Mathematical methods and its applications

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Lecture - 42
Initial and final value theorems for Z – transforms

Hello friends, welcome to my lecture on initial and final value theorems for Z transforms. In practical applications often we need the values of u n for n equal to 0 and n tends to infinity without actually requiring the complete knowledge of the sequence u n. We know that the sequence u n is related to the Z transform u z. So, by using that relation relationship, we can find the values of u n for n equal to 0 and n tends to infinity without actually knowing the sequence u n. So, suppose the Z transform of u n sequence is known it is given by u z, then it tells out that the value of u n for n equal to 0 is given by limit of U z as z tends to infinity.

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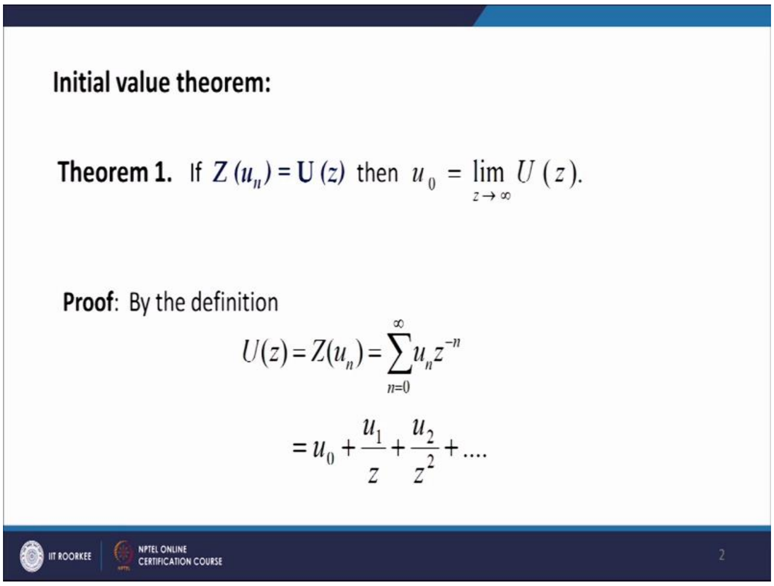

The proof is quite simple. We can see the definition of the z transform, U z is given by which is Z transform of the sequence u n is given by sigma n equal to 0 to infinity u n z to the power minus n which when expressed as u naught plus u 1 z plus u 2 y z square and so on taking the limit as z tends to infinity gives us the value of u naught as limit z tends to infinity u z.

So, when we take the limit of this equality as z tends to infinity, we get limit z tends to infinity U z as u naught.

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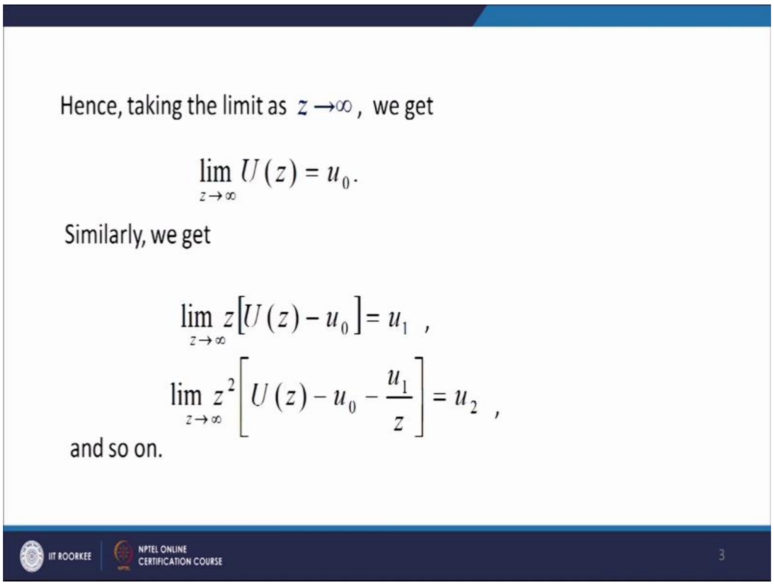

So, this limit of U z as z tends to infinity gives us the value of u naught. Now similarly we can see that limit z tends to infinity z times z minus u naught is equal to u 1.

