Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 47
Fourier Series and its Convergence – I

Welcome to the lecture series on Mathematical Methods and their Applications. In the last few lectures we have seen that what Laplace transforms are what are their properties and how can we solve an ordinary differential equation or a partial differential equation ordinary equations using Laplace transforms. Now the next topic is Fourier series. Now in this lectures, we will see that what Fourier series are and what is the convergence theorem for Fourier series and what are the applications of Fourier integrals. Now first few introduction, first introduction on Fourier series it is named after a French mathematician and a physicist Jacques Fourier who was the first to use Fourier series in his work.

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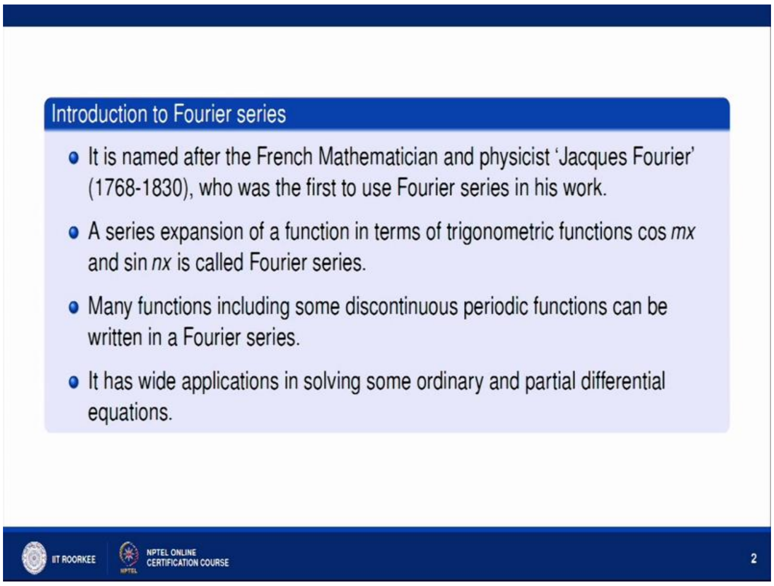

A series expansion of a function in terms of a trigonometric functions cos m x and sin n x is called a Fourier series. So, Fourier series is nothing but if you express a function in terms of sine and cosines then we call such a series as Fourier series. Many functions including some discontinuous periodic functions can be written in the Fourier series. And hence it has a wide application in solving ordinary and partial differential equations.

Now, let us start with this set, set is 1 cos pi x by l cos 2 pi x by l and so on, sine pi x by l sine 2 pi x by l and so on. Now what is an orthogonal set of functions, how do we define it? Suppose we have some functions suppose functions are f 1, f 2, f 3, and so on. And it is defined suppose defined in a comma b.

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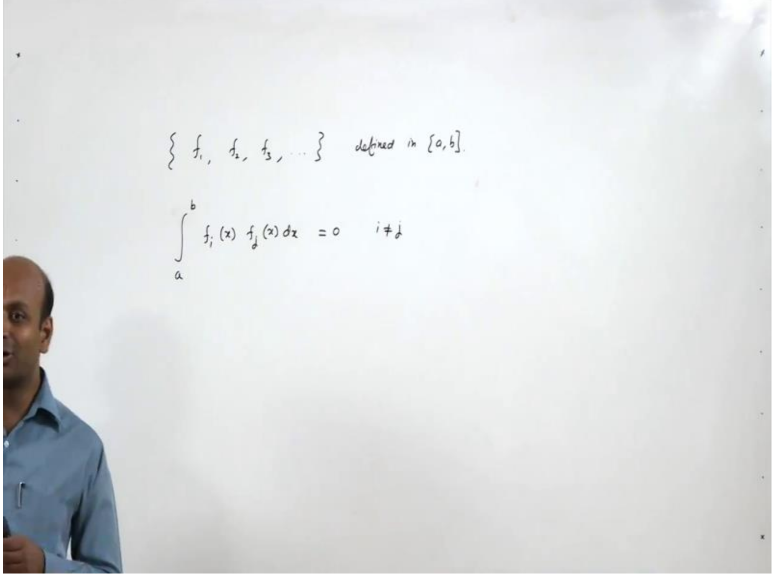

Then we say that this set of function is orthogonal, if in this interval if, integral a to b f i x into f j x d x is equals to 0, for all i not equal to j. So, if this condition holds I for set of functions f 1 f 2 up to f 3 and so on. Which is defined in the interval a comma b then we say that the set of functions are orthogonal. Now consider this set 1 cos pi x by l cos 2 pi x by l and so on, sin pi x by l sin 2 pi x by l and so on.

