Mathematical methods and its applications
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Lecture - 45
Applications of Z – Transforms – I

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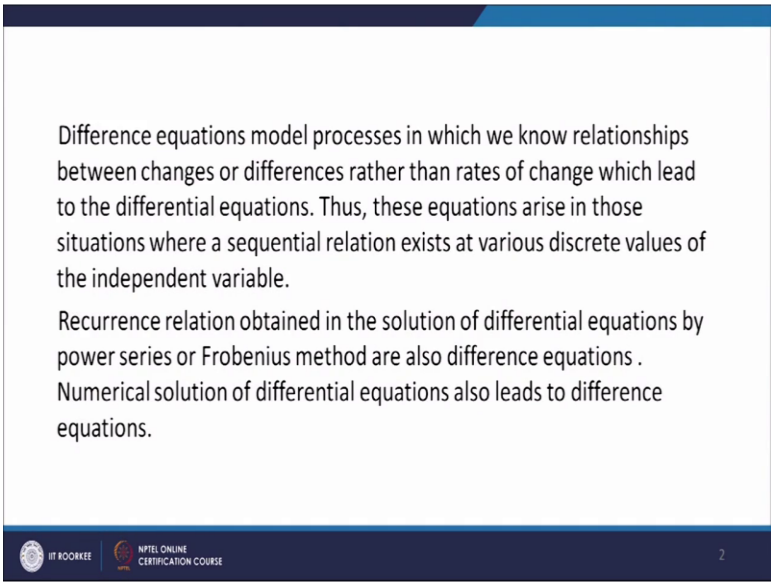

Hello friends welcome to my lecture on applications of Z-transforms. So, this we will cover in two lectures, this is first lecture on application of Z-transforms. We know that the difference equations model those processes in which we know relationships between changes and or differences rather than the rates of change which leads to the differential equations. Now, these difference equations arise in those situations where sequential relation exists at various discrete values of the independent variable, the recurrence relation that we obtained while solving the difference equation by power series or Frobenius method are also difference equations. Similarly, numerical solution of differential equations also leads to difference equations.

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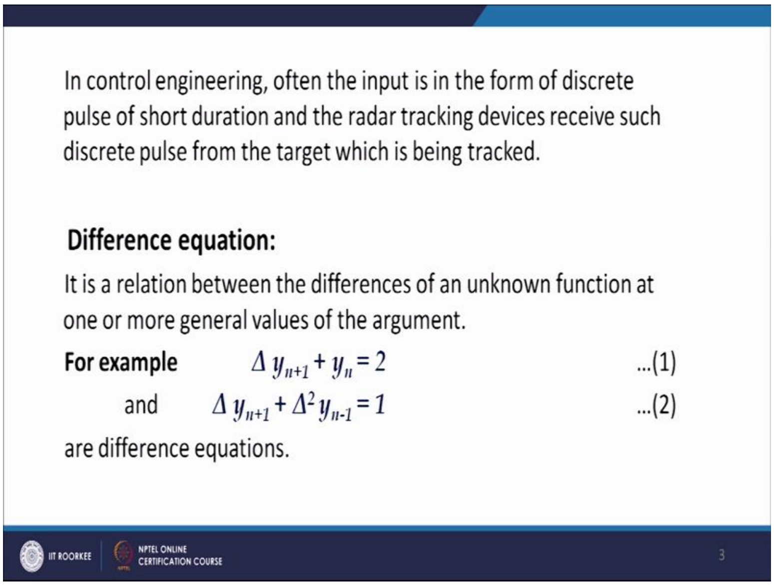

Now, in control engineering, often the input is in the form of discrete pulse of short duration and the radar tracking devices receive such discrete pulse from the target which is being tracked. So, difference equations is a relation between the differences of an unknown function at one or more general values of the argument.

