Mathematical methods and its applications
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Lecture - 44
Convergence of Z – transform

Hello friends, welcome to my lecture on convergence of Z transform. Before discussing the convergence of Z transform.

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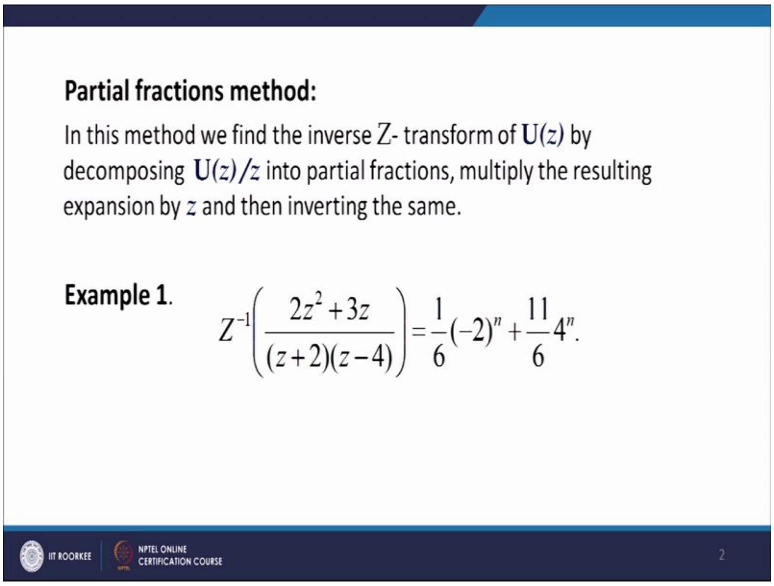

Let us look at some more methods where, we can find the Z transform, we can find the inverse Z transform of U z. The partial diffraction method we have discussed earlier also, here we have one more case where we can discuss the, we can find the inverse Z transform of U z. So, in this method as I have said in my previous lecture when we want to use the partial fraction method we consider, U z by z and break U z by z into partial fractions and then multiply the resulting expansion by z and then we invert that. So, suppose we have the example of U z equal to 2 z square plus 3 z over z plus 2 z minus 4 then, and we have to find the inverse Z transform of this function of z. So, then what we will do is.

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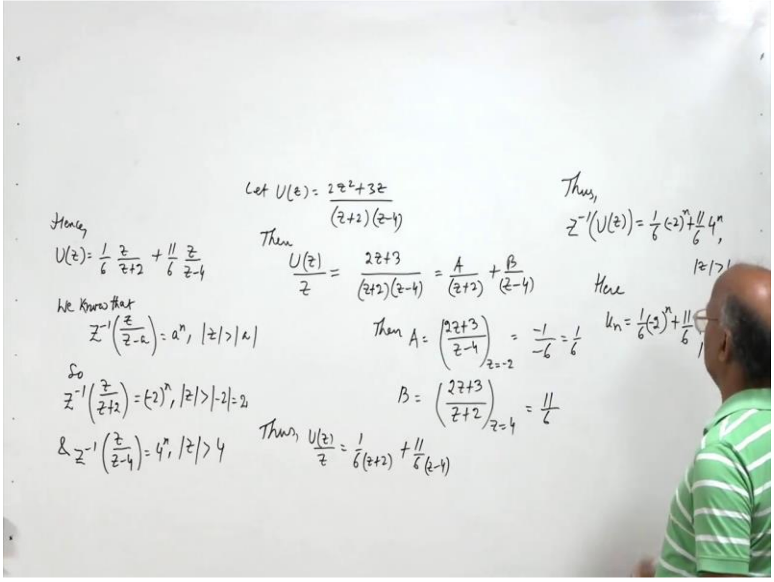

Let us say, let U z equal to 2 z square plus 3 z divided by z plus 2 z minus 4; so then we consider, U z by z. Let us write U z by z this will be 2 z plus 3 divided by z plus 2 z minus 4. We will break this 2 z plus 3 over z plus 2 z minus 4 into partial fractions.

So, let us write this as, A over z plus 2 and B over z minus 4 then, the values of A and B can be found easily. So, then A will be equal to 2 z plus 3 divided by z minus 4 evaluated at z equal to minus 2 and this will be giving you minus 4 plus 3. So, minus 1 divided by minus 2 minus 4. So, that is minus 6. So, we will get 1 by 6 and B will be equal to; similarly, 2 z plus 3 divided by z plus 2 and we will have to evaluate this at z equal to 4. So, when we put z equal to 4 here, 4 into 2; 8 plus 3 is 11 divided by 4 plus 2 is 6. So, B is 6. So, thus U z by z is equal to 1 by 6 z plus 2 and then 11 by 6 into 1 over z minus 4.