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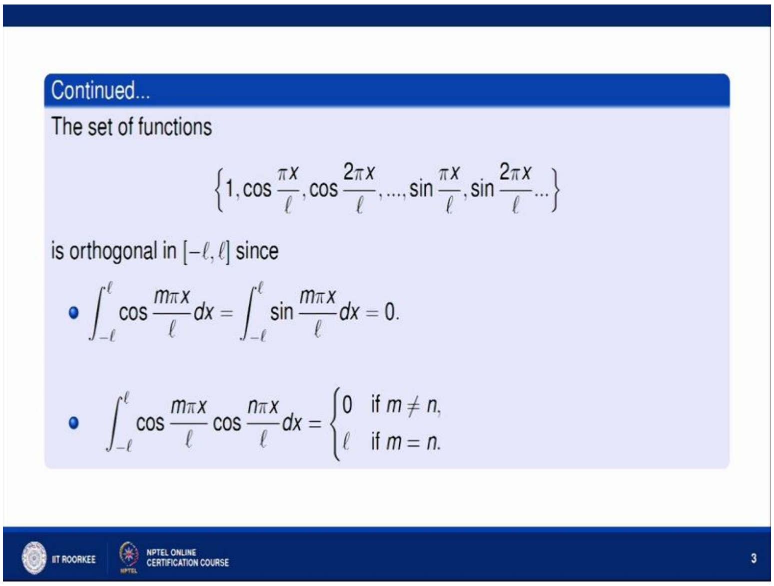

Now, in the interval minus l to l if we carefully see then minus integral minus l to l, cos m pi x by l, d x and this equal to 0. Again the second inequality holds the second inequality also hold it is 0, when m not equal to n. The next inequality minus l to l sin m pi x by l and sin n pi x by l, d x equal to 0 when m not equal to n and is l, when m equal to n; so this we can derive very easily, and minus l to l cos pi m pi x by l into this is also 0 for all m and n.

Hence, if we take any 2 multiples in this set, in this set if we take any 2 different functions, and integrate from minus l to l, it is always 0. So, we say that this set of this set is an orthogonal set of functions. Because it satisfies this condition because integral a to b f i x, f j x is 0 for all i not equal to j and this set satisfies this property. Hence we say that this set is nothing but an orthogonal set of functions.

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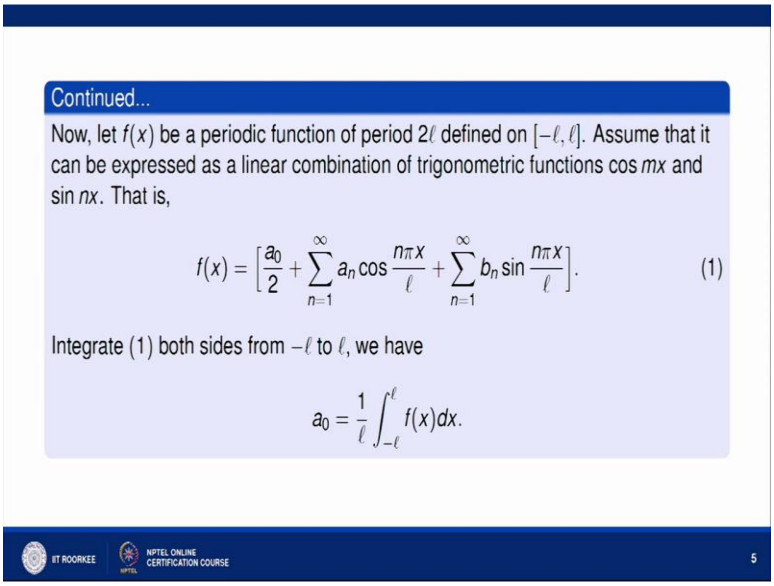

Now, let f x be a periodic function of period 2 n. And define an interval minus l to l. Assume that now we are assuming that it is it can be expressed as the linear combination of trigonometric functions of cos m x and sin m x.

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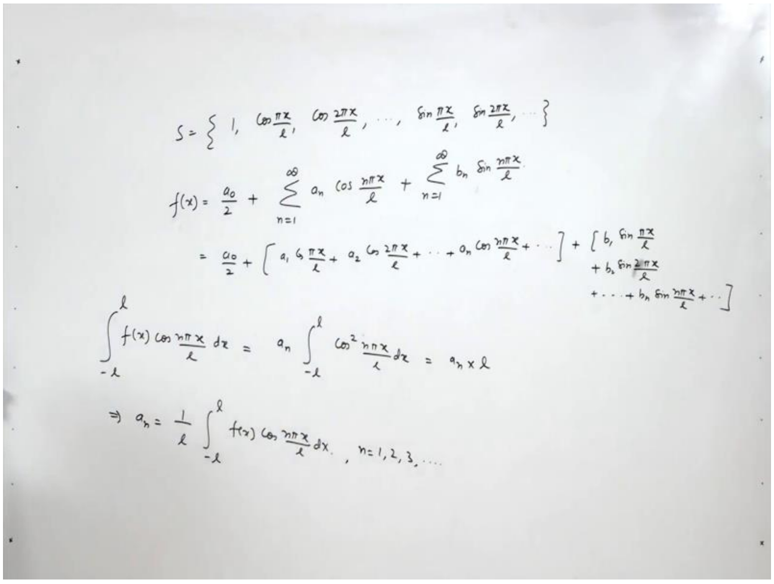

