**Mathematical methods and its applications**
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**Lecture - 46**
**Applications of Z – transforms – II**

Hello friends, welcome to my lecture on applications of Z transforms. We have seen that the Z transform of the sequence u n equal to 1 is z over z minus 1 when mod of z is greater than 1.

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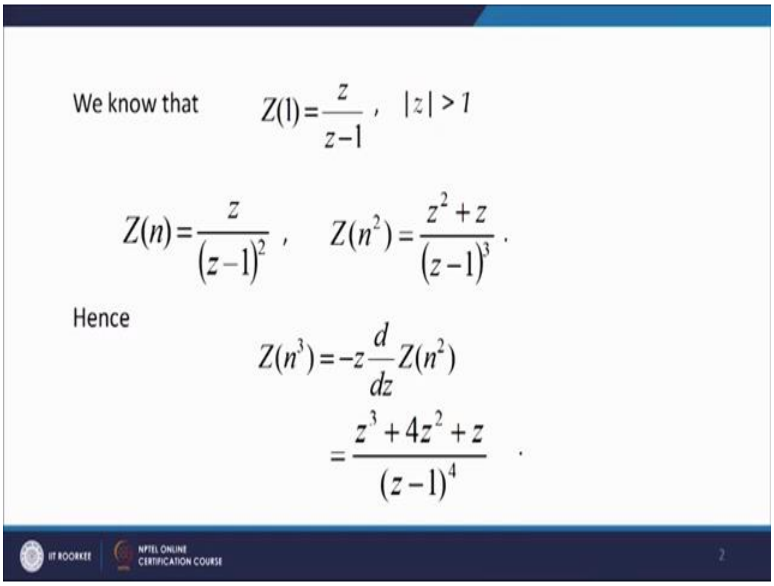


And the Z transform of the sequence u n equal to n is z over z minus 1 whole square when mod of z is greater than 1 and z of Z transform of the sequence u n equal to n square is z square plus z over z minus 1 whole cube and mod of z is greater than 1. Now we can also find the Z transform of the sequence u n equal to n cube by using the recurrence relation Z transform of n to the power p, Z transform of this equals n to the power p equal to minus z d over dz of Z transform of n to the power p minus 1 where p is a positive integer.

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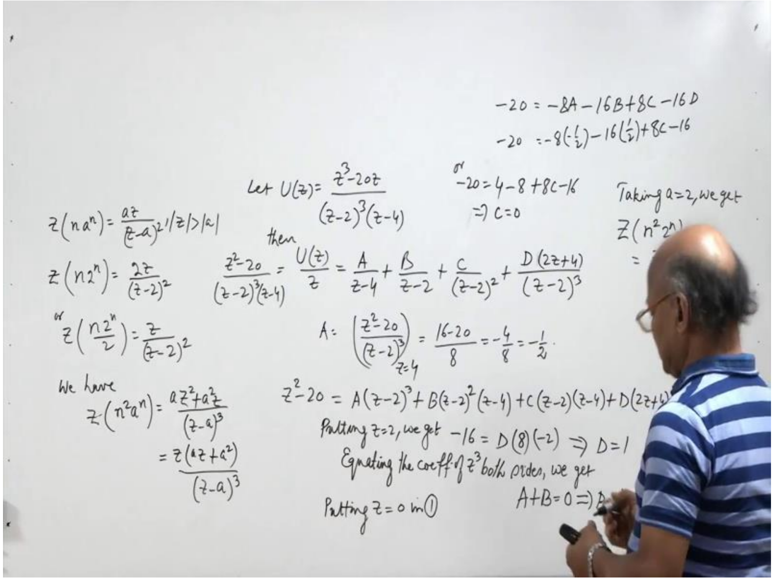


So, here Z transform of n cube where we have taken n equal to 3 here. So, Z transform of a n cube is minus z d over dz; dz of Z transform of n square.

So, we write the Z transform of a n square that is z square plus z over z minus 1 cube here and then differentiate with respect to z and multiply by z we arrive at z cube plus 4 z square plus z over z minus 1 to the power 4 when mod of z is greater than 1. This Z transform, we will need when we solve difference equations. So, we have to drive this Z transform of n cube. Now let us; what I have done is that we are taking Z transforms inverse Z transforms of some expression functions.