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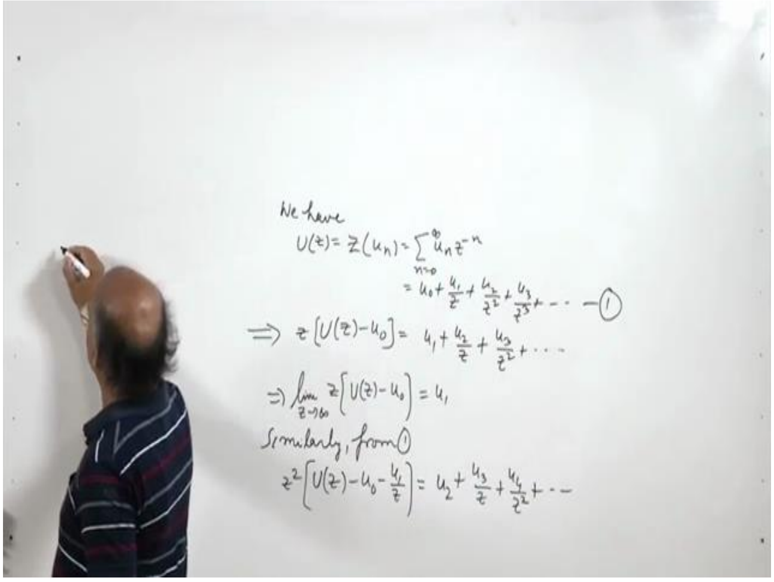

So, in order to see this we have U z equal to z of u n sequence which is sigma n equal to 0 to infinity u n z to the power minus n which is u naught plus u 1 by z plus u 2 by z square and so on.

So, from here we see that U z minus u naught multiplied by z, this is equal to u 1 plus u 2 by z plus u 3 by z square and so on and so taking the limit as z tends to infinity z times U z minus u naught gives us the value of u 1. Similarly from equation 1, we can see that U z minus u naught minus u 1 by z; U z minus u naught minus u 1 by z when it is multiplied by z square gives us u 2 plus u 3 by z plus u 4 by z square and so on.

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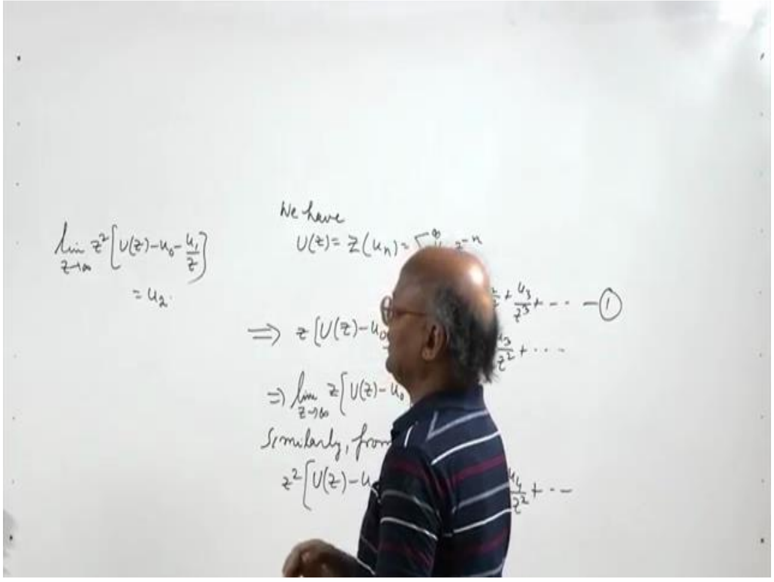

So, taking the limit again as z tends to infinity, we get limit z tends to infinity z square times U z minus u naught minus u 1 by z equal to u 2 and so on. So, this is how we can get the initial value of the sequence u n by its z transform.