So, what I want to say. So, first I construct a orthogonal set of functions, and that set is nothing but this set 1 cos pi x by l, cos 2 pi x by l and so on, sin pi x by l, sin 2 pi x by l and so on. This is an orthogonal set of function. And a function which is periodic in the interval minus l having a period 2 l defined on minus l to l which is given can be expressed as a linear combination of these functions, that is some multiple of this we are taking for convenience as a naught by 2 plus a 1 into this a 2 into this and so on, b 1 into this b 2 into this and so on. So, which can be written as summation n from 1 to infinity a n cos n pi x by l plus summation n from 1 to infinity b n, sin n pi x by l.

This function we can write as a linear combination of these trigonometric functions, sine and cos. Now if you want to find out the values of a naught a n b n and so on, So, how can you find these values. So, what is this expression basically this is a naught by 2 plus a 1, cos pi x by l, plus a 2 cos 2 pi x by l and so on, plus a n cos n pi x by l, and so on. This is what we are having here and here it is b 1 sin pi x by l, plus b 2 sin pi x sin 2 pi x by l, sin 2 pi x by l and so on plus, b n sin n pi x by l and so on.

So, basically the linear combination of these functions is this thing or this thing. Now to find out the value of a naught let us integrate both the sides from minus l to plus l. So, if we integrate from minus l to l in this side it is minus l to l f x d x, is equals to a naught by 2 integral minus l to plus l d x plus, now integral minus l to l cos n pi x by l for every n is 0 and integral minus l to l sine n pi x by l d x is 0, for any n so; that means, these all are 0. So, this implies a naught is nothing but 1 by l times minus l to l f x d x.

So, that is how we can compute a naught 1 by l minus integral minus l to l f x d x. Now suppose you want to compute a n, a n, n may be 1 2 3 and so on. Suppose you want to compute this a n. So, you multiply both sides by cos n pi x by l. So, when you multiply both sides by cos pi x by l, it is f x into cos n pi x by l, integrate from minus l to plus l d x is equal to, now here if you multiply and integrate from minus l to l, cos pi x by l is 0 for any n. Now if it will be not 0 cos into cos is not equal to 0 only when m equal to n, that is n equal to n, and all in all other cases it will be 0. And cos and sine multiple is always 0 for any m and n this is always 0.

Here what we have, here it is nothing but a n integral minus l to l, cos square n pi x by l into d x. And we have already seen that this value is nothing but l a n into l because when m equal to n. So, this value is nothing but l this we have already seen. So, from here we can say that a n is nothing but 1 by l times integral minus l to l f x cos n pi x by l into d x. So, this is how we can find out a n, and what is n, n may be one may be 2 may be 3 and so on clear.

Now, now suppose you want to find out b n. In the same way you multiply both the sides by sin n pi x by l, this is integral minus l to l f x into sin n pi x by l into d x, integrate both the sides here integral minus l to l sin n pi x by l is 0. And the multiple of sine and cos for any m and n integral minus l to l is always 0.

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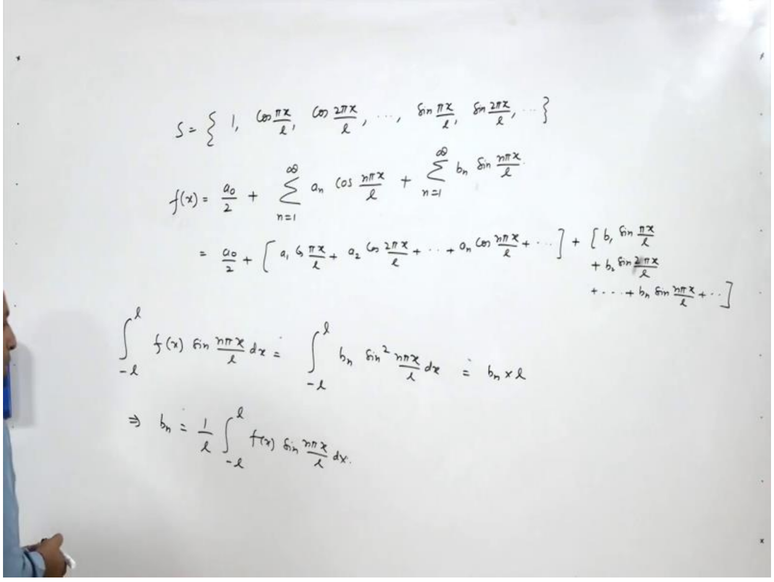