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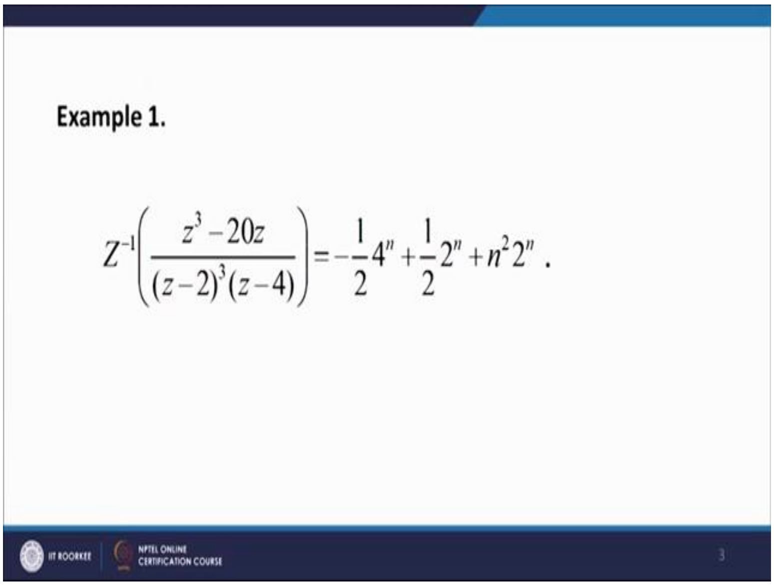


Where we shall see that the; when we write the partial fractions of U z over z, we have to write U z over z. The partial fractions of U z over z according to the inverse z formulas that is we will have to use while inverting will have to use the Z transforms of special sequences. So, we will have to write in the U z over z also in that form.

So, for example, here when we say z the inverse of z cube over 20 z over z minus 2 whole cube equal to z minus 4, let us see how we write the U z by z into partial fractions. So, what we see do is let us say that U z equal to z cube minus 20 z divided by z minus 2 whole cube into z minus 4 then U z by z by in the partial fractions method we have to write U z by z into partial fractions. So, we write A over z minus 4, B over z minus 2, C over z minus 2 whole square see we have z minus 4 power 1. So, we will have 1 partial fraction corresponding to the factor z minus 4, but correspond to z minus 2 whole cube, we will have 3 fractions; 1 with z minus 2 with the other 1 z minus 2 whole square and the third one with z minus 2 whole cube.

Now, let us see how we will have to write these partial fractions while inverting we will write U z; U z will be a times z over z minus 4. So, while inverting the inverse Z transform of z over z minus 4 will be 4 to the power n. So, we will be able to invert and here we will get z over z minus 2 inverse Z transform of z over z minus 2 will be 2 to the power n. So, that is also possible to invert and here we will have z over z minus 2 whole square let us recall that Z transform of n into a to the power n is a z upon z minus a whole square when mod of z is greater than mod of a. So, here if you take because here if z minus 2 whole square. So, you take a equal to 2. So, Z transform of n into 2 to the power n will be equal to 2 z upon z minus 2 whole square and. So, we can say that Z transform of n 2 to the power n divided by 2 will be z upon z minus 2 whole square.

So, inverse Z transform of z over z minus 2 whole square will be n times 2 to the power n by 2. So, we will have to we will be able to invert this term also z over z minus 2 whole square, but here when we come to z minus 2 whole cube let us recall the formula for n square a to the power n we have Z transform of n square a to the power n this is equal to a z square plus a square z divided by z minus a whole cube. So, this is actually z times a z plus a square divided by z minus a whole cube now if I take a equal to 2 taking a equal to 2 we get Z transform of n square into 2 to the power n equal to z into 2 z plus 4 divided by z minus 2 whole cube.

So, z while writing U z this z we will get come here. So, here we have z; z here and then 2 z plus 4 is also needed. So, what we do is we write here D times 2 z plus 4. Now when we will be able when we will when we will find the values of A, B, C and D, this term we can also invert because this term will be z into 2 z plus 4 divided by z minus 2 whole cube.

So, it is in its inverse Z transform will be n square into 2 to the power n. Now the only thing that remains is to do here is that we have to find the values of A, B, C, D. So, A can be found easily, A corresponds to z square minus 20 because U z by z is z square minus 20 divided by z minus 2 whole cube z minus 2 whole cube. So, this is when you put z equal to 4. So, you put z equal to 4. So, we get 16 minus 20 divided by 4 minus 2 that is 2 whole cube. So, that is 8 and we get here minus 4 by 8. So, we get minus half we can then write this is let me say that we write it as z square minus 20 divided by z minus 2 whole cube into z minus four. So, equating both sides we can also find the value of D directly, what you do is let us write from this i equality, we write z square minus 20 equal to a times z minus 2 whole cube plus B times z minus 2 whole square into z minus 4 plus

C times z minus 2 into z minus 4 and D times 2 z plus 4 into z minus 4.