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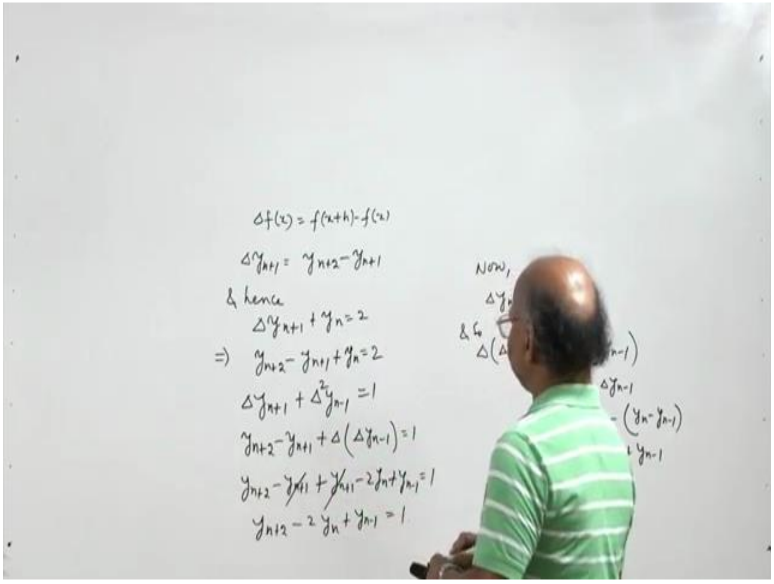

For example, let us consider delta y n plus 1 plus y n equal to 2, this delta is the forward difference, and then you can also consider delta y n plus 1 plus delta square y n minus 1 equal to 1; both of these equations are difference equations. These two equations can be put in the following form. We know that delta of f x the forward difference delta of f x is f x plus h minus f x. So, using this definition delta y n plus 1 will be equal to y n plus 2 minus y n plus 1. And hence delta y n plus 1 plus y n equal to 2 gives us y n plus 2 minus y n plus 1 plus y n equal to 2 which is; so alternately the difference equation 1 can be expressed like this.

And similarly the equations 2, the equations 2 is delta y n plus 1 and plus delta square y n minus 1 y n minus 1 equal to 1. So, here we can write it as delta y n plus 1 is first order forward difference. So, we get y n plus 2 minus y n plus 1, and delta square y n minus 1 here is delta square second order forward difference. So, this will be delta of delta y n minus 1 equal to 1.

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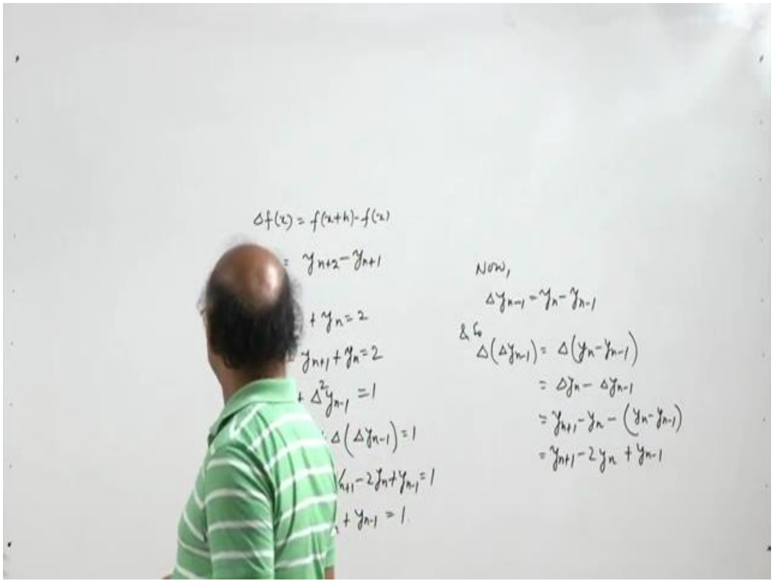

Now, delta y n minus 1 is equal to y n minus y n minus 1. And so delta of delta y n minus 1 is equal to delta of y n minus y n minus 1 which is equal to delta y n minus delta y n minus 1, but delta y n is y n plus 1 minus y n, and here this is y n minus y n minus 1.

So, we will get y n plus 1 minus 2 y n plus y n minus 1. So, let us substitute the value here, y n plus 2 minus y n plus 1 and then plus n plus 1 minus 2 y n and then plus y n minus 1 equal to 1. So, this cancels y n plus 1 cancels and we get y n plus 2 minus 2 y n plus y n minus 1 equal to 1. So, the equations one and two can be alternately written in this forms. So, the equations 1 and 2 can be also written as y n plus 2 minus y n plus 1 plus y n equal to 2 and y n plus 2 minus 2 y n plus y n minus 1 equal to 1. Now, let us discuss how can we determine the order of a difference equation.