So, we will then multiply this equation by z. So, multiplying by z we get U z as, z over 1 over 6 into z over z plus 2 and then 11 over 6, z over z minus 4. Now let us recall that we know that Z inverse of z over z minus a, this is equal to a to the power n provided mod of z is greater than mod of a. So, Z inverse of z over z plus 2 will be equal to minus 2 to the power n provided mod of z is greater than mod of minus 2, which is 2. And similarly Z inverse of z over z minus 4 will be equal to 4 to the power n provided mod of z is greater than 4 and hence Z inverse of U z will be equal to 1 by 6, Z inverse of z over z plus 2 which is minus 2 to the power n and plus 11 by 6 z over z minus 4. In Z inverse of z over z minus 4 which is 4 to the power n and the common region is mod of the in case of, Z inverse z over z plus 2, the region of convergence was mod of z greater than 2 and here it is mod of z greater than 4. So, we will have this is valid when mod of z is greater than 4.

So, here u n is equal to 1 by 6, 2 to the power minus 2 to the power n and plus 11 by 6, 4 to the power n where, mod of z is greater than 4. So, this is how we can find the inverse Z transform in this case.

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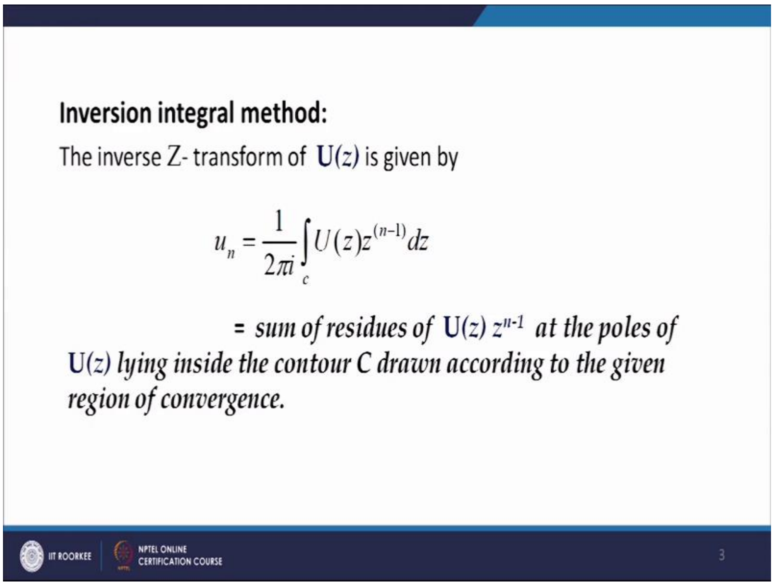

Let us, now there is another method which is known as inversion integral method. Here we will need analysis of complex analysis, knowledge of complex analysis that is the contour integration. So, I think I hope that you are familiar with this contour integration. So, if you know contour integration, you can use this method also to find the inverse Z transform. So, suppose we have sequence u n the inverse Z transform we know the Z transform of a sequence u n, it is given as U z. Then the inverse Z transform of U z is given by u n equal to 1 over 2 pi i integral over c U z into z to the power n minus 1, d z. Where, c is the contour which encloses the similarities of the function U z.

So, to evaluate u n, we calculate the sum of residues of the function U z into z to the power n minus 1 at the poles of U z which lie inside the contour c according to the given region of convergence. So, by the given region of convergence, we decide how many poles of U z lie inside the contour c and then calculate the residues at those poles of U z and take their sum to get the sequence u n.

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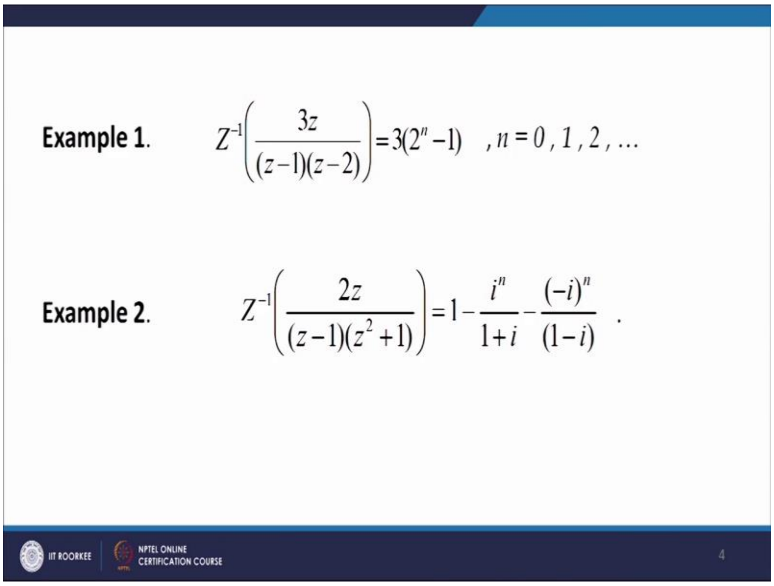

