Mathematical methods and its applications

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Lecture – 43
Convolution theorem for Z – transforms

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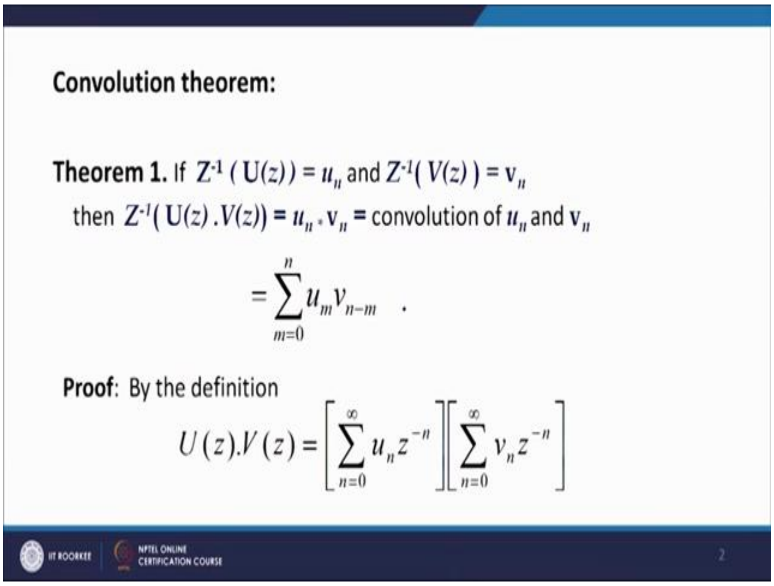

Hello friends, welcome to my lecture on convolution theorem for Z transforms. We will begin with the convolution theorem which says that if we know the Z transforms of 2 sequences say u n and v n, Z inverse U z is equal to u n and Z inverse V z is equal to v n. Then the theorem says that Z inverse of U z into V z is equal to the convolution of u n and v n and the convolution of u n and v n is denoted by u n star v n and this is defined as sigma m equal to 0 to n v n minus m.

A similar theorem on convolution is there in the Laplace transform where we say that if we know the Laplace transforms of 2 functions then Laplace transform of their convolution is equal to the a product of their Laplace transform. So, here also if we have 2 sequences u n and v n then Laplace transform Z transform of their convolution is equal to product of their Z transform. So, to prove this let us by definition of the Z transform.

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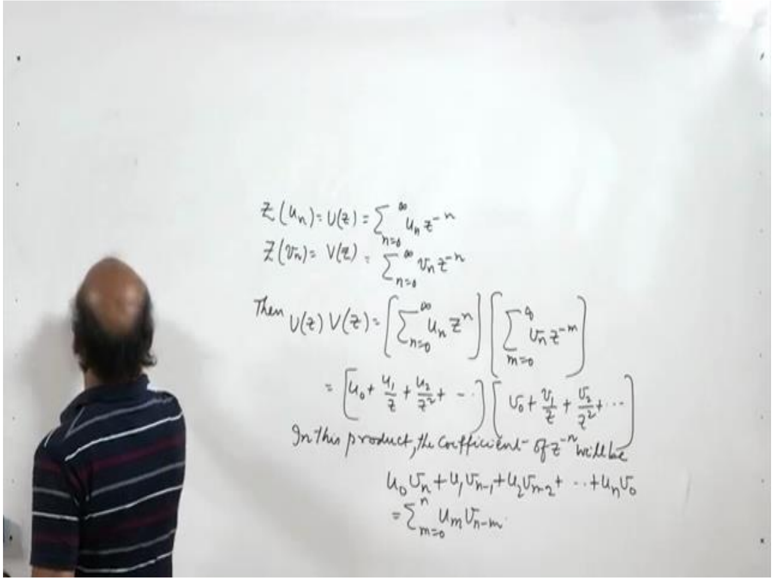

Since U z is the transform of u n Z transform of u n sequence is equal to U z, we are given and Z transform of the sequence v n where given as V z. So, we can see that this is equal to sigma n equal to 0 to infinity u n z minus z to the power minus n and this is sigma n equal to 0 to infinity v n z to the power minus n.

So, when you multiply U z and V z, what we will have then U z into V z will be equal to sigma n equal to 0 to infinity U z u n z to the power minus n. this is U z and V z over the in V z we will take the summation over some other index let us say m equal to 0 to infinity v n z to the power minus n. So, this is U z into V z and then we can write it in the expanded form. So, this is u naught plus u 1 by z plus u 2 by z square and so on and here it is v naught plus v 1 by z plus v 2 by z square and so on. Now when you multiply; these 2 are infinite series. So, when you multiply we can collect the coefficient of z to the power minus n, collecting the coefficient of z to the power minus n what we will have is the coefficient of in this product, the coefficient of z to the power minus n will be u naught v n plus u 1 v n minus 1 u 2 v n minus 2 and so on u n v naught. This can be written as sigma m equal to 0 to m v n minus m.