Now, you can see that when you take z equal to 2, this term is 0, this term is 0, this term is 0 and this term is non 0. So, we will get the value of D. So, putting z equal to 2, we have 2 square means 4 minus 20. So, we get minus 16 equal to D times 2 into 2 that is 4. 4 plus 4 is 8 and then we get 2 minus 4 that is minus 2. So, minus 16 D equal to minus 16, this gives you D equal to 1 and the next thing that we can do is let us equate the coefficients of various powers of z. So, equating the coefficient of z cube on both sides here we have term in z cube is 0.

So, the coefficient of z cube is 0 there here the coefficient of z cube is a plus B. So, equating the coefficient of z cube both sides, we get a plus B equal to 0. Now A; we have already found A is equal to minus half. So, B is equal to half, then we can equate the constants on both sides which means that we can put z equal to 0 in this equation. So, putting z equal to 0 we get let me call this equation as equation number 1. So, putting z equal to 0 in 1, what we get is I will write here. So, minus 20 equal to A times minus 2 whole cube. So, minus 8 into A and then we get minus 2 whole square which is 4, 4 into minus 4 that is minus 16.

So, minus 16 B and then we will get C times minus 2 and minus 4. So, that is 8 times C and here we shall have z equal to 0 means D times 4 into minus 4. So, minus 16 D, let us use the values of A, B, C; A, A, D, B and D. So, A is equal to minus half. So, minus 8 into minus half and then we will have the value of the D B equal to half and then we have to find the value of C minus 16, D is equal to 1. So, how much is this or minus 20 is equal to? Now this is 4, 8 by 2 is 4, here we get minus 8, here we get 8, C minus 16.

So, this is 4 minus 24 that is minus 20. So, C is equal to 0 now. So, we get C equal to 0 and thus we have the following fractions. So, hence U z by z, A is equal to; A is equal to minus half. So, minus 1 by 2 z 1 over z minus 4 then we have B equal to half. So, we get 1 by 2 z minus 2 and C is 0 D is 1. So, we get 2 z plus 4 divided by z minus 2 whole cube, now let us multiply by z. So, r U z equal to minus half z over z minus 4 plus half z over z minus 2 and then 2 z square plus 4 z divided by z minus 2 whole cube. Now let us take inverse Z transform both sides. So, taking inverse Z transform we get u n equal to minus 1 by 2 4 to the power n then 1 by 2 2 to the power n and this is n square into 2 to the power n and for these transforms mod of z is greater than 2, but here mod of z is greater than 4. So, we will take mod of z greater than for the common region. So, this is the inverse Z transform of z cube minus 20 z over z minus 2 to the power 3 over z minus.

So, while writing the partial fraction of U z by z, we have to write them in such a way that while inverting we get the standard sequences. So, standard results we have we will have to use now the again next in the next example we have taken this difference equation and we have to find the response of the system that is we have to find the sequence y n such that y naught equal to 0 y 1 equal to 1 and here we are given that u n is equal to unit step sequence u n equal to 1 for n equal to 0, 1, 2, 3 and so on and which is nothing but the unit step sequence and for the unit step sequence if you remember we have seen that the Z transform is z over z minus 1 where mod of z is greater than 1.

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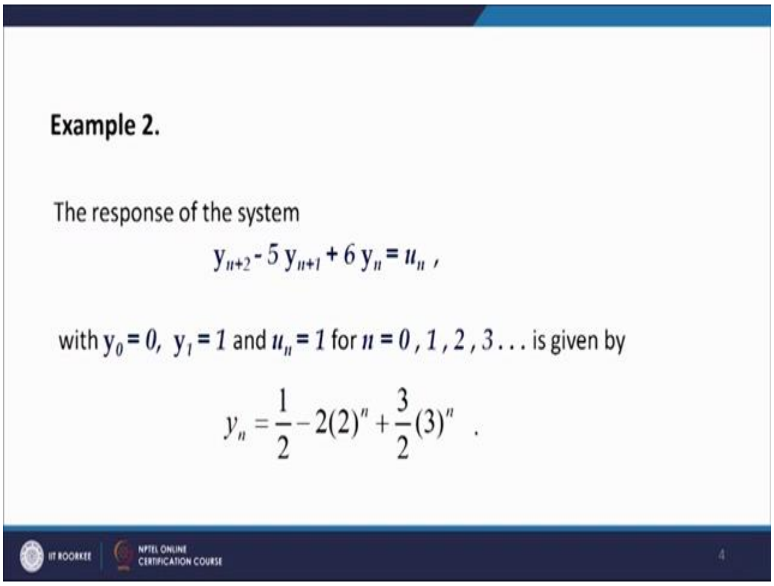


So, here we shall see that the sequence y n is equal to 1 by 2 minus 2 times 2 to the power n plus 3 by 2 3 to the power n.