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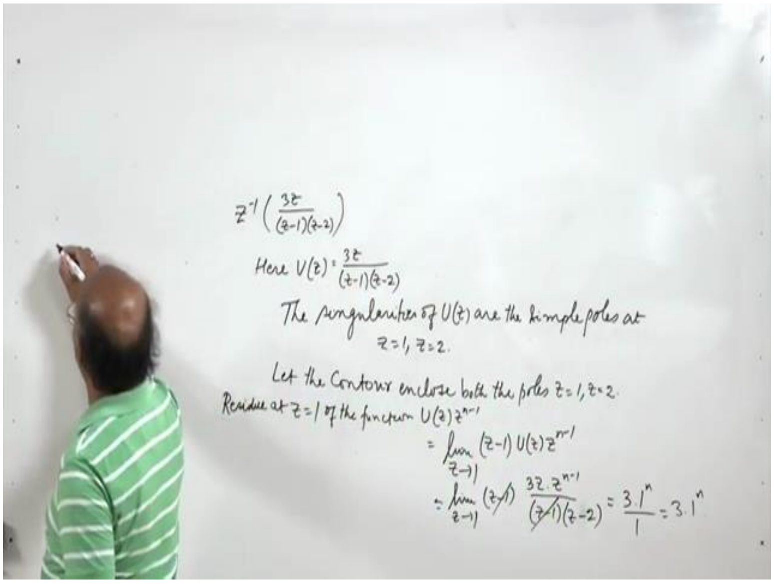

So, let us look at this method. Suppose we want to determine the inverse Z transform of 3 z over z minus 1, z minus 2. So, Z inverse of 3 z over z minus 1 z minus 2. Here let us say U z is equal to 3 z over z minus 1, z minus 2. So, let us find the poles of U z, the similarities of U z are the simple poles at z equal to 1 and z equal to 2. Now we are not given the region of convergence. So, we shall assume that the region of convergence is such that it encloses all the similarities. So, z equal to 1, z equal to 2 are enclosed inside the contour c. So, let us see the contour c enclose both the poles z equal to 1, z equal to 2 then, we have to find the residue at z equal to 1. So, residue at z equal to 1 of the function U z into z to the power n minus 1 is equal to limit z tends to 1, z minus 1 into U z into z to the power n minus 1.

We know that, when he function F z in complex analysis has a simple pole at the point z equal to a and we want to determine the residue there, then we multiply F z by z minus a and take the limit as z tends to a. So, here we multiply U z into z to the power n minus 1 by the factor z minus 1 and find the limit at z tends to 1. So, this is equal to limit z tends to 1, z minus 1 and then, U z is 3 z into z to the power n minus 1 divided by z minus 1, z minus 2.

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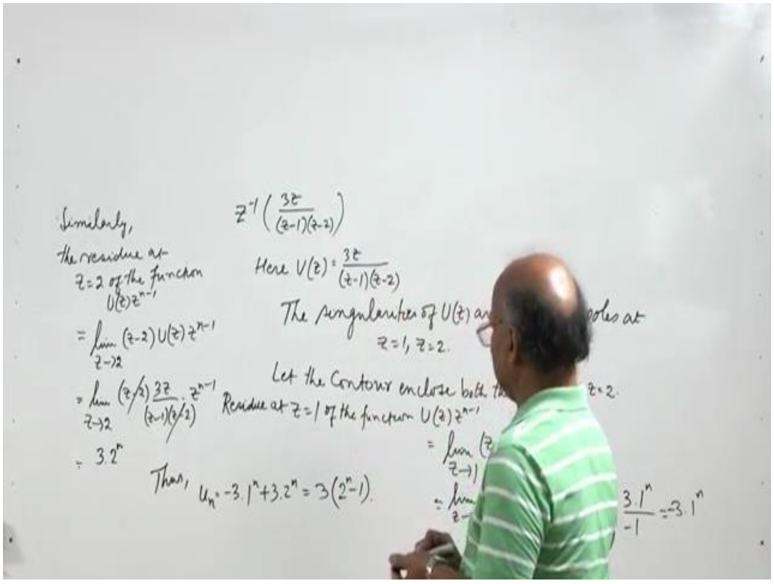

So, this z minus 1, we can cancel and when z tends to 1, it is 3 into 1 to the power n divided by 3 minus 2. So, 1, this is 3 into 1 to the power n and similarly limit z tends to 2, similarly the residue at z equal to 2 of the function U z into z to the power n minus 1 is equal to limit z tends to 2, z minus 2 into U z into z to the power n minus 1, which will be equal to limit z tends to 2, z minus 2 into 3 z divided by z minus 1, z minus 2 into z to the power n minus 1. So, this will cancel with this and when z tends to 2, what we get is 3 into 2 to the power n. So, thus we get, thus u n by the formula u n equal to sum of residues at the poles z equal to 1 and z equal to 2. So, 3 into 1 to the power n plus 3 into 2 to the power, here we are getting 1 minus 2 minus 1 we are getting. So, this will be minus. So, this is minus here, minus 3 into 1 to the power n plus 3 into 2 to the power n. So, we can write it 3 times 2 to the power n minus 1.