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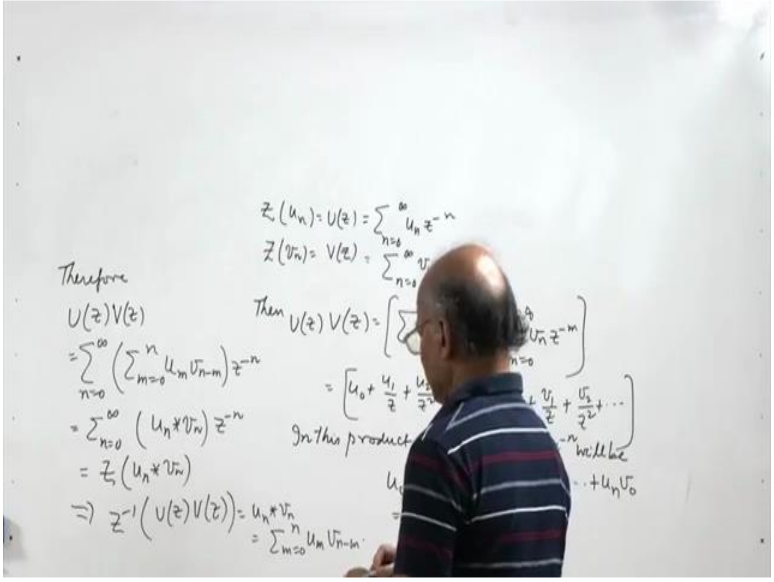

So, this is the coefficient of z to the power minus n when you multiply these 2 infinite series and with this we can write it as, therefore, U z into V z; U z into capital V z may be written as summation n equal to 0 to infinity and then this expression sigma m equal to 0 to n u n b n minus n into z to the power minus n. So, the coefficient of z to the power minus n is this sum which is given by the summation notation sigma m equal to 0 n u n b n minus m and by our definition this sum actually denotes the convolution of the sequences u n and v n. So, this is Z transform of u n and v n. So, this from here we can say that Z inverse of U z into V z is equal to u n star v n and which is equal to summation m equal to z to n v n minus m. So, when we know the Z transforms of 2 sequences u n and v n, the Z transformer their convolution is equal to product of their Z transforms. So, this is the convolution theorem. Now let us apply this theorem to find the Z transforms.

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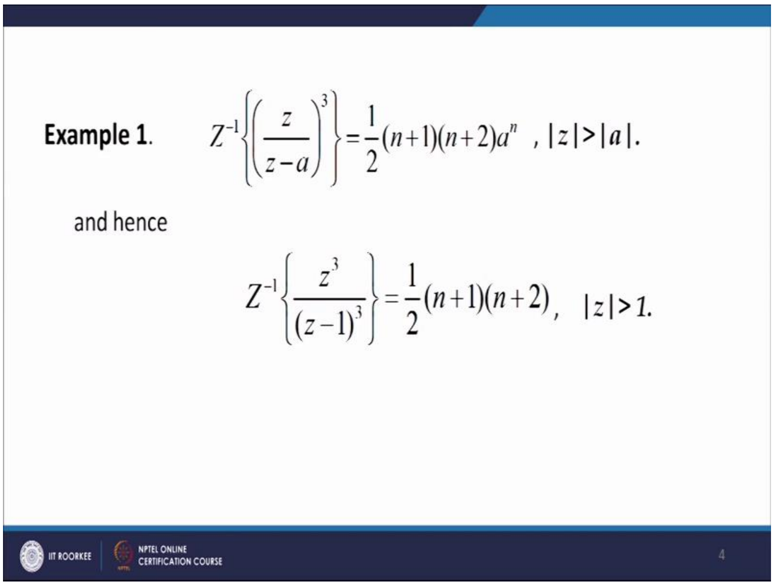