So, all these terms are 0, and here we have and here when m is not equal to n it is 0. We left with only b n. So, integral minus l to l, b n into sin square n pi x by l into d x. And this value is nothing but l when m equal to n. So, it is b n into l. So, this implies b n is nothing but 1 by l integral minus l to l, f x sin n pi x by l into d x. So, these formulas are called basically Euler's formula, to find a naught a n and b n. So, if we assume that if we express function periodic function f x, as a linear combination of cos trigonometric functions of cos and sine, then a naught a n and b n can be find using these expressions.

Now, let us solve some problems based on this, that how we can find Fourier series of periodic function. So, basically if we have a periodic function f x of period 2 l defined in minus l to l.

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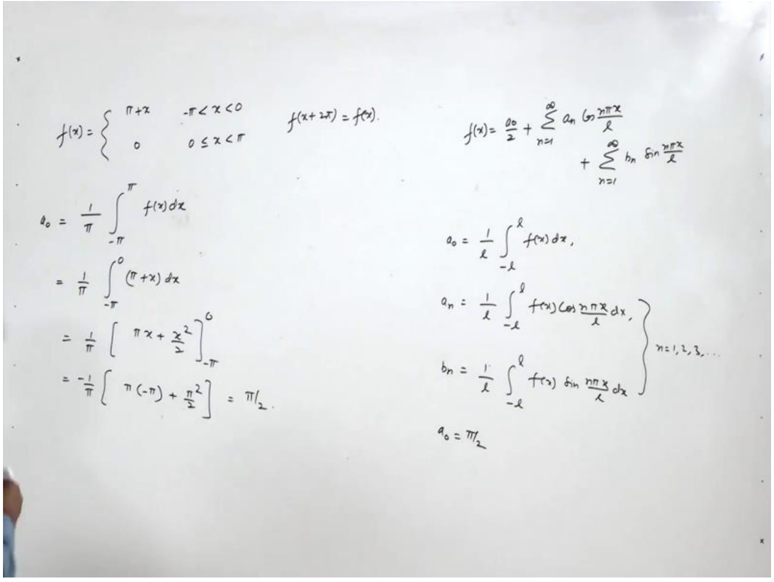

So, what will be f x what we have learnt, that f x is nothing but a naught by 2 plus summation n varying from 1 to infinity, a n cos n pi x by l plus summation n, from 1 to infinity, b n sin n pi x by l. Now what are what is a naught a naught is nothing but 1 by l integral minus l to l f x d x, this we have derived, a n is nothing but 1 by l integral minus l to l f x, cos n pi x by l into d x, and what is b n, it is 1 by l integral minus l to l f x sin n pi x by l into d x. Here in these 2 expressions n is 1 2 3 and so on. So, now, let us solve this problem, here it is given that function is periodic and has a period 2 l, sorry 2 pi and what is function what is nature of function is pi plus x when x is varying from 0 to pi, 0 to minus pi and it is 0 when 0 less than equal to x less than pi. And f x plus 2 pi is f pi f x, because period is 2 pi. So, what is the shape of the function when it is minus pi? So, it is 0 and when it is 0 it is pi. So, here it is like this and from 0 to pi suppose pi is here from 0 to pi it is 0.

Again from pi to 2 pi, it is this function say I am having the same height and from pi to 3 pi is again 0. Then 3 pi to 4 pi again it is this line, and 4 pi to 5 pi it is 0. So, this function is something like this is the shape of this I mean graph of this function. Now how can we find the Fourier series of this function? So we will express this function in this form a naught by 2 plus summation a n cos n pi x by l plus b n plus summation b n sin n pi x by l. Now we first compute a naught and a n, b n. What is a naught? A naught is nothing but 1 by l. Now instead of l here we have pi.

Because 2 l is 2 pi, that means, l equal to pi it is 1 by pi integral minus pi to plus pi f x d x. So, this is nothing but 1 by pi integral from minus pi to 0 it is pi plus x otherwise it is 0. So, it is nothing but 1 by pi, and this term is nothing but pi x plus x square upon 2 minus pi to 0, upper limit minus lower limit, when you apply the lower limit it is nothing but negative of pi into minus pi, plus pi square upon 2. And it is minus pi square by 2 that is pi by 2.