So, this is the, this how we can find the sequence u n by using the inversion integral method.

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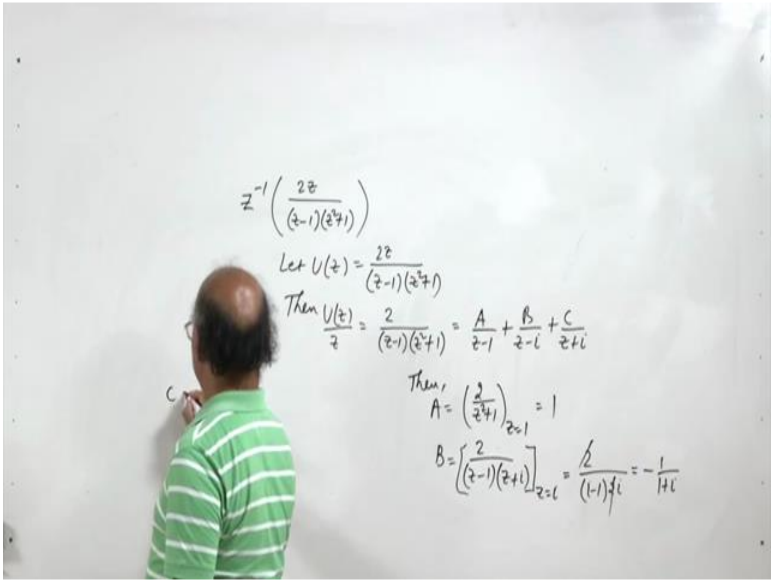

So, let us discuss one more example, Z inverse of 2 z over z minus 1, z square plus 1. Let us say U z be equal to 2 z over z minus 1 z square plus 1. So, then we write U z by z. So, then U z by z will be equal to 2 over z minus 1 z square plus 1, which we can write as A over z minus 1, B over z minus i and C over z plus i because z square plus 1 can be broken into linear factors z minus I, z plus i. Now, so then A will be equal to put z equal to 1 and 2 over z square plus 1. So, in 2 over z square plus 1 when, we put z equal to 1 we will get 2 over 2, so we get 1 then, we can find. So, 2 over z minus 1 and here z minus 2 we shall leave z plus i we will take.

So, this evaluated at z equal to I, this will give you 2 over i minus 1 into 2 i. So, we shall get, this 2 will cancel with this 2 and we shall get i square which is minus 1; so minus 1 minus i. So, minus 1 upon 1 plus i and similarly C will be equal to 2 over z minus 1 into z minus i at z equal to minus i.

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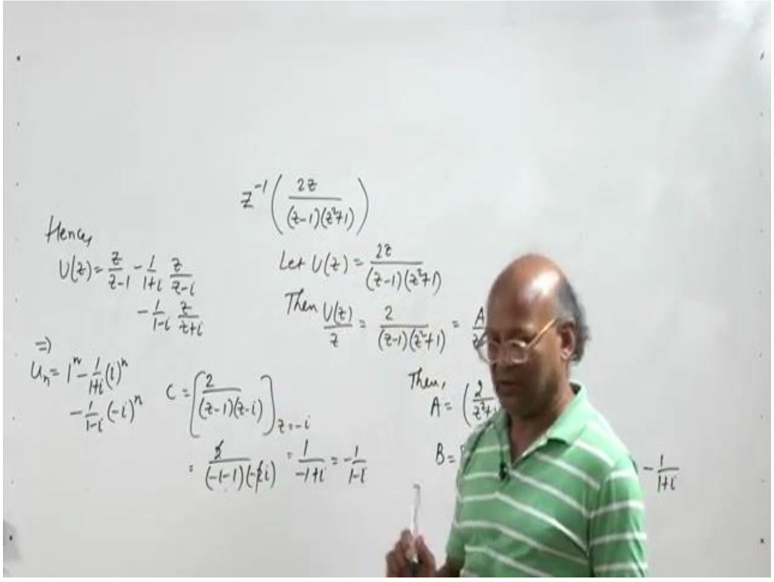

So, this will be equal to 2 over minus i minus 1 and then minus 2 i. So, 2 will cancel with 2 when, we multiply by minus i here. So, minus i into minus i is i square which is minus 1. So, minus 1 and then we get plus i. So, we get minus 1 upon 1 minus i and hence U z will be equal to z, we multiply in this equation and A is 1. So, z over z minus 1 we have and then we have B is equal to minus 1 over 1 plus i into z over z minus i and then C is minus 1 over 1 minus i. So, minus 1 over 1 minus i into z over z plus i and this implies u n, the inverse Z transform of U z will be z over z minus 1 means 1 to the power n because inverse Z transform of z over z minus i is a to the power n and then minus 1 over 1 plus i inverse Z transform here will be i to the power n.