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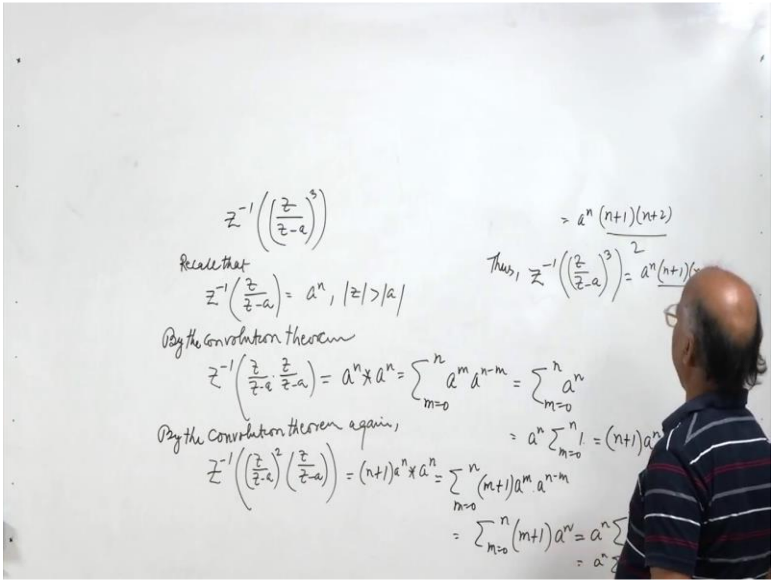

So, let us see suppose we want to find the Z transform of Z inverse of z over z minus a whole cube Z inverse of z over z minus a whole cube. This is our aim; we have to find the inverse Z transform of z over z minus a whole cube. So, let us recall that Z inverse of z over z minus a, this is equal to a to the power n provided mod of z is greater than mod of a. So, we know Z transform inverse Z transform of z over z minus a is a to the power

n. So, by the convolution theorem, Z inverse of z over z minus a into z over z minus a. So, this is U z this is V z; Z inverse of U z into V z is convolution of u n and v n. So, a n is star a n convolution of a n be the a n which is equal to sigma m equal to 0 to n um is a to the power m into b n minus m.

So, a to the power n minus m. this is this is b m, we have here; here we have b n minus m and this is equal to sigma m equal to 0 to n. So, a to the power n now a to the power n is independent of the index m. So, a to the power n can be written outside this is sigma m equal to 0 to n applied to 1. So, this is 1 summing 1 is summed n plus 1 times. So, this is and plus 1 a to the power n now. So, this is inverse Z transform of z over z minus a whole square we have to again apply the convolution theorem. Now by the convolution theorem again Z inverse of z over z minus a whole square into z over z minus a it is convolution of n plus 1 a to the power n and a to the power n n plus 1 a to the power n is the inverse Z transform of z over z minus a whole square and a to the power n is inverse Z transform of z over z minus a.

So, this is equal to summation m equal 0 to n we can we can write it as this we can write as a am and this we can write as a n minus we can choose any sequence to write am. So, m plus 1 a to the power m and into a n minus m. So, this is b n minus m. So, a to the power m a to the power n m will cancel and we will have summation m equal to 0 to n m plus 1 a to the power n a to the power n is independent of m. So, a to the power n can be written outside then summation m equal to 0 to n plus 1. Now replacing m plus 1 by j i can write it to summation a to this I can write as a to the power n summation j equal to 1 to n plus 1 n sorry j. So, this is summation of j when j runs from 1 to n plus 1 and we know the value of this a to the power n sigma j equal to 1 to n j is n into n plus 1 by 2. So, it is n plus 1 into n plus 2 by 2.

So, that is the Z inverse Z transform of. So, thus Z inverse of z over z minus a whole cube this is equal to a to the power n and plus 1 plus 2 by 2 when mod of z is greater than mod of a now in particular inverse Z transform of z cube over z minus whole cube can be obtained by taking a equal to 1 here. So, inverse Z transform of z cube over z minus 1 whole cube is 1 by 2 n plus 1 into n plus 2 provided mod of z is greater than 1 let us take some more examples on convolution theorem suppose we want to find inverse Z transform of z square over z minus 1 z minus 3.

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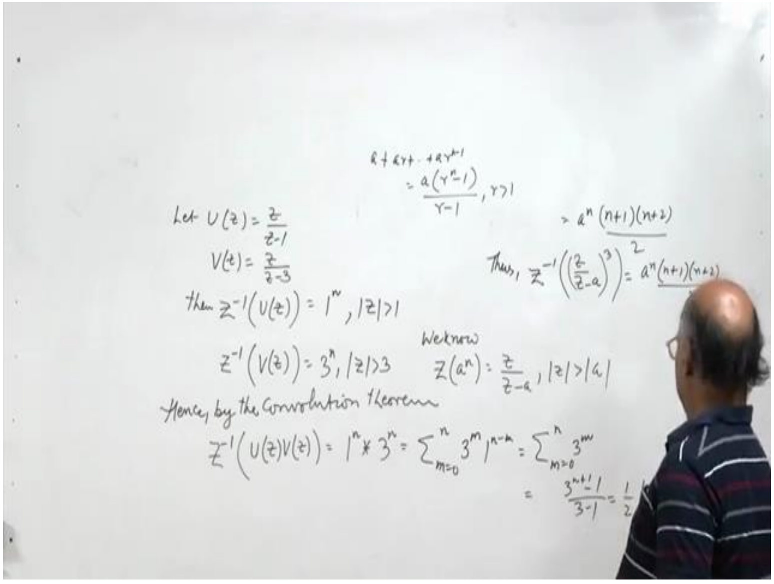