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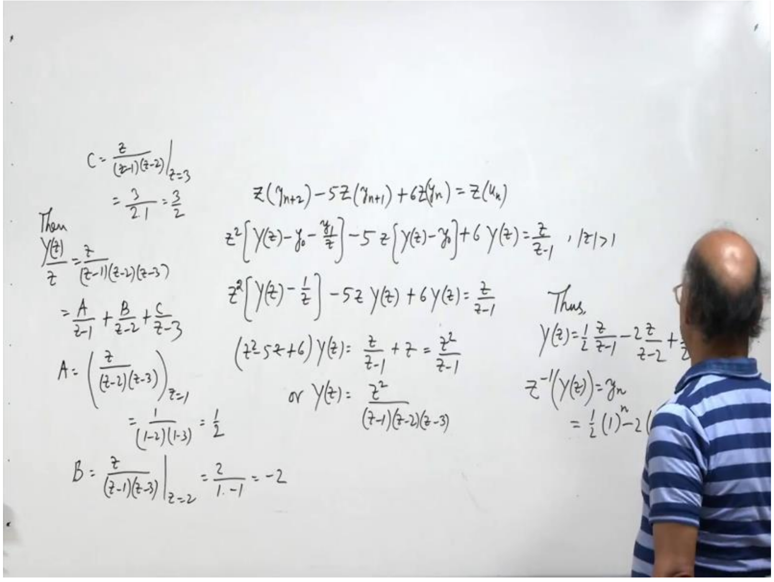

Let us see, how we get the solution of this it is very simple difference equation. So, as usual let us take the Z transform of the given difference equation y n plus Z transform of y n plus 2 and then we have minus 5 times Z transform of y n plus 1 then we have 6 times Z transform of the sequence y n equal to z 6 times Z transform of this sequence y n and then we have the Z transform of the unit step sequence u n Z transform y n plus 2 we can find by the shifting property. So, this is z square times Y z where Y z is the transform of the sequence y n minus y naught minus y 1 by z minus 5 times z of y n plus 1 is z times Y z minus y naught plus 6 times Z transform y n we have taken as Y z equal to unit Z transform of the unit step sequences z over z minus 1.

So, here we have taken mod of z greater than mod of 1. Now let us use the values of y naught and y 1, y naught is 0 y 1 is 1. So, z square times Y z minus 1 by z because y naught is 0 y 1 is 1 minus 5 times z into Y z because y naught is 0 plus 6 Y z equal to z over z minus 1, this is how much? Let us collect the coefficient of Y z here. So, z square minus 5 z plus 6 times Y z equal to z over z minus 1 plus z which is z square divided by z minus 1 now the factors here are z minus 2 and z minus 3. So, we can write Y z equal to z square divided by z minus 1 z minus 2 z minus 3.

So, as usual, let us write Y z by z. So, then Y z by z will be z over z minus 1 z minus 2 z minus 3 which we shall write as A over z minus 1 because z minus 1 z minus 2 z minus 3 all occur in power 1. So, for each one, we will have 1 fraction. So, B over z minus 2 C over z minus 3 and we can see that while writing Y z each term here will be multiplied by z. So, and we will be able to determine the inverse Z transform of z over z minus 1 z over z minus 2 z over z minus 3. So, we just have to find values of A, B, C. So, A; you can see A is equal to z over z minus 2 z minus 3 at z equal to 1 which comes out to be 1 over 1 minus 2 1 minus 3. So, this is minus 2 this is minus 1. So, we have 1 over 2 and similarly B will be z over z minus 1 z minus 3 at z equal to 2. So, we get 2 over 2 minus 1 is 1, 2 minus 3 is minus 1. So, we get minus 2 and then C similarly will be z over z minus 1 z minus 2 evaluated at z equal to 3. So, we shall have 3 over 3 minus 1 is 2 3 minus 2 is 1.

So, we get 3 by 2 and thus Y z will be equal to let us multiply by z the value of a we have found as half. So, half z over z minus 1 the value of B we have found to be minus 2. So, minus 2 times z over z minus 2 and then the value of C we have found 3 by 1. So, 3 by 2 z over z minus 3, inverse Z transform of Y z which we will be equal to the sequence y n this is equal to half 1 to the power n minus 2 times 2 to the power n and then we get 3 by 2 3 to the power n for this mod of z greater than 3 for this mod of z greater than 2 and for this mod of z greater than 1.