So, this is a naught. So, a naught for this problem is pi by 2. So, I am writing here a naught is nothing but pi by 2 for this problem. So, a naught is pi by 2. Now let us find a n, a n is nothing but 1 by pi integral minus pi to plus pi f x and it is cos n pi x by l, l is pi into d x.

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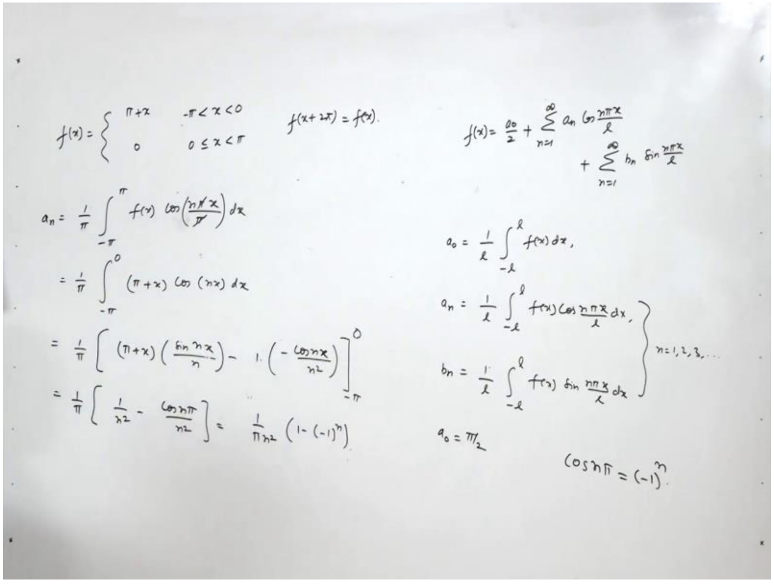

So, pi, pi cancels out it is nothing but 1 by pi integral minus pi to 0, pi plus x cos, n x d x. Which is nothing but 1 by pi, now you will integrate this apply integration by parts. So, it is first as it is integration of second sin n x upon n, minus this derivative is 1, into integration of this. That is minus cos n x upon n square from minus pi to 0. So, this is nothing but 1 by pi, when x equal to 0 sine is 0, and when x equal to pi or minus pi sin n pi is 0. So, this term is 0 from the lower limit and the upper limit.

So, this term is 0. When x is 0 cos 0 is 1, and when x is minus pi it is cos n pi. And cos n pi is minus 1 for negative n for odd, n I mean what is cos n pi cos n pi is nothing but minus 1 k to the power n. When n is odd it is minus 1 when n is even it is 1. So, this is nothing but minus plus, it is 1 by n square minus cos n pi by n square. So, this is nothing but 1 by pi n square 1 minus minus 1 k to the power n. So, this is a n for this problem. So, what is a n? A n will be nothing but 1 by n square pi 1 minus minus 1 k to the power n.

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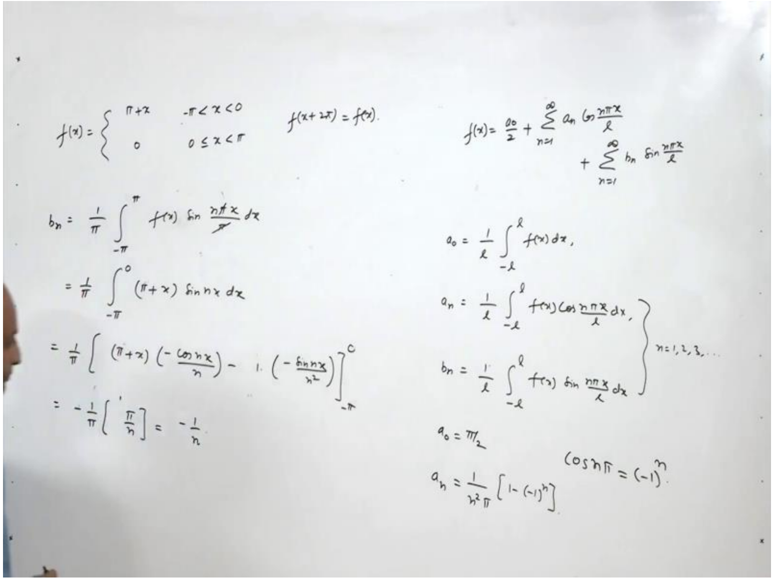

Now, let us compute the last term that is b n, b n will be nothing but again 1 by pi integral minus pi to pi. It is f x, sin n pi x by l, l is pi. So, this is 1 by pi again integral minus pi to 0, it is pi plus x by the definition of the function. Now again you will integrate by parts here, pi plus x sin n x integration of this. So, integration of this is nothing but minus cos n x by n minus derivative of first integration of second that is minus sin n x, by n square from minus pi to 0.