So, let us say; let U z be equal to z over z minus 1 and V z be equal to z over z minus 3 then we know that inverse Z transform of U z is equal to 1 to the power n provided mod of z is greater than 1 because we know that Z transform of a to the power n is z over z minus a provided mod of z is greater than mod of a.

So, taking a equal to 1, here we get that Z inverse of z over z minus 1 is 1 to the power n and here and similarly Z inverse of V z will be equal to Z inverse of V z will be equal to this is z over z minus 3. So, we will take a equal to 3 here. So, this is 3 to the power n provided mod of z s greater than 3. So, we know the inverse Z transforms of z over z minus 1 and z over z minus 3. So, inverse Z transform of hence by the convolution theorem inverse Z transform of U z into V z will be equal to convolution of u n with v n. So, the required Z inverse Z transform that is Z inverse of the z into V z is their convolution here. So, sigma m equal to 0 to n 1 to the power you can write 3 to the power m, this I can write as 3 to the power m that I can write it as b n minus m, 1 to the power n minus m.

So, 1 to the power n minus m which is summation m equal to 0 to n 3 to the power, now this is a geometric series and here the first term is 1. So, the total numbers of terms are n plus 1. So, and the geometric ratio is 3. So, we apply the formula we know that a plus a r and so on, a r to the power n minus 1. This has some a times r to the power n minus 1 divided by r minus 1 when r is greater than 1. So, here we have r is 3. So, this is a; a is equal to 1. So, we have 3 to the power n plus 1 minus 1 divided by 3 minus 1. So, this is half of 3 to the power n plus 1 minus 1. So, that is inverse Z transform of z square over z minus 1 z minus 3, 1 by 2 z 3 on and the reason of convergence here will be the region that is common to both the inverse Z transforms inverse Z transform of U z has common the region of convergence has mod of z greater than 1 here have mod of z is greater than 3. So, the common region is mod of z greater than 3. So, the region should be mod of z greater than 3. Now let us take up another question, here we have 2 sequences 1 sequence u n is 1 by n factorial that there is sequence v n is also 1 by n factorial and we have to show that the convolution of u n with v n or you can say u n with v n is gives you 2 to the power n by y factorial.

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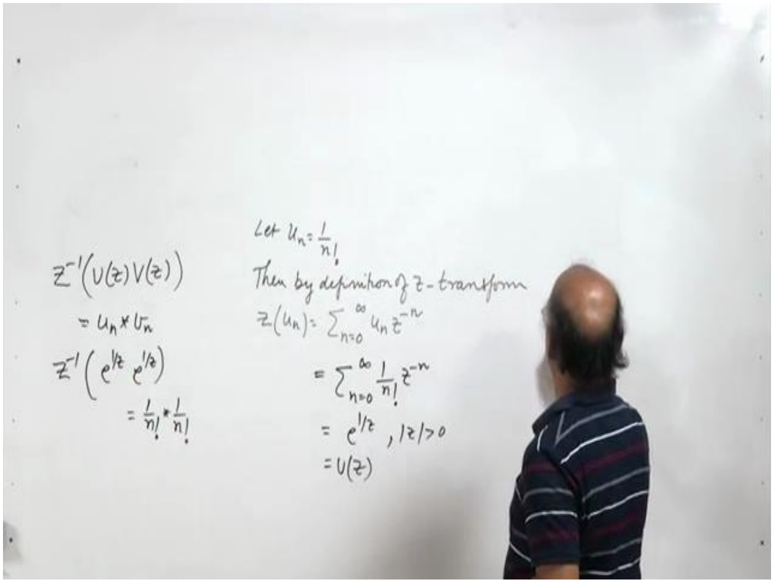

So, let us say; let u n be equal to 1 by n factorial then we know that by def y z by definition of Z transform; the Z transform of u n is equal to sigma n equal to 0 to infinity u n z to the power minus n. So, this will give you a summation n equal to 0 to infinity, u n is 1 by 1 factorial.