So, we have mod of z greater than 3 which is the common portion. So, he inverse Z transform is half minus 2 into 2 to the power n plus 3 by 2 into 3 to the power n now let us go to this problem where we shall see that we have to use Z transform of n cube which we have found earlier.

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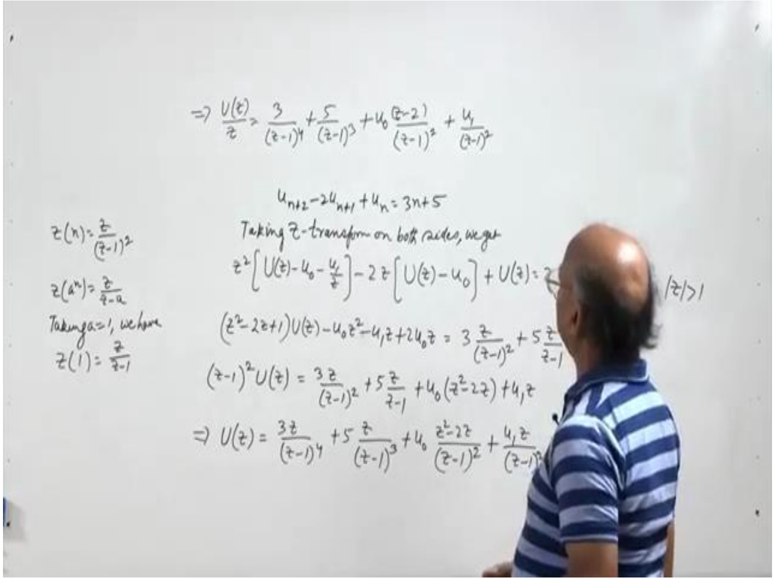


So, let us consider the difference equation u n plus 2 minus 2 u n plus 1 2 u n plus 1 plus u n equal to 3 n plus 5, you can see that here we are not given any conditions on the sequence u n. So, the u naught and u 1, u 2 which we will occur which will occur while taking the Z transform of this will be taken as some arbitrary constants we will see that there will occur u naught and u 1. So, u naught and u n will be taken as some constants and writing the solutions solution of this problem we shall see that we have to choose C naught as u naught and C 1 as u 1 minus u naught to arrive at this sequence the solution of the problem.

So, let us take Z transform of this. So, taking Z transform on both sides we get. So, Z transform will be equal to z square U z minus u naught minus u 1 by z and then minus 2 times z into U z minus u naught the Z transform of u n plus 1 plus Z transform of u n will be U z z trans 3 times Z transform of n let us recall that Z transform of n is equal to z over z minus 1 whole square where mod of z is greater than 1. So, this is z over z minus 1 whole square and Z transform of 5, 5 is a constant.

So, we need to know the Z transform of 1 Z transform of a to the power n was equal to z over z minus a provided mod of z is greater than A. So, taking a equal to 1 here you have Z transform of 1 is equal to z over z minus 1 so, provided mod of z is greater than 1. So, this is z over z minus 1 provided mod of z is greater than 1 for these to be true. So, now, let us take the coefficient of U z. So, z square minus 2 z plus 1 into U z and here we shall have minus u naught into z square here we shall have u 1 z and here we will get 2 u naught z equal to 3 times z over z minus 1 whole square plus 5 times z over z minus 1. So, let us collect let us find the value of U z. So, U z will be equal to; U z will be equal to this is z minus 1 whole square.

So, let me write it as z minus 1 whole square into U z and let us take these terms with the other side we have 3 z over z minus 1 whole square plus 5 times z over z minus 1 and then we have here u naught times z square minus 2 z u naught times these terms when go to the other side will give you u naught times z square by 2 z plus u 1 z plus divide by z minus 1 whole square to get the U z. So, U z is equal to 3 z over z minus 1 to the power 4 plus 5 z over z minus 1 to the power 3 then u naught times z minus square minus 2 z divided by z minus 1 whole square and then u 1 z over z minus 1 whole square now let us write U z by z from here and then we shall write the fractions of U z by z in such a way that we can easily get the inverse Z transform. So, U z by z is equal to 3 by z minus 1 to the power 4 5 upon z minus 1 whole cube and then we have here u naught into z minus 2 divided by z minus 1 square then we have u 1 upon z minus 1 whole square.

Now, we can see that in the denominator we have z minus 1 power of z minus 1 as 2, 3, 4. So, when we write the partial fractions corresponding to this U z by z, we have to write 4 fractions corresponding to z minus 1 to the power 4.