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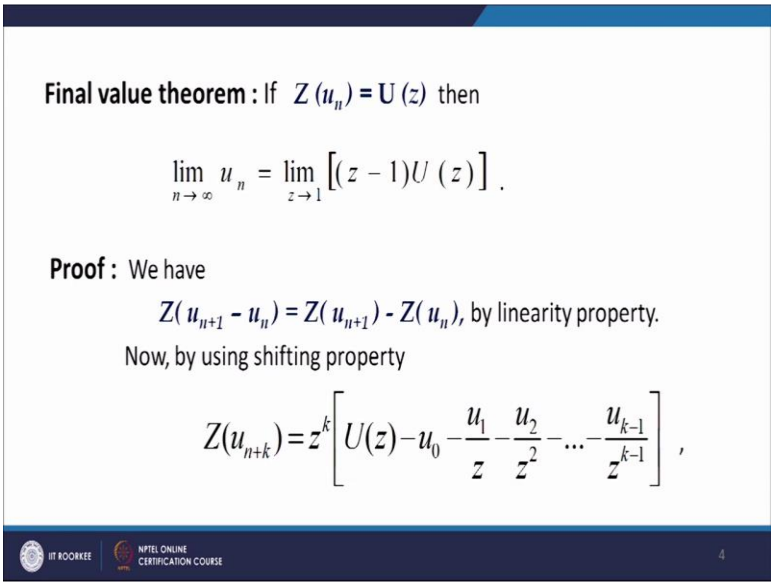

So, let us now talk about the final value theorem, here we will get the value of u n as n tends to infinity. So, if the Z transform of the sequence u n is known say it is U z then this result tells us that the final value of the sequence u n that is u n as n tends to infinity is given by the limit of z minus 1 U z as z tends to 1. Let us look at the proof of this final value theorem. So, we can take the Z transform; suppose we take the Z transform of the sequence u n plus 1 minus u n then by linearity property, we can write it as Z of u n plus 1 minus Z of u n and then we use the shifting property here, shifting to the right shifting of the sequence u n to left. So, Z of u n plus k is z to the power k into U z minus u naught minus u 1 by z minus u 2 by z square and so on minus u k minus 1 z to the power k minus 1.

If we use this shifting property then z of u n plus 1 will be equal to z times U z minus u naught. So, we will get z of u n plus 1 equal to z times U z plus u naught.

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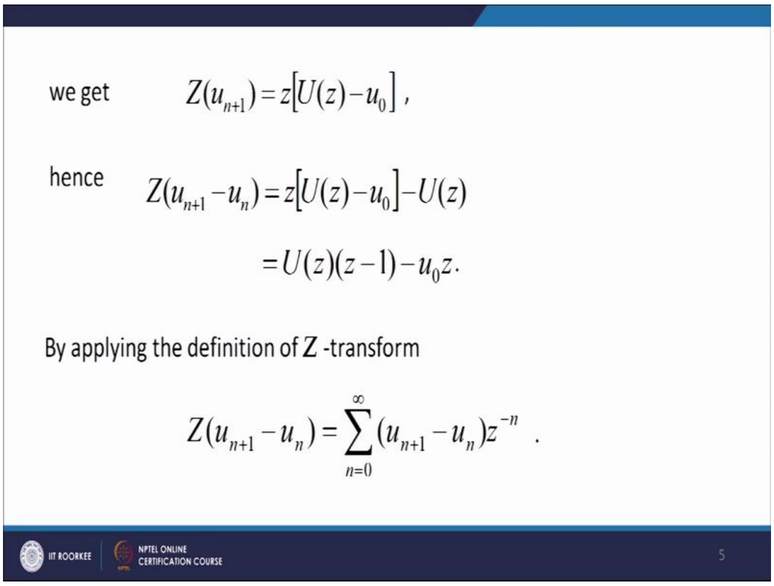

So, let us substitute the value of z of u n plus 1 here. So, z of u n plus 1 minus u n is z times U z minus u naught minus U z and then we can take; write this further as z times z minus 1 minus u naught z.

Now, by definition of Z transform; Z transform of the sequence u n plus 1 minus u n is equal to sigma n equal to 0 to infinity u n plus 1 minus u n into z to the power minus n as z tends to 1 then we will have limit z tends to 1 z minus 1 U z minus u naught as limit z tends to 1 n equal to 0 to infinity u n plus 1 minus u n z to the power minus n.