So, 1 by n factorial z to the power minus n and this is nothing, but e to the power 1 by z provided mod of z is greater than 0. So, the region of convergence here is mod of z greater than 0 and e to the power 1 by z the exponential function e to the power 1 by z is expanded into the Maclaurin series n equal to 0 to infinity into the Maclaurin series n equal to 0 to infinity 1 by n factorial z to the power minus n now let us find that Z inverse of U z into V z. So, this is U z now Z transform Z inverse of U z into V z we know, this is equal to u n star v n. So, let us take V z to be same as U z. So, Z inverse of U z U z is e to the power 1 by z, V z also is e to the power 1 by z, this is equal to u n into star v n. So, 1 by n factorial star 1 by n factorial and now we see that Z inverse of e to the power 2 by z; Z inverse of e to the power 2 by z.

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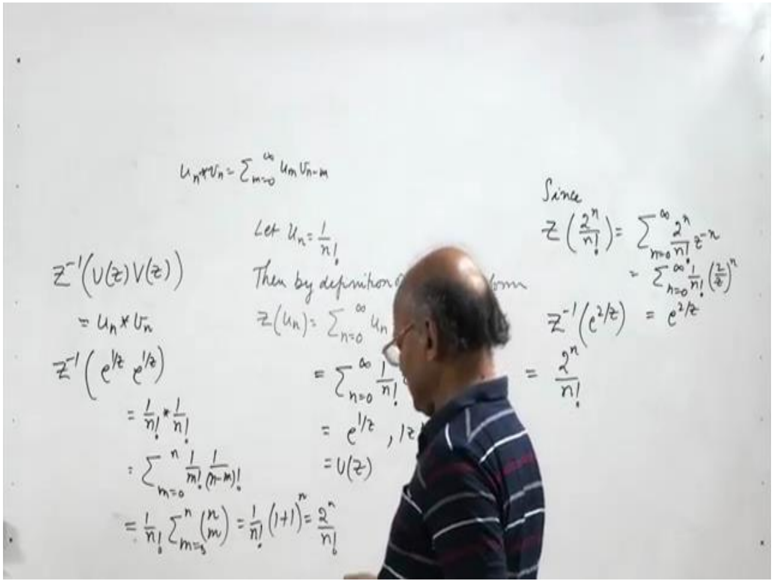

So, it is z of 2 to the power this is 2 to the power n by n factorial we have because since Z inverse z of 2 to the power n by n factorial if we take this sequence this is summation n equal 0 to infinity 2 to the power n by factorial z to the power minus n.

So, I can write it as summation n equal to 0 to infinity of 1 by n factorial 2 by z to the power n and therefore, it is e to the power 2 by z. So, Z inverse e to the power 2 by z is nothing, but 2 to the power n by n factorial. So, we have we have seen here that since e to the power 1 by z was equal to sigma n equal to 0 to infinity, 1 by n factorial z to the power minus n from here one can guess that e to the power 2 by z will be sigma n equal to 0 to infinity 1 by n factorial 2 by z to the power minus n and therefore, this sequence that we will get and taking the inverse Z transform of e to the power 1 by z will be 2 to the power n by factorial.

So, that gives us an idea about the inverse Z transform of e to the power 2 by z hat we can see from here also the by definition of convolution u n star v n is sigma m equal to 0 to n which is 1 by n factorial into v n minus m. So, n minus m factorial by definition of convolution we can see his we know that u n star v n this is summation m equal 0 to infinity v n minus m. So, I have just put the values here now I can multiply and divide by n factorial.

So, if we do that 1 by n factorial summation m equal to 0 to n n c m we can write multiplying and dividing by n factorial and this is nothing, but the sum of the binomial coefficients n c naught plus n c 1 plus n c 2 and so on, n c n is given by 1 plus 1 to the power n which is 2 to the power n by n factorial. So, the convolution of 1 by n with 1 by n 1 by n factor 1 by n factorial will give you 2 to the power n by n factorial from the inverse Z transform when we apply inverse Z transform here we know that u; u to the power 1 by z is I think I did not; I should not have done that rather we can see that here yeah Z inverse u n star I u n star v n is 1ne by n factorial start 1 by n factorial and this we did; we do not have to do this that is Z inverse of U z, V z, U z into V z and which is e to the power 2 by z and Z inverse e to the power 2 by z is nothing, but 2 to the power n by n factorial. So, this is 2 to the power n by n factorial we get. So, this is what we have and with that I would like to conclude my lecture.