Now again this term, when you take x equal to 0 it is 0 when you take x equal to minus pi it is 0. And here you take you can take minus outside 1 by pi. When you substitute x as minus pi it is 0. When you take x equal to 0 it is pi by n. When you take x equal to 0, it is it is pi, pi by n. And when you take x equal to minus pi it is 0. So, it is nothing but minus 1 by n. So, what will be? So, b n is nothing but minus 1 by n. So, what will be the Fourier series of this function?

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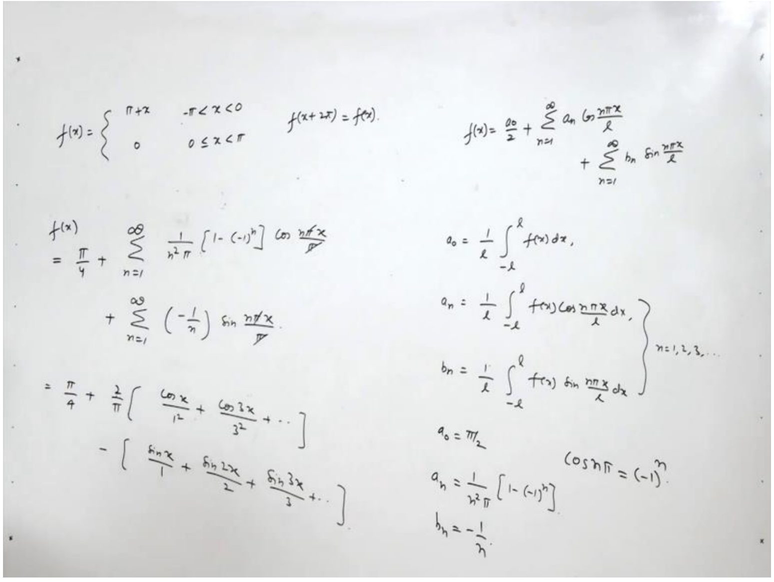

So, Fourier series representation of this function f x is nothing but f x can be written as a naught by 2. What is a naught? A naught is pi by 2, a naught by 2, that is pi by 4 pi by 4 plus summation n from 1 to infinity, a n, a n is nothing but this expression. So, it is 1 by n square pi 1 minus minus 1 k to the power n, into cos n pi x by l, plus summation n from 1 to infinity, it is b n sin n pi x by l. What is b n minus 1 by n.? It is minus 1 by n sin n, pi x by l. So, which if you expand this it is nothing but pi by 4 plus, when n is minus when n is 1, it is nothing but 0 when n is 2, when n is 1 it is nothing but 2 for odd for even values of n it is 0.

So; that means, from here only odd values come it is nothing but 2 will be outside 2 upon pi. And it is nothing but and this l is nothing but pi sorry. This l is nothing but pi pi pi cancels out, because l is nothing but pi. So, you substitute when you substitute x as 1. So, when x is 1 it is 2. So, 2 I am taking outside pi I am taking outside, it is cos x upon 1 square plus cos 3 square and so on. And minus it is sin x upon 1 plus sin 2 x upon 2, plus sin 3 x upon 3. So, this will be the Fourier series representation of this function f x.

Now, let us solve one more problem based on this Fourier series representation of this function. And hence reduce this series. So, let us solve this problem also. So, here period is again 2 pi. So, l equal to pi and function is x minus x square. So, directly compute a naught a n and b n, after calculating a naught a n and b n, we substitute it over here find f x and hence the Fourier series representation of this function and then substituting some values of x we will try to obtain this series. So, what is f x here f x is x minus x square.

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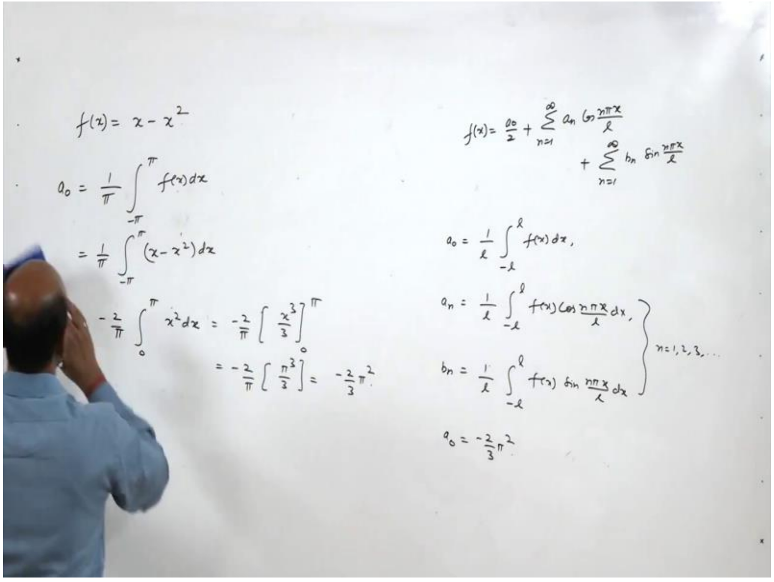