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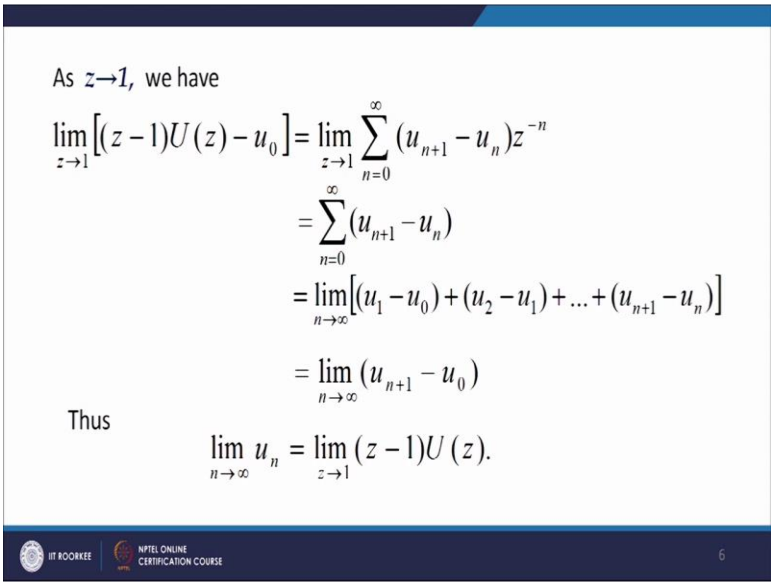

Because z minus 1 into U z minus u naught z is equal to this. So, from this equation and this equation, we arrive here as z tends to 1, limit of this expression is equal to limit z tends to 1 this. So, when z tends to 1, it will tend to sigma n equal to 0 to infinity u n plus 1 minus u n which can be expressed as limit n tends to infinity the its partial sum and partial sum. So, u 1 minus u naught minus u 2 minus u 1 and plus u 1 u n minus u n plus 1 minus u n and this will reduce further to u n plus 1 limit n tends to infinity, u n plus 1 minus u naught which is equal to limit of u n as n goes to infinity minus u naught because u naught is independent of n. So, then we can cancel and the left hand side is limit z tends to 1 z minus 1 U z minus u naught because u naught is independent of z. So, we can cancel u naught on both sides and then we get limit of u n as limit z tends to 1 z minus 1 u z. So, this is the proof of the final value theorem.

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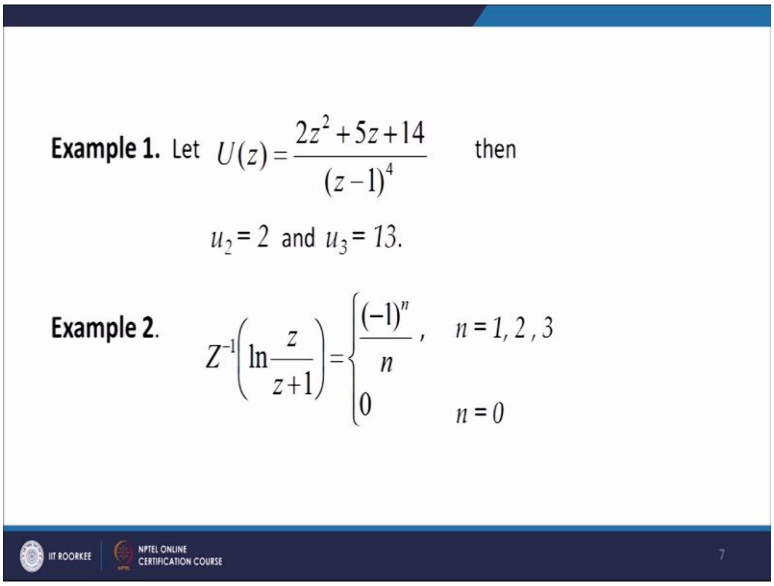

Let us use this initial value theorem to find the initial values of the sequence u n when u its Z transform is known. So, let say U z is known to us and we want to determine the value of u n for n equal to 2 and n equal to 3.