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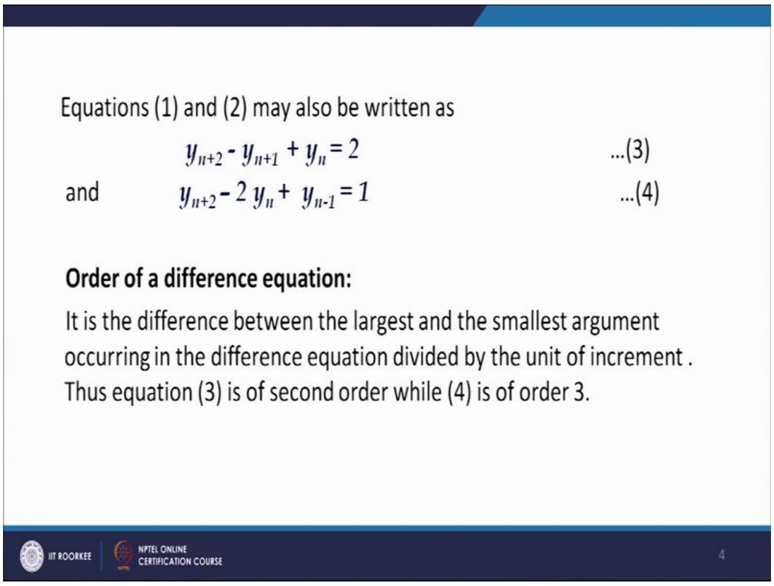

So, the order of a difference equation defined as the difference between the largest and the smallest argument occurring in the difference equation divided by the unit of increment. So, let us see here this equation, which is alternate form of the first equation number one and this one; and this equation, which is the alternate form of this equation. So, here we see that the largest argument is n plus 2. So, let me call it as equation number this is equation number 3 and this is equation number 4. So, in 3 largest argument is n plus 2 and then the smallest argument is n divided by the unit of increment, unit of increment is 1 we have n, n plus 1 n plus 2, so this is 2.

So, this is of second order. And here the largest argument is n plus 2, the smallest argument is n minus 1 divided by the unit of increment is 1. So, we get here 3. So, equation 1 is of second order, while equation 2 is of third order or we can say equation 3 is of second order and equation 4 is of order three. Now solution of difference equation, so when we have a difference equation the solution is defined as an expression for y n which satisfies the given difference equation. The general solution of a difference equation is that in which the number of arbitrary constant is equals to the order of the difference equations.

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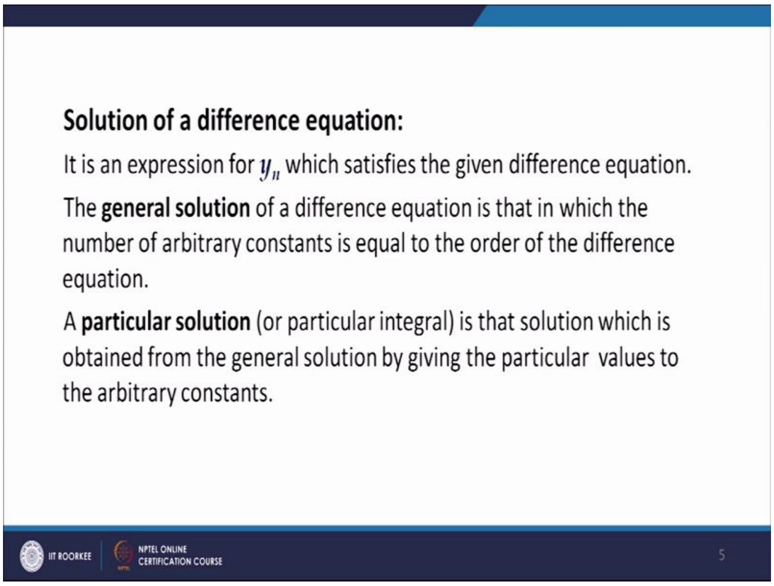

So, suppose we want to call this difference equation 3, then its general solution must involve two arbitrary constant because here there are where the order of the difference equation is 2. Similarly, the equation 4 is of order three, so the general solution must involve three arbitrary constants. So, general solution will be that solution of the difference equation in which the number of arbitrary constants is equal to the order of the difference equation. Now, particular solution of the difference equation will be obtained from the general solution by assigning arbitrary by assigning the particular values to the arbitrary constants.