Now, what will be a naught, again 1 by pi minus pi to plus pi, f x d x. It is 1 by pi minus pi to plus pi, x minus x square by 2 d x. Now this x is a odd function. From minus pi to pi it is 0. And this x square is an even function. So, it will be 2 times. So, we can easily write it is minus 2 by upon pi 0 to pi x square d x. And it is nothing but minus 2 upon pi, it is x cube upon 3 0 to pi which is nothing but minus 2 upon pi pi cube by 3. And it is equal to minus 2 by 3 pi square. So, this is a naught for this problem. So, I am writing a naught over here. So, a naught is minus 2 by 3 pi square. Now let us compute a n for this function. So, what will be a n, a n is 1 by pi integral minus pi to plus pi f x, cos n pi x by l into d x.

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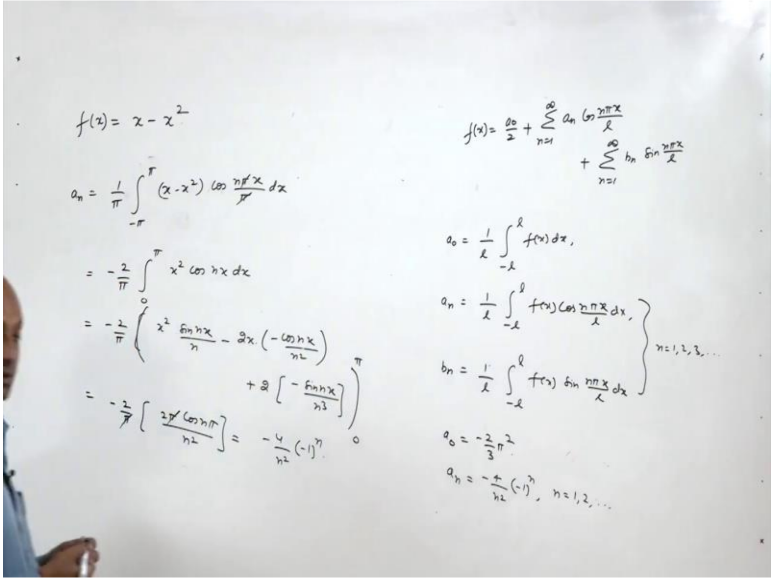

Now, again x into this is odd function. It will be 0 from minus pi to plus pi. And x square into this is an even function. So, it will be 2 times. So, we can write it here minus 2 upon pi integral 0 to pi, x square cos n x d x. So, this is nothing but minus 2 upon pi you will integrate by parts, x square sine n x upon n minus 2 x, this is minus cos n x upon n square, then plus 2 this is minus sin n x upon n cube, and the whole expression from 0 to pi. Now whatever terms of sine from 0 to pi is 0, because upper limit is 0 sin n pi is 0, and sin 0 is 0 this is 0.

This is 0 now only this term left. Now it is also when x is 0 is 0. So, only the above limit left. It is minus 2 upon pi it is 2 pi and it is cos n pi upon n square which is nothing but pi pi cancels out. So, it is minus 4 by n square into minus 1 k to the power n. So, a n is nothing but minus 4 upon n square minus 1 k to the power n when n varying from 1 2 3 and so on.

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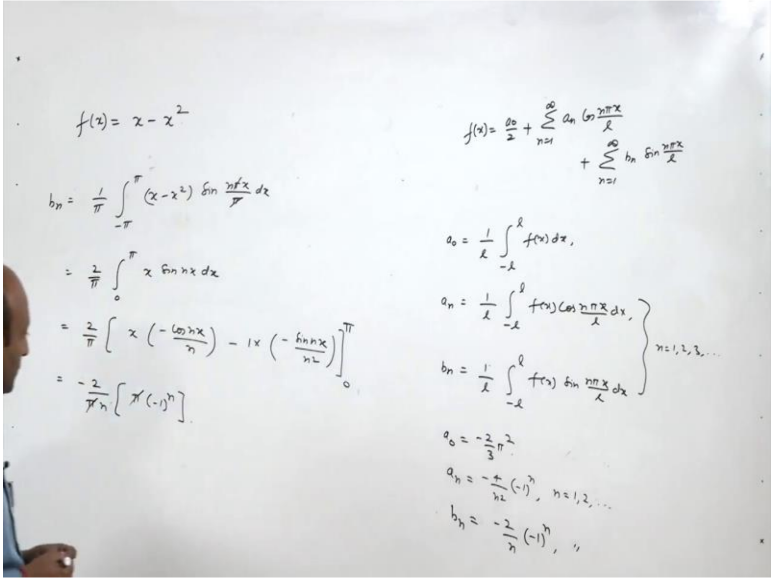