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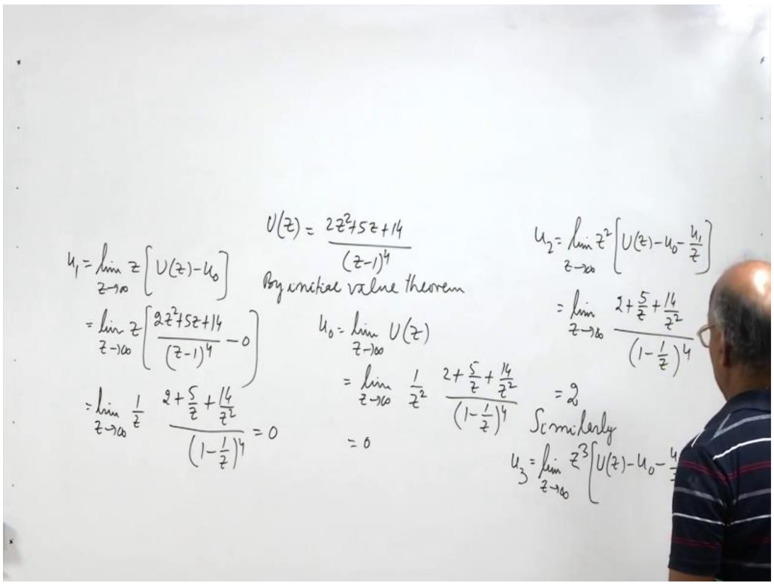

So, let us see how we get this U z is equal to 2 z square plus 5 z plus 14 divided by z minus 1 to the power 4 and the initial value by initial value theorem initial value theorem u naught, the initial value of the sequence u n naught is limit z tends to infinity; z tends to infinity u z. So, this is equal to now U z can be written as in the numerator. We have polynomial of degree 2 in z and the denominator has a polynomial in z of degree 4. So, I can write it as 1 by z square 2 plus 5 by z plus 14 by z square divided by 1 minus 1 by z to the power 4 on taking z square common from the numerator and z to the power 4 common from the denominator.

So, this will get we will get this now as z tends to infinity his expression tends to 2 this tends to 1. So, his whole thing tends to 2 and this goes to 0. So, the limit is 0. So, we get the value of u naught as 0. Let us now find the value of u 1, u 1 is equal to limit z tends to infinity, z times U z minus u naught. So, this is limit z tends to infinity z times 2 z square plus 5 z plus 14 by z minus 1 to the power 4 minus 0. So, U z can be expressed in this form when we multiply this form by z, what we get is 1 by z. So, this z cancels with 1 z here and we get 1 by z 2 plus 5 by z plus 14 by z square divided by 1 minus 1 by z to the power 4 again this expression tends to 2 and this then goes to 0. So, we get the limit as 0.

Now, let us find u 2. So, u 2 is limit as z tends to infinity, z square times U z minus u naught minus u one by z u naught and u one both are 0s and when we multiply U z by z square what we get is limit z tends to infinity 2 plus 5 by z plus 14 by z square 5 by z divided by 1 minus 1 by z to the power 4 and when z goes to infinity this limit clearly is 2. So, we get the value of u 2 as 2 we can similarly find u 3.

So, similarly u 3 is equal to limit z tends to infinity; z cube times U z minus u naught minus u 1 by z minus u 2 by z square. Now we have obtained u naught as 0, u 1 as 0, u 2 equal to 2 by z; u 2 equal to 2. So, we can see find the limit.