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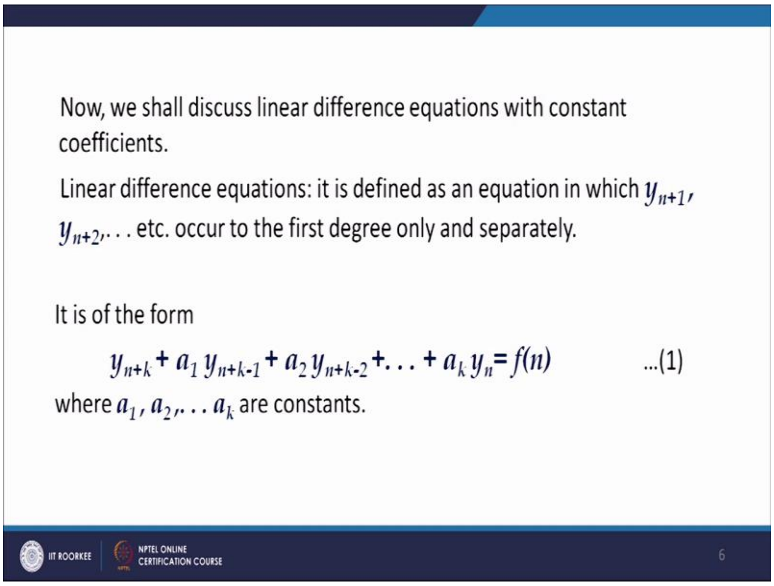

Let us now discuss how we can solve linear difference equation with constant coefficients. A linear difference equation is defined as an equation in which y n plus 1 y n plus 2 etcetera that is all y n plus all the sequences arguments y n plus 1 y n plus 2 and so on, they occur in the first degree and they occur separately. Like linear difference equation with constant coefficients in general will be of this form y n plus k plus a 1 y n plus k minus 1 a 2 y n plus k minus 2 and so on a k by n equal to f n where a 1, a 2, a k are constants. So, here you can see this is kth order difference equation because n plus k is the highest or the largest argument and n is the smallest argument. So, n plus k minus n divided by one unit of increment is 1. So, it is k th order difference equation, and it is a linear difference equation because y n plus y n plus k y n plus k minus 1 and so on y n they all occur in first degree and separately.

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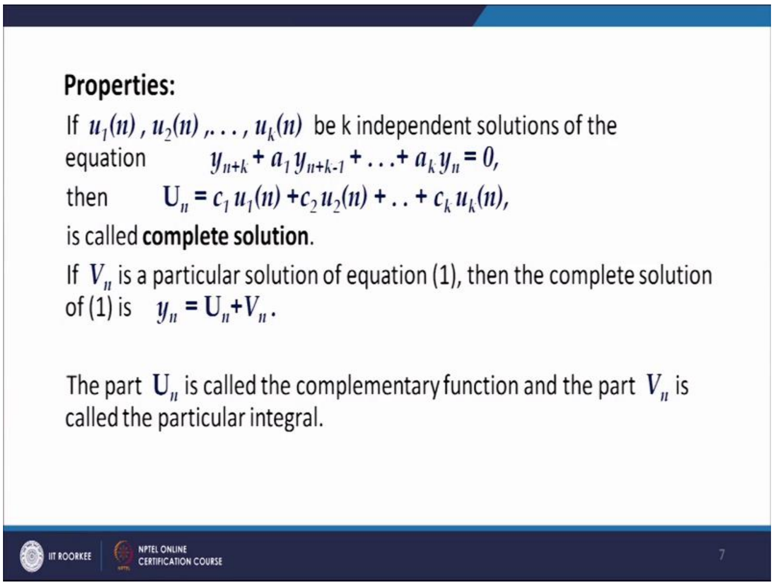