So, here mod of z is greater than 1, here mod of z is greater than mod of I; mod of i is also 1. So, region of convergence is same mod of z greater than 1 and here we have minus 1 over 1 minus i into minus i to the power n and the region of convergence is mod of z greater than mod of minus I, which is again mod of z greater than 1. So, we get mod of z greater than 1. So, we have 1 minus i to the power n over 1 plus i minus i to the power n over 1 minus i where mod z is greater than 1. So, this is the solution of example 2. Now let us go to two sided Z transform. So, far we considered the Z transform of which was one sided Z transform where, one sided Z transform means we consider the values of n to be non negative, n equal to 0, 1, 2, 3 and so on and we considered U z equal to sigma n equal to 0 to infinity u n z to the power minus n, but n can take negative values also. So, when n takes values from minus infinity to plus infinity, the Z transform will be defined as sigma n equal to minus infinity to u n z to the power minus n.

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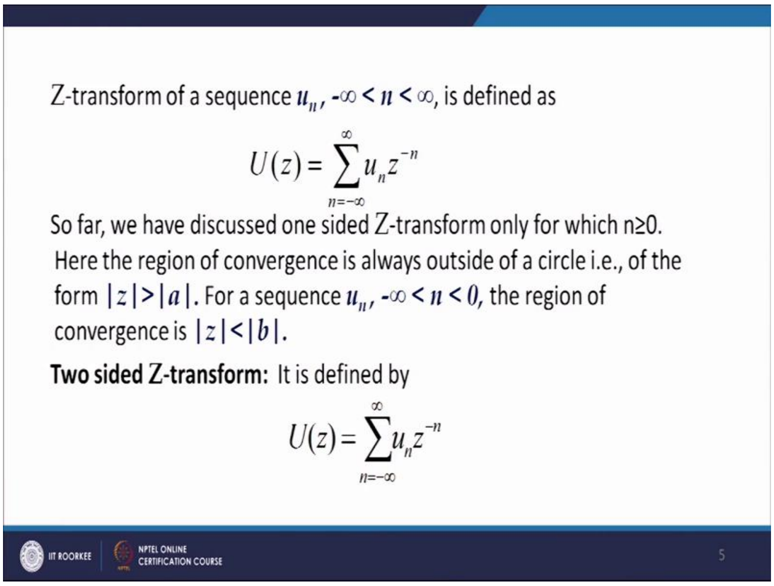

And this transforms then called as two sided Z transform. In the case of one sided Z transform where, we had taken n greater than or equal to 0. The region of convergence as we have seen is always of the form mod of z greater than mod of a which is the region outside the circle mod of z equal to mod of a, The circle where the center is at the origin and a dash is mod of a.

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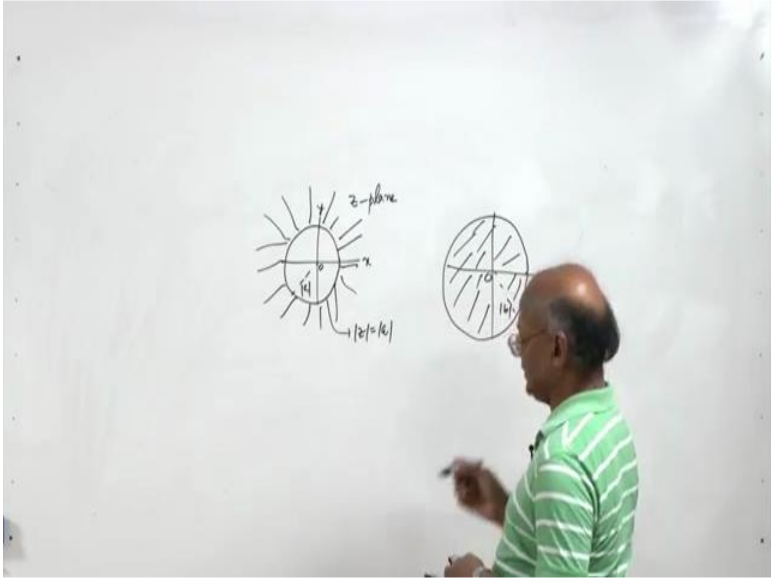

So, the region of convergence will look like this. If this is the circle with mod of with radius mod of a in the z plane then, the region outside the circle mod of z equal to mod of a is this shaded region. So, the region of convergence is always the circle is mod of z equal to mod of a.