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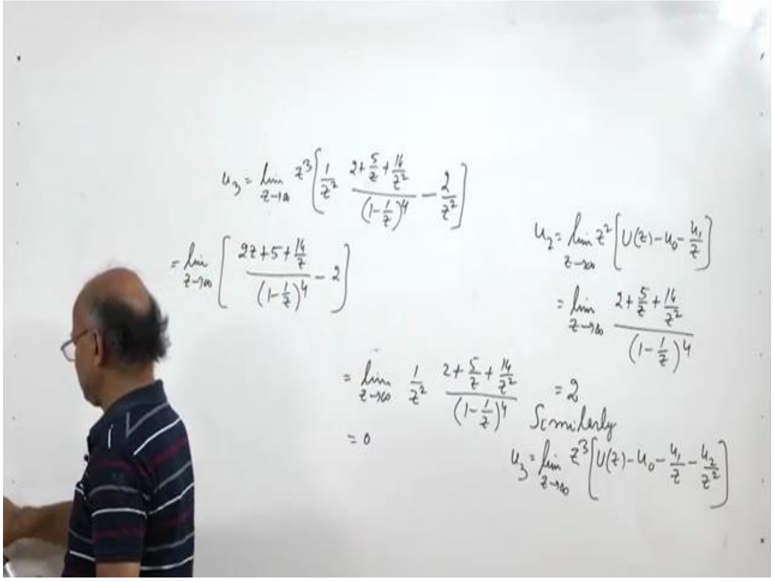

So, u 3 is equal to limit z tends to infinity; z cube times U z is 1 by z square 2 plus 5 by z; 5 by z plus 14 by z square divided by 1 minus 1 by z to the power 4 minus this 0, this 0, this 2 by z square. So, 2 by z square, z cube can be multiplied. So, this is limit z goes to infinity z cube when multiplied gives you 2 z plus 5 plus 14 by z divided by 1 minus 1 by z to the power 4 minus 2 and we can simplify it further to find the limit.

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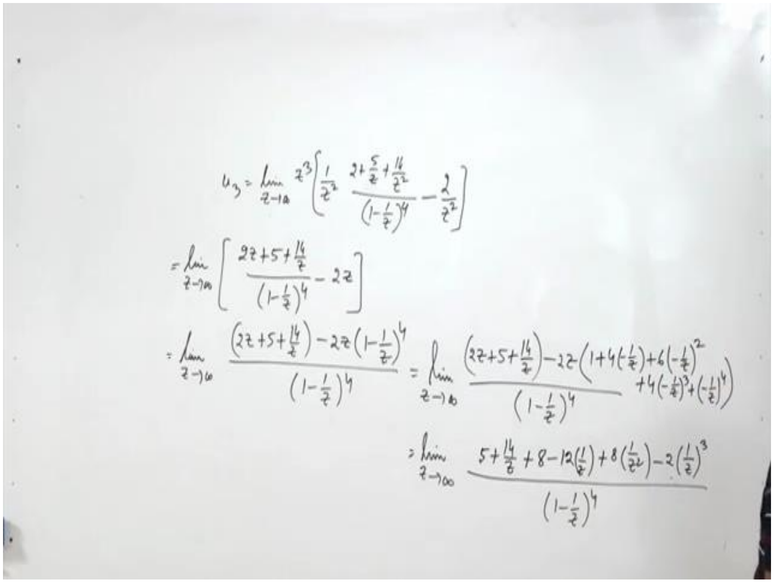

So, this will give you this z; z cube when multiplied by this is 2 z. So, 2 z times this, now we can write it further as 1 minus 1 by z to the power 4 can be expanded by binomial expansion. So, we have 1 then 4 times minus 1 by z then 6 times minus 1 by z square then we have 4 times minus 1 by z cube and then we have minus 1 by z to the power 4. So, we can do this, now we can see when you multiply 2 z here and what does then happens? So, 2 z minus 2 z; 2 z cancel out, we have then 5. So, we 2 z; 2 z cancel out, we get 5 plus 14 by z and here what we get this z; this z cancel out, we get minus 4 into minus 2. So, plus 8 and then this z when multiplied here to z square what we get is minus 12 into 1 by z and then we get here minus 1 by z cube. So, we get plus 8 times 1 by z square and then we get minus 2 times one over z to the power 3.

So, when z goes to infinity, this goes to 0, this goes to 0, this goes to 0 and what we get is 13. So, this is equal to 13. So, this is how we get the value of u 3 now. So, there is example one; and example 2 we have take an example of inverse Z transform.