Now, b n can be calculated in the similar way. So, what will be b n, b n will be 1 by pi integral minus pi to plus pi, x minus x square, sin n pi x by l into d x. Now x into this is an even function, and this is an odd function this will be 0, and we left with only 2 upon pi integral 0 to pi x sin n x d x.

Now, again integration by parts it is first as it is integral second it is minus cos n x upon n minus 1 into integral of this, that is minus sin n x upon n square, from 0 to pi. Now this will be 0 when x is pi and when x is 0 only this will be left. So, it is minus 2 upon pi n I am taking outside. It is only existing when x is pi. So, it is pi into minus 1 k to the power n, pi pi cancels out b n is nothing but minus 2 by n into minus 1 k to the power n.

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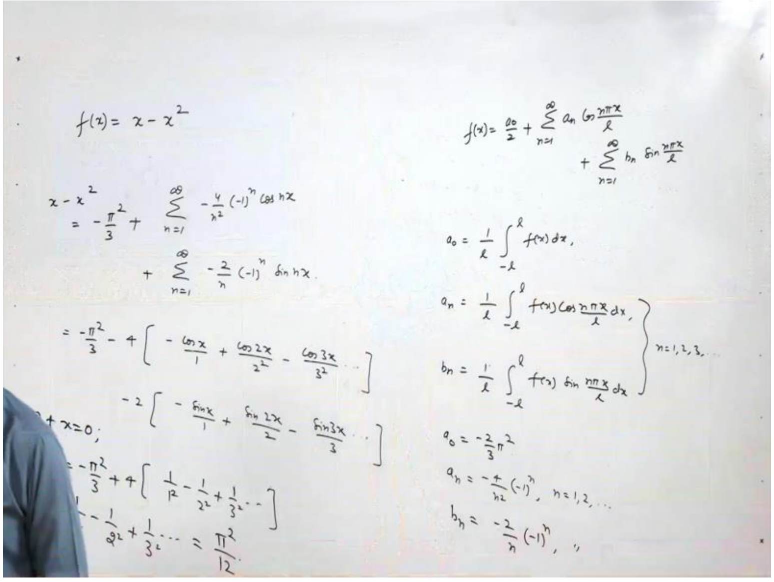

So, what is a Fourier series representation of this function? What is the function, x minus x square? So, f x will be equal to a naught by 2, a naught is this term, a naught by 2 is minus pi square upon 3 plus summation n from 1 to infinity a n, a n is minus 4 by n square minus 1 k to the power n cos n x, plus summation n from 1 to infinity b n b n is minus 2 k to the power 2 upon n minus 1 k to the power n sin n x. So, what the series is this, is minus pi square upon 3 minus 4 times when you take n equal to 1 when n is 1 it is minus cos x, when n is 2 it is plus cos 2 x upon 2 square. When n is 3 it is minus cos 3 x upon 3 square and so on. And it is minus 2 times when n is 1 it is minus sin x upon 1, plus when n is 2 it is sin 2 x upon 2, when n is 3, it is minus sin 3 x upon 3 and so on.

So, now we want to formulate this series. This is the Fourier representation of Fourier representation of this function. The first part is over now we want to deduce this equal 2 pi square by 12. Now it is 1 by one square 1 by 2 square with alternate signs 1 by 3 square 1 by 4 square with alternate sign. And we do not want any term from this series. So, this we can obtain putting x equal to 0.

When you put x equal to 0 all these terms are cos x cos 2 x cos 3 x all will be 1, and all these will be 0. So, put x equal to 0; if we put x equal to 0 the left hand side is 0 it is minus pi square by 3, take minus common from here it is 4 times. It is 1 by 1 square minus 1 by 2 square plus 1 by 3 square and so on. And this is 0. So, what will be the series of this it is nothing but one upon 1 square, plus 1 by 2 square, plus 1 by 3 square and so on is equal to pi square by 3 will go here divide by 4. So, it is pi square by 12. So, hence this result can be obtained.

So, in the next class we will see that, how we can say that what is the convergence theorem for the Fourier series; that we will see.

Thank you very much.