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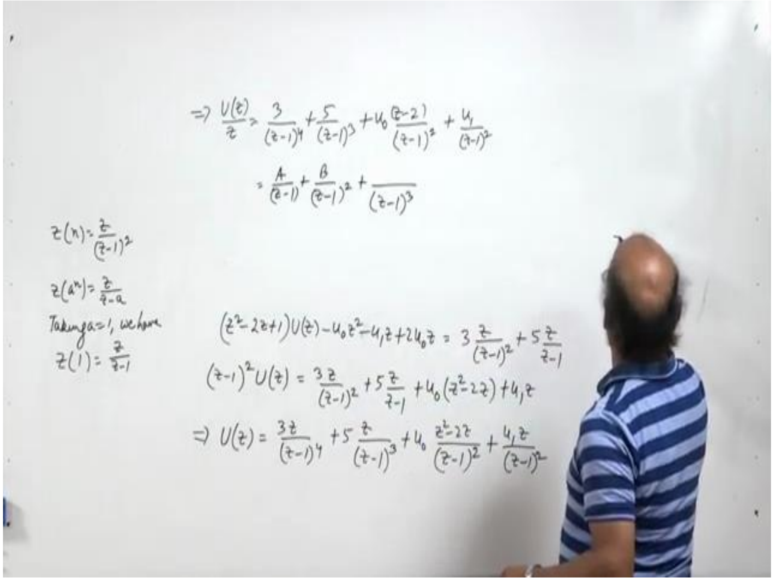

So, this we shall write as a over z minus 1 B over z minus 2 sorry a over z minus 2 1 B over z minus 1 square now. So, far when you will write U z you will multiply this y A and B y z. So, z over z minus 1 can be inverted z over z minus 1 whole square can also be inverted, but here when you will multiply by z the term corresponding to z minus 1 to the power 3.

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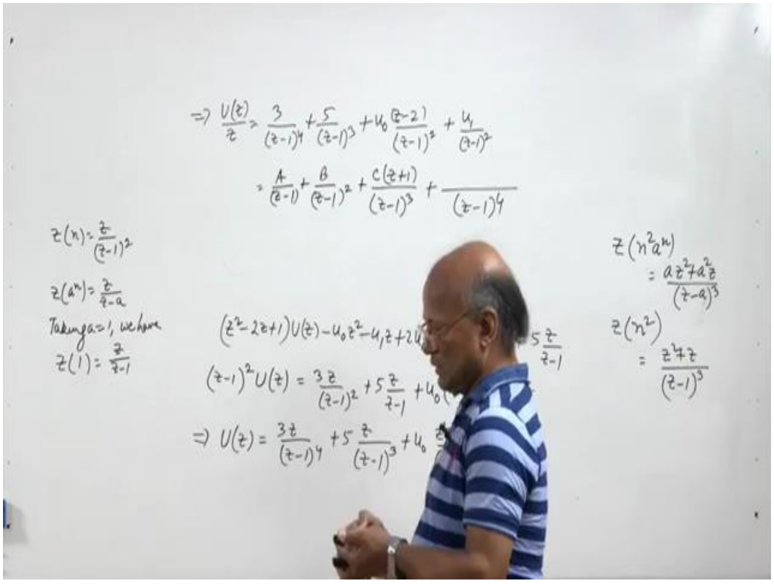

So, let us recall that Z transform of n square a to the power n this was equal to A z square plus A square z divided by z minus A to the power 3.

So, corresponding to z minus 1 cube we will have to have in the numerator term like this. So, what we will do let us take A equal to 1. So, z of n square will by taking A equal to 1 is how much; z square plus z divided by z minus 1 whole cube z minus 1 whole cube. So, we will have. So, this z is to be multiplied to z plus 1 in order to get z square plus 1. So, we get here C times z plus 1. So, that when we multiply by z we get C times z square plus z divided by z minus 1 whole cube whose inverse Z transform will be n square into a to the power n, but a is equal to 1. So, n square, this is how much we this is how we write the term corresponding to z minus 1 whole cube then the term corresponding to z minus 1 to the power 4 we have to write.

Now, let us recall the Z transform of n cube which we had found earlier. So, this is what we have. So, corresponding to z minus 1 to the power 4 in the numerator we must have z cube plus 4 z square plus z 1 z will come from U z by z. So, we need to have z square plus 4 z plus 1. So, we write here D times z square plus 4 z plus 1. So, this how we write U z by z and now later our aim will be to find out the values of A, B, C, D. So, which we can do easily we shall write. So, this is U z by z U z by z come is equal to let me; I think I have omitted that.