So, in the case of the values of n greater than or equal to 0, the region of convergence is always outside the circle. It is of the form mod z greater than mod of a, but when n takes the negative values that is from minus infinity to 0, the region of convergence we shall see is of the form mod z less than mod of b. So, it will be of this form. Suppose this is your mod z equal to mod of b circle with center at origin and radius mod of b then, it is interior of this circle. The region of convergence lies to the interior of the circle. So, in the case of two sided Z transform for the negative values of the negative powers of z we get the region of convergence which is outside the circle for the power of z, which are positive we the region of convergence will look like, it is inside the circles. So, for the two sided Z transform, the region of convergence will be of the form mod of a less than mod of z less than mod of b that is a the region of convergence lying between two concentric circles.

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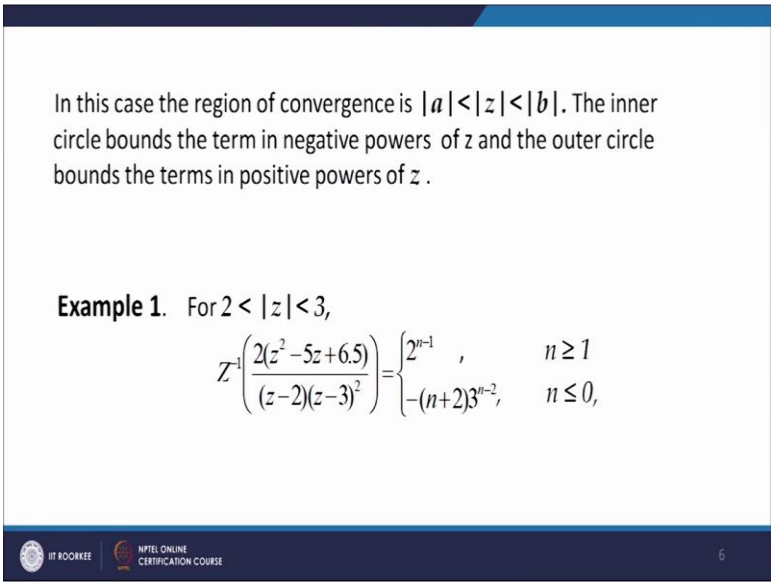

So, such a region of convergence is known as annular region and the inner circle than inner circle is mod of z equal to a mod of z equal to mod of a. So, it bounce the terms in negative powers of z and the outer circle which is mod of z equal to mod of b, it bounds the terms in the positive powers of z. So, for example, let us consider this 2 less than mod of z less than 3.

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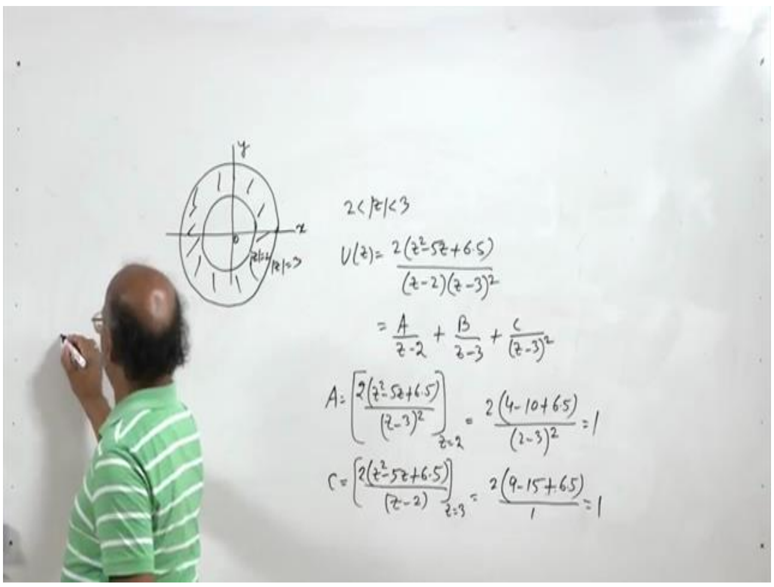

So, it is an annual region, the region concentric to two circles mod of z equal to mod of 2, mod of z equal to 2 and then mod of z equal to 3.