Now, if u 1, u 2, u 1 if u 1, u 2, uk n are k independent solutions of this equation y n plus k a 1 y n plus k minus 1 a k by n equal to 0. So, here we say that k is the order of the difference equation k, k is the order of the difference equation. So, let us consider the corresponding homogeneous part. So, y n plus k plus a one y n plus k minus n plus ak y n equal to 0, if u 1 u 2 u 1 u k and v k independent solutions of this homogeneous equation then their linear combination c 1 u 1 plus c 2 u 1 c k u k is called the complete solution of this homogeneous equation. Let us, denoted by say U n. If V n is a particular solution of equation 1, if V n is a particular solution of this non-homogeneous equation then the complete solution of equation 1 will be given by y n equal to n plus V n, where U n is called the complementary function, and V n is called the particular integral. Now thus the general solution of the difference equation one is the sum of the two that is complementary function and the particular integral.

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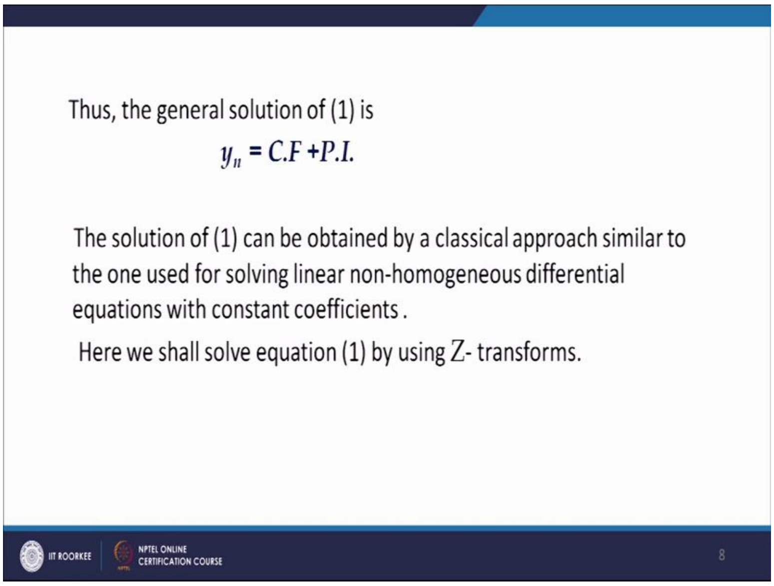

The solution of one equation one solution of one can be obtained by the classical approach like we just has been solve the linear differential equations with constant coefficients. So, these linear difference equations with constant coefficient can be solved in a analogous manner by the classical approach, but here we will solve them by using the Z-transforms. So, we are solving these linear non-homogeneous difference equations with by using Z-transforms.

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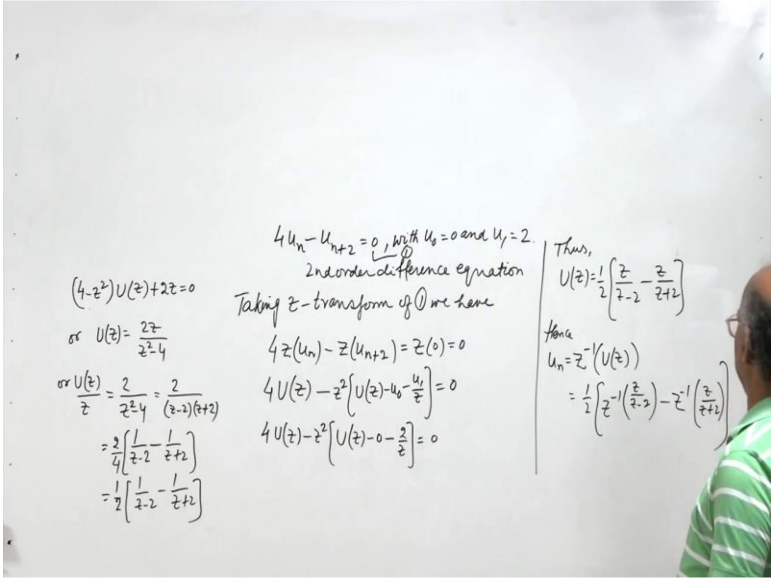

Let us see how we do this. So, suppose we consider, so let us consider 4 U n minus U n plus 2 equal to 0 with u naught equal to 0, and u 1 equal to 2. So, we are given this difference equation. You can see the largest argument here is n plus 2, the smallest argument is n. So, and the unit of increment is 1. So it is the second order difference equation second order difference equation. And they are given two initial conditions the conditions are u naught equal to 0, so u 1 equal to 0. So, we will get with these two initial conditions we shall get a particular solution of this difference equation.