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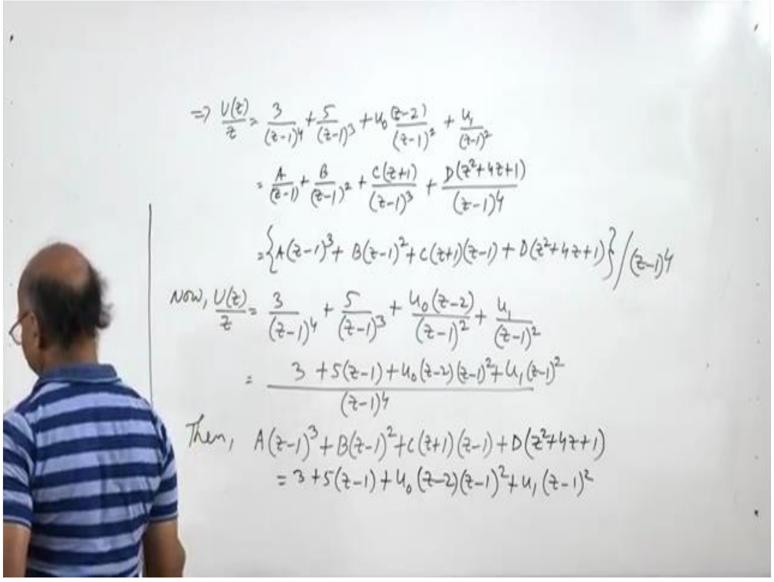


So, this is equal to a times z minus 1 to the power 3 plus B times z minus 1 to the power 2 plus C times z plus 1 into z minus 1 and then we have D times z square plus 4 z plus.

Now, U z was equal to now U z by z we have seen U z by z is equal to 3 over z minus 1 to the power 4 and then we have 5 over z minus 1 cube and then we have u naught times z minus 2 upon z minus 1 to the power 2 plus u 1 upon z minus 1 whole square. So, this is U z by z is equal to this. So, let us equate the both equate the both their side this is this is divided by z minus 1 to the power 4. Here also we can write z minus 1 to the power 4. So, this is z minus 1 to the power 4. So, 3 plus 5 times z minus 1 plus u naught times z minus 2 into z minus 1 whole square plus u 1 times z minus 1 whole square.

So, this way this is equal to this. So, we have a times z minus 1 whole cube plus B times z minus 1 whole square plus C times z plus 1 into z minus 1 plus D times this equal to the numerator here. So, then plus this will be equal to u 1 times z minus 1 whole square now from here if when we solve this we can get the value we get the from the equations that we get we shall be able to determine the values of A, B, C, D. Let us see if we put we can easily see that the A is equal to u naught how we get A is equal to u naught here by taking z equal to if we z q the term z q. So, here we see that let us go equate the coefficient of z q both sides. So, equating here the coefficient of z q is a here the co efficient of z q s u naught.

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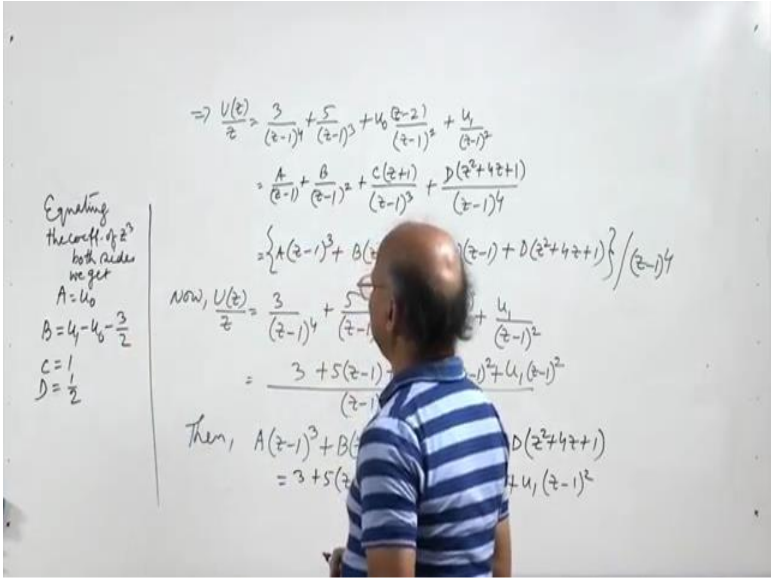


So, we get a equal to u naught and we will similarly we can equate the coefficient of z square the coefficient of z the constants we will get the questions and then we can see that a comes out to be u naught B comes out to be u 1 minus u naught minus 3 by 2 and C comes out be equal to 1 D is equal to half. So, A is equal to u naught B is equal to u 1 minus u naught minus 3 by 2 C is equal to 1 D is equal to half. So, we put these values here and then take the inverse Z transform.

