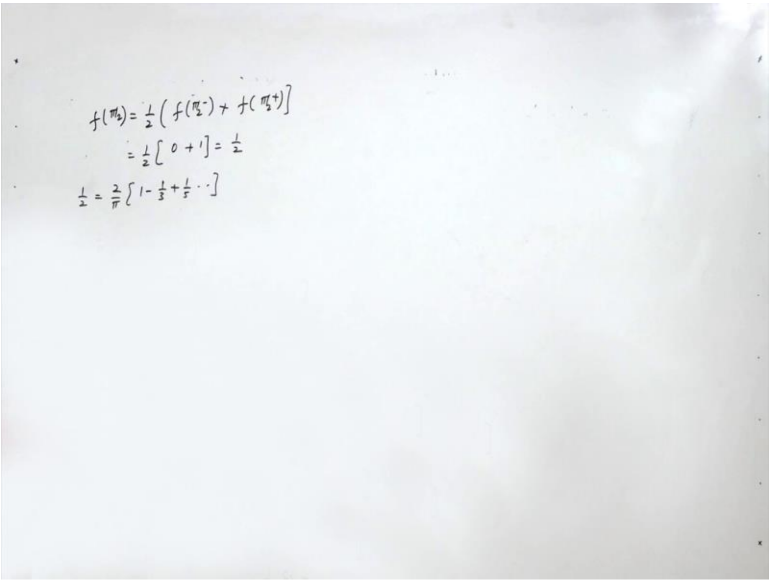

So, f at pi by 2 will be nothing but 1 by 2 f of pi by 2 minus plus f of pi by 2 plus. So, it is nothing but 1 by 2 f of pi by 2 minus is 0 and f of pi by 2 plus is 1. So, it is nothing but 1 by 2. So, we substitute x equal pi by 2 both the sides, on the left hand side we have 1 by 2; and the on the right hand side, we have two upon pi one minus 1 by 3 plus 1 by 5 and so on. So, value of this is nothing but 1 by 2. So, in this way, if you want to find out Fourier series, we first have to find out a naught, a n and b n.

And if we want a series expression value of the series expression, and we have to substitute x as a point where function is discontinuous then you first find f x naught minus f x naught plus and the average of these two will give the value of function at that point.

Thank you very much for this lecture.