So, this region is the given region of convergence. So, here 2 is less than mod of z less than 3; let us find the inverse Z transform of the function U z equal to here. So, U z is equal to 2 times z square minus 5 z plus 6.5 divided by z minus 2, z minus 3 whole square. So, you can put it as A over z minus 2; let us break it into partial fractions, B over z minus 3 and then C over z minus 3 whole square. So, we can determine the values of A, B, C very easily; A is equal to 2 times z square minus 5 z plus 6.5divided by z minus 3 whole square evaluated at z equal to 2. So, let us put z equal to 2 here. So, then 2 times we have 4 minus 10 plus 6.5 divided by 2 minus 3 whole square. So, this is 1 and here we have 10.5 minus 10. So, we get 0.5; 0.5 into 2 is equal to 1 and similarly, we can find directly the value of C. C will be 2 times z square minus 5 z plus 6.5 divided by z minus 2 and let us put z equal to 3 in this. So, that we will get 2 times 9 minus 15 and then plus 6.5 divided by 3 minus 2 is 1.

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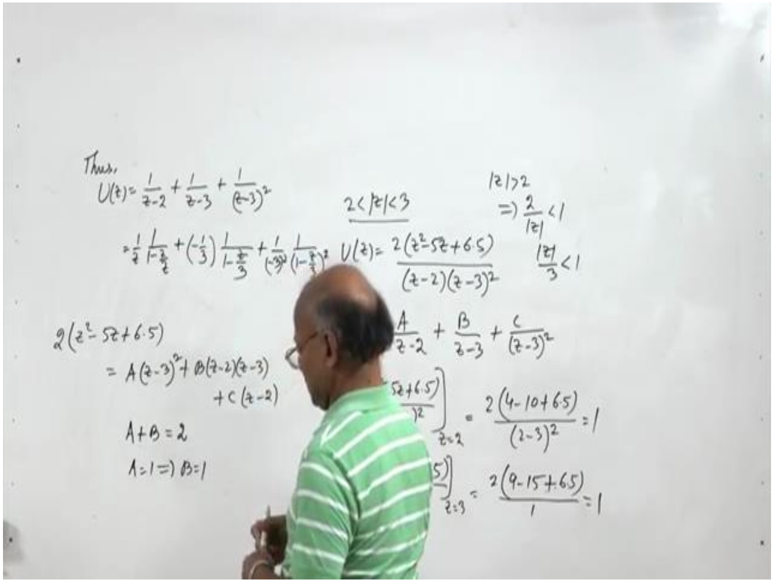

So, we get here 15.5 minus 15 is point 0.5; 0.5 into 2 is equal to 1. Now to determine B we have to write the identity. So, 2 times let us take the LCM, 2 times z square minus 5 z plus 6.5, this is equal to, I am taking LCM, this is equal to A times z minus 3 whole square plus B times z minus 2, z minus 3 plus C times z minus 2. Now let us equate the terms containing z square on both sides. So, here the term containing z square is A and then here B. So, A plus B, the coefficient of z square on the right side is A plus B and the left side it is 2. So, A plus B equal to 2, A we have got already equal to 1. So, A equal to 1 gives us B also equal to 1. So, A, B, C here are all equal to 1. So, thus we have. So, thus U z is equal to z over A, sorry 1 over z minus 2 and then 1 over z minus 3 and then 1 over z minus 3 square.

Now we are given the region of convergence as 2 less than mod of z less than 3. So, here we can see that mod of z greater than 2 implies, 2 over mod of z; 2 over mod of z is less than 1. So, what we will do here to expand 1 over z minus 2 by using the binomial theorem; this I can write as 1 over z times 1 over 1 minus 2 over z and mod of 2 over z is less than 1. So, we can expand it by binomial theorem and here this is mod of z is less than 3. So, mod of z divided by 3 is less than 1. So, what I will do is I will take minus 1 over 3 out and write it as 1 minus z upon 3.

So, that mod of z less than 3 gives us the infinite series expansion of 1 over minus z by 3. So, here again here we shall write 1 over minus 3 whole square and then I will write, 1 over 1 minus z by 3 whole square. So, now, let us expand by binomial theorem.

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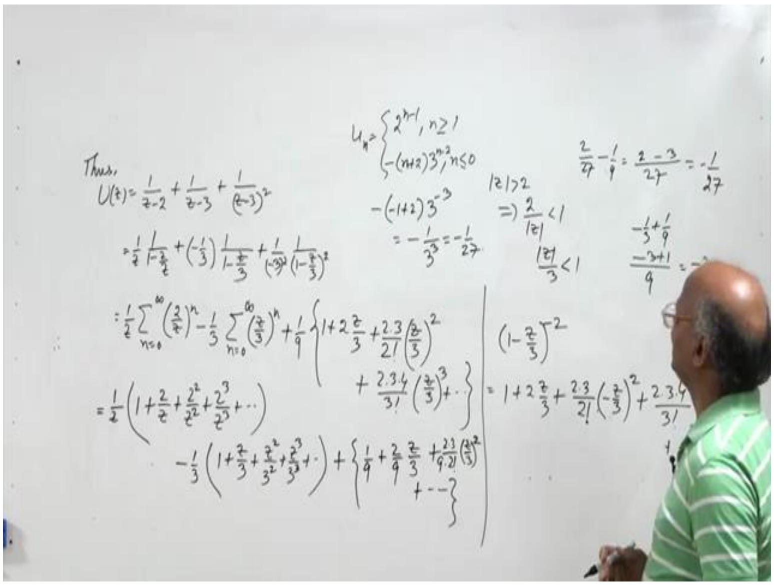

So, if we do that I will write it as 1 over z. Since mod of 2 by z is less than 1, I will write it sigma n equal to 0 to infinity 2 by z raise to the power n and then minus 1 by 3 again this is mod of z by 3 is less than 1.