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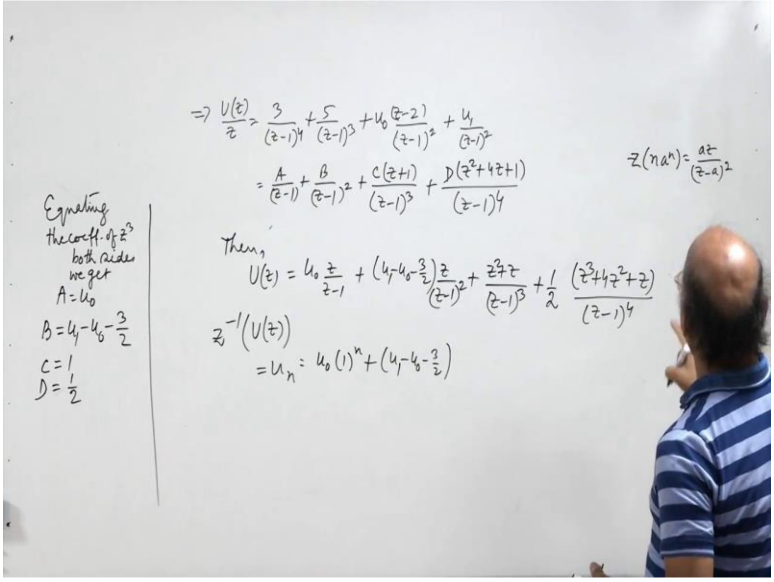

So, we shall have, using these values of a B C D we then get U z equal to a times z over z minus 1. So, u naught times z over z minus 1 and then B is u 1 minus u naught minus 3 by 2 in to z over z minus 1 whole square and we will get C is equal to 1. So, z plus 1, z square plus z divided by z minus 1 whole cube and then we will get D times D is equal to half. So, D equal to half will give us a 1 by 2 times z cube plus 4 z square plus z divided by z minus 1 to the power 4.

Now, take the inverse Z transform. So, z inverse U z which is equal to u 1 sequence will be equal to u naught in to 1 to the power n u 1 minus u naught minus 3 by 2 and then we get here Z transform of n into a to the power n is a z over z minus a whole square. So, taking inverse transform take n is a equal to 1 and then take the inverse transform the inverse transform of z over z minus 1 whole square will be n. So, we get n here then we get here Z transform of n square. So, the Z transform of n square. So, we get n square here and then we get half of n cube.

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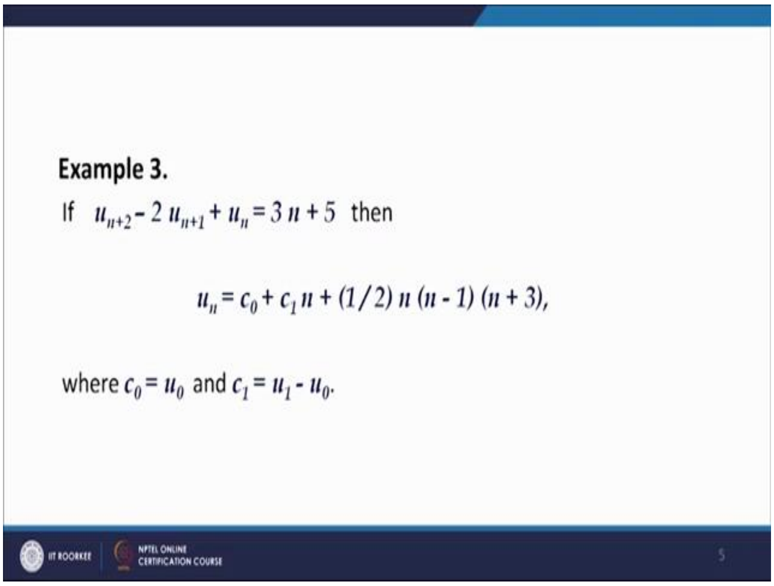


So, if we simplify this, we will get u n equal to C naught plus C 1 and plus half of n into n minus 1 n plus 3. We can easily check that where we are chosen C naught equal to u naught and C 1 equal to u 1 minus u naught. So, this is how we will find the solution of the difference equation given in an example 3. I think with those examples, how to deal with the solution of a difference equation must be very clearly to you.

Thank you very much for your kind attention.