So, what we do in order to solve this difference equation what we do is we take the Z-transforms of both the sides of the difference equation and then solve this solve the resulting equation for the value of u z, where u z is the Z-transform of this sequence U n. And then we divide u z by z and break u z by z into partial fractions in such a way that when we write the corresponding expression for a u z, it comes in terms of the known Z-transforms, so that while inverting u z we get the sequence U n easily.

So, let us take the Z-transform, taking Z-transforms of both the sides, let me call it equation 1. So, this equation 1, so taking Z-transform of equation 1, we have 4 times z is a linear property. So, 4 times z of U n minus z of U n plus 2 equal to z of 0 which is 0. So, now, 4 z of U n is four of u z, u z is the Z-transform this sequence U n. And here while writing Z-transform of U n plus 2, let us recall the shifting to the left shifting of U n to the left. So, we know that Z-transform of U n plus k where k is greater than 0 is equal to z to the power k and then we have u z minus u naught minus u 1 by z and so on minus u k minus 1 over z to the power k minus 1.

So, let us apply this formula. So, z square into u z minus u naught minus u 1 by z equal to 0. Let us make use of the given conditions u naught equal to 0 U n equal to 2. So, 4 u z minus z square and then we get u z minus 0 minus 2 by z. So, the coefficient of z is u z is 4 minus z square into u z and we get this term minus z square into minus 2 y z, so plus 2 z is equal to 0. We can write it as u z equal to 2 z upon z square minus 4. Now let us write u z by z we break u z by z into partial fraction. So, 2 over z square minus 4 which is 2 over z minus 2 into z plus 2. The partial fraction of this expression is 2 times 1 over z minus 2 minus 1 over z plus 2 divided by 4. So, this is 1 by 2 1 over z minus 2 minus 1 over z plus 2.

Now, we can write u z from here. So, u z now, thus u z is equal to 1 by 2 z over z minus 2 minus z over z plus 2. And hence U n which is z inverse of u z is equal to 1 by 2 z inverse of z over z minus 2 m z inverse of z over z plus 2. Let us recall that Z-transform of a to the power n is z over z minus a provided mod of z is greater than mod of a. So, z inverse of z over z minus a will be a to the power n. So, here we have 2 and therefore 1 by 2 2 to the power n and for this mod of z must be greater than 2 and then from here we get minus 2 raise to the power n.

So, here also mod of z must be greater than 2. So, mod of z must be greater than 2 for this two we true. So, 1 by 2 2 raise to the power n minus minus 2 to the power n we can simplify it further two to the power n minus 1 and then I can write it as plus or minus 2 I can take from here. So, minus 2 to the power n minus 1 minus minus becomes plus. So, this is minus 2 to the power n minus 1. So, this is the sequence U n equal to 2 to the power n minus 1 plus minus 2 to the power 2 n minus 1 which is the solution of example 1.

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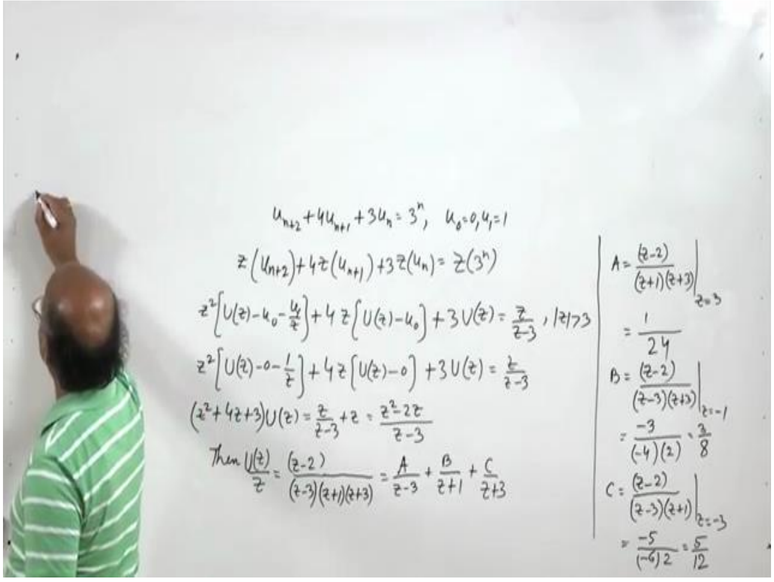