So, sigma n equal to 0 to infinity z by 3 raise to the power n and here plus 1 by 9 summation n equal to 0 to infinity, I think I should write like, first I would write it in expanded form. So, I should write it as this is 1 minus z by 3 raise to the power minus 2. So, I will get here 1 plus 2 times z by 3 plus 2 into 3 by 2 factorial and then I will have z by 3, minus z by 3 whole square and then 2, 3, 4 divided by 3 factorial minus z by 3 whole cube and so on. Here, sorry minus, minus, minus and minus becomes plus. So, I will get this here. So, this will be equal to so this will be 1 plus 2 times z by 3 and then 2 into 3 by 2 factorial z by 3 whole square and then 2 into 3 into 4 divided by 3 factorial z by 3 to the power 3 and so on.

Now let us see at this, further let us expand this. So, 1 by z this is what, when n is equal to 0; first term is 1, then 2 by z we have, then we have 4 by z, 2 square by z square 2 cube by z cube and so on. And so these terms give us negative powers of z while these two series are in the positive powers of z. So, minus 1 by 3 and then here we have 1 plus z by 3 plus z square by 3 square, z cube by 3 cube and so on. This, the series in the second term; in the third term we get 1 by 9, I can multiply 1 by 9 or 1 by 9 into plus 2 by 9 into z by 3 and then we have 2 into 3 divided by 9 into 2 factorial z by 3 whole square and so on.

Now, let us determine the sequence u n from here. So, for n greater than or equal to 0 we will get the value of u n from this. So, because sequence u n, we have to find u n for the values of n greater than or equal to 0. So, when n is z to the power minus n, when n is equal to 1 and onwards. So, n greater than or equal to 1 gives you because when n is equal to 1, we get 1 here’ when n is equal to 2 here, we get 2 to the power 2 minus 1. So, we get 2 here, the coefficient of z to the power minus 2 is 2 and the coefficient of z to the power minus 3 is 2 square. So, u n is 2 to the power minus n and we get here, minus n plus 2 into 3 to the power n minus 2. So, this is for n less than or equal to 0. Let us take for n equal to 0, what do we get? When n is equal to 0, we have here minus 2 over 3 square. So, this is minus 2 over 3 square; minus 2 over 3 square means minus 2 over 9 and here what do we get, minus 1 over 3 plus 1 over 9.

So, this is 9 minus 3 plus 1, so minus 2 over 3 square. So, here also we get minus 2 over 3 square and the coefficient of then z, z here is minus 1 by 3 square and here it is 2 by 3 cube. So, coefficient of z is 2 by 27 minus 1 by 9, which is we can say it is 6 minus 1; 6 minus 3. So, we get 3 by 27 or 1 by 9 and what we get here? I think minus 1 by 9 and here we get 2 by 27, 2 by 27 minus 1 by 9. So, we get here 27, this will be 2 here and here we will get minus 3. So, we will get minus 1 by 27 and here what we get? So, when n is equal to minus 1, it will be minus minus 1 plus 2 and 3 to the power minus 3. So, this will be minus 1 over 3 cube which is minus 1 by 27. So, the terms which contain positive powers of z there u n is minus n plus 2, 3 to the power n minus 2 and when the negative powers of z give you, u n equal to 2 to the power n minus 1 where, n is bigger than or equal to 1.

So, in the case of this annular region which, we will get the two sided Z transform and the sequence un then is two sided. For n greater than or equal to 1, we get 2 to the power n minus 1 and for n less than or equal to 0, we get minus n plus 2, 3 to the power n minus 2. We shall be applying the knowledge Z transform to the difference equations, as we know that the Laplace transforms are very useful in solving linear differential equations, we shall show in our next two lectures. The Z transforms are also very useful to solve liner difference equations. When we have a discrete system, the performance of the discrete system is expressed by a difference equation and Z transforms are very useful in the analysis and representation of discrete time systems. So, the solution of the difference equation is required to determine for frequency response of such a discrete system. So, in our next two lectures, we shall be discussing, how we can solve a difference equation by applying the Z transforms. With that I would like to conclude my lecture.

Thank you very much for your attention.