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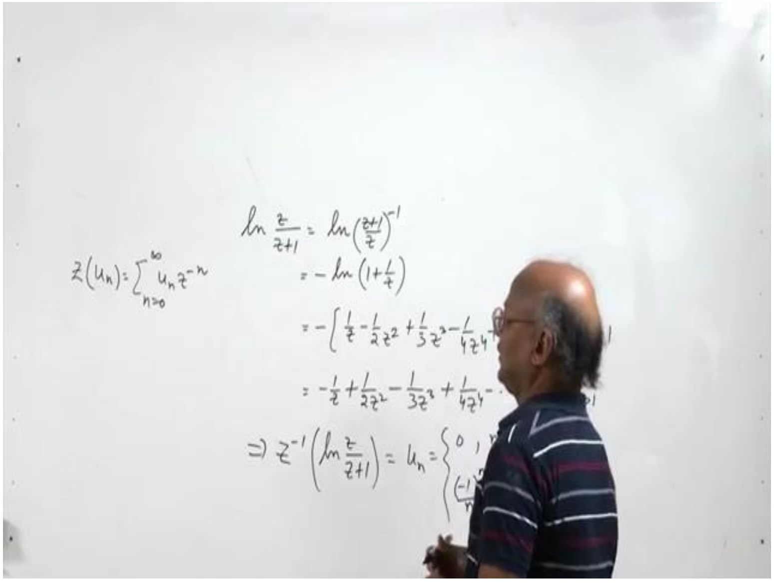

Let us find inverse Z transform of l n z over z plus 1. So, we shall apply the power series method to determine the inverse Z transform, let us see we l n z over z plus one is equal to l n z plus one over z to the power minus 1 and this is l n or I can say this is minus l n 1 plus 1 by z.

Now, if mod of z is greater than 1 then I can write it as minus 1 over z minus 1 over 2 times z 1 by z square plus 1 by 3 times 1 by z cube and so on provided mod of z is greater than 1. So, I can write it as minus 1 by z plus 1 by 2 z square minus 1 by 3 z cube plus 1 by 4 z to the power 4 and so on.

Now, comparing this series in finite series with sigma, we note that here there is no term which is free from z. So, we get z inverse, if, Z inverse of l n z over z plus 1. This is equal to u n sequence then u n is equal to 0 when n is equal to 0 and so here when n is equal to 1, the coefficient is minus 1 when n is equal to 2, the coefficient is 1 by 2. So, we get minus 1 to the power n divided by n when n is equal to 1, 2, 3 and so on.

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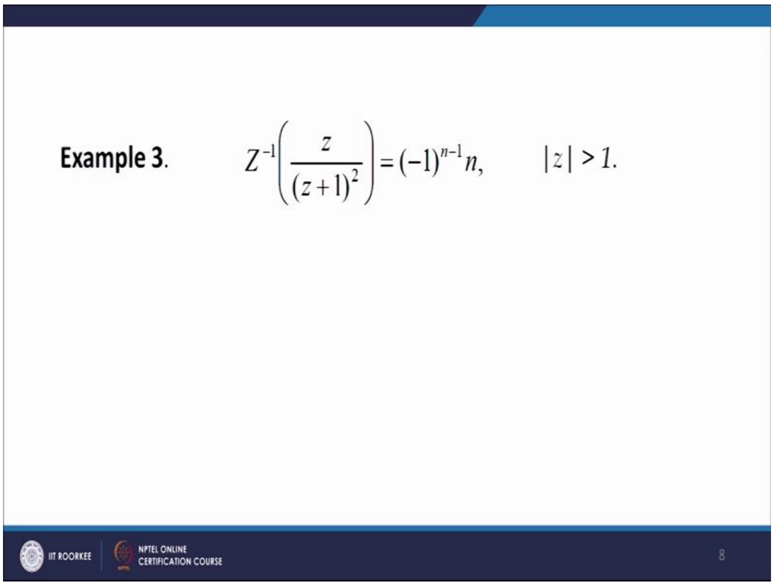

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