Now let us take the example 2. So, here we have this sequence U n plus 2 plus 4 U n plus 1 plus 3 U n equal to 3 to the power n. Again let us see n plus 2 is the largest argument and n is the smallest argument here. So, n plus 2 minus n divided by the unit of increment is 1, this means the order of this difference equation is n plus 2 minus n divided by 1, so that is 2, this is the order of the difference equation. So, this is second order difference equation. Let us solve this. Here we are given the initial conditions u naught equal to 0, and u 1 equal to 1. So, the general solution of this difference equation will involve two arbitrary constants, to determine two arbitrary constants we are given two conditions u naught equal to 1 and u naught equal to 0 and u 1 equal to 1. So, this will give us a particular solution of the difference equation.

Now, let us take Z-transform here. So, Z-transform is a linear operation. So, z of U n plus 2 z of U n plus 2 plus 4 times z of U n plus 1 plus 3 times z of U n equal to Z-transform of 3 to the power n. Now again let us apply this shifting of U n to the left. So, z of U n plus 2 will be z square u z, where u z is Z-transform of the U n sequence minus u naught minus u 1 by z plus 4 times z of U n plus 1 will be z times u z minus u naught plus 3 times Z-transform of U n is u z equal to z to the power z of 3 to the power n is z over z minus 3 provided mod of z is greater than 3.

Now, let us use the values of u naught and u 1 which are given to us, u naught is given to be 0. So, u z minus 0 minus u 1 is 1 1 over z plus 4 z times u z minus 0 plus 3 u z equal to z over z minus 3. Let us collect the coefficient of u z. So, z square plus 4 z plus 3 into u z we have, and then we have minus z square by z, so minus z minus z when goes to the other side becomes plus z, so z over z minus 3 plus z. And this will give you z square minus 2 z divided by z minus 3.

Now, let us find u z by z from here. So, then u z by z, it will be equal to z minus 2 divided by z minus 3 into z square plus 4 z plus 3, if you factorize you get z plus 1 z plus 3. So, z plus 1 and z plus 3, now, we have to break this u z by z into partial fractions and then determine the inverse Z-transform. So, u z by z, so let us write it as a over z minus 3 b over z plus 1 c over z plus 3. The values of a, b, c can be obtained, now a is equal to z minus 3, so z minus 2 divided by z plus 1 z plus 3 evaluated at z equal to 3. So, this will be equal to 3 minus 2 that is 1 and 3 plus 1 is 4, 3 into 3, 3 plus 3 – 6, so 24. So, it is 1 by 24.

B will be equal to z minus 2 divided by z minus three into z plus 3 evaluated at z equal to 1 minus 1. So, we get this yes minus 3 divided by minus 4 and here we have 2. So, 3 by 8 and c we will have as z minus 2 over z plus z minus 3 in to z plus 1 evaluated as z equal to minus 3. So, this will be minus 5. And here we will have minus 3 minus 3 minus 6 into 2, so 5 by 12.

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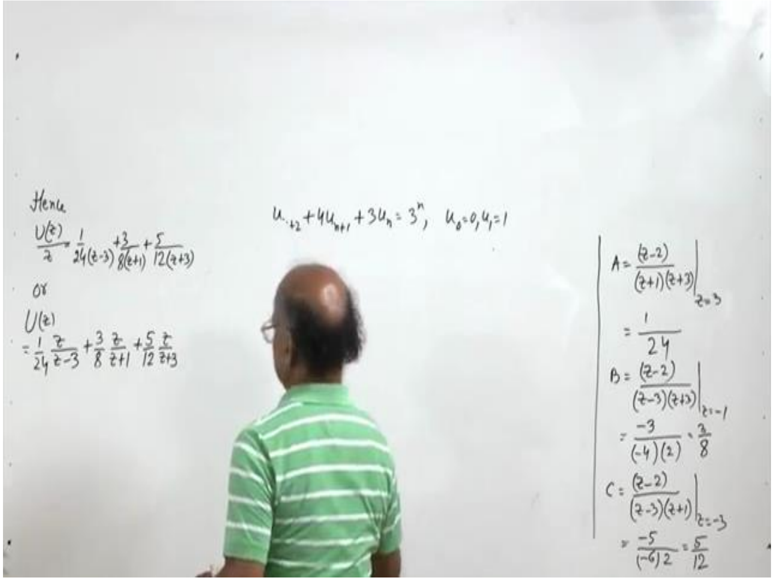

And hence u z by z is equal to 1 by 24 z minus 3, b is 3 by 8 z plus 1 and then we have 5 by 12 z plus 3. Now, we can multiply by z. So, this gives u z is equal to 1 by 24 z over z minus 3 plus 3 by 8 z over z plus 1 and 5 by 12 z over z plus 3. Now we can easily invert this.

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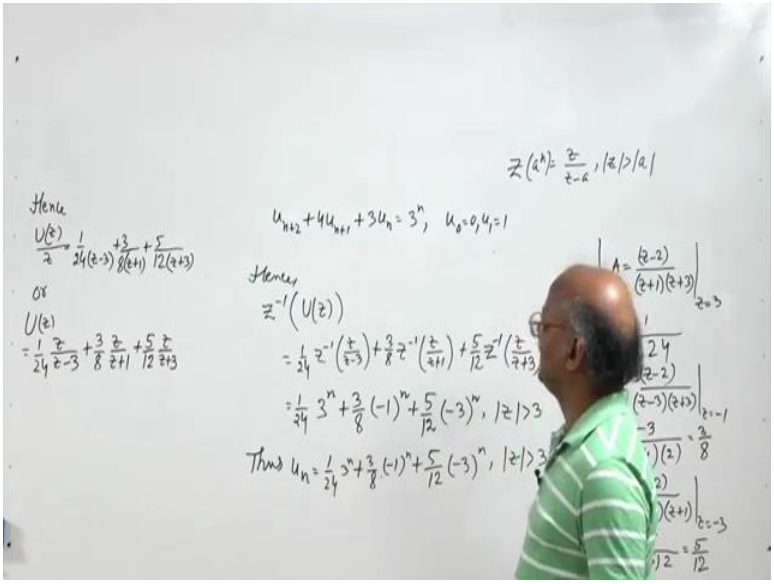

So, let us take the inverse Z-transform. So, this was U n plus 2 here. So, z then hence z inverse of u z is equal to 1 by 24 z inverse of z over z minus 3 plus 3 by 8 z inverse of z over z plus 1 and then we have 5 by 12 z inverse of z over z plus 3. Again let us recall that u z sorry z of a to the power n is z over z minus a, where mod of z is greater than mod of a let us apply this formula. So, z inverse of z over z minus 3, so this is 1 by 24 3 to the power n, where mod of z is greater than 3 and then 3 by 8 here minus 1 to the power a is minus 1 and here mod of z is greater than 1. So, mod of z is greater than 3 here, here mod of z is greater than 1, then 5 by 12 minus 3 to the power n. So, here a is minus 3 and therefore, mod of z is greater than 3. So, the common reason is mod of z greater than 3 and z inverse of u z is U n. So, thus U n is 1 by 24 3 to the power n 3 by 8 minus 1 to the power n plus 5 by 12 minus 3 to the power n where mod of z is greater than 3.

So, this is the answer to this difference equation by taking Z-transform, Z-transform takes care of the initial conditions, we do not have to use initial conditions after we have found this solution like we do in this classical approach. So, here this how we apply the transform techniques, first we take the Z-transform of the given difference equation. Collect the coefficient of u z, rest of the terms be tend to the right side divide by the coefficient of u z, and then break then you write u z by z then these expression of u z by z is then broken in to partial fractions. In such a way that, we while multiplying by z, we get the expressions in terms of the Z-transform of known Z-transforms. So, that while inverting we get easily the sequence U n. So, U n here can be obtained like this.

In our next lecture, we shall discuss some more problems, where we will have to be very careful while writing u z by z because there in the problems the expressions, we will have to arrange of u z by z, so that while inverting we get the sequence even very easily. So, we will do that in the next lecture. With that, I would like to conclude my lecture.

Thank you very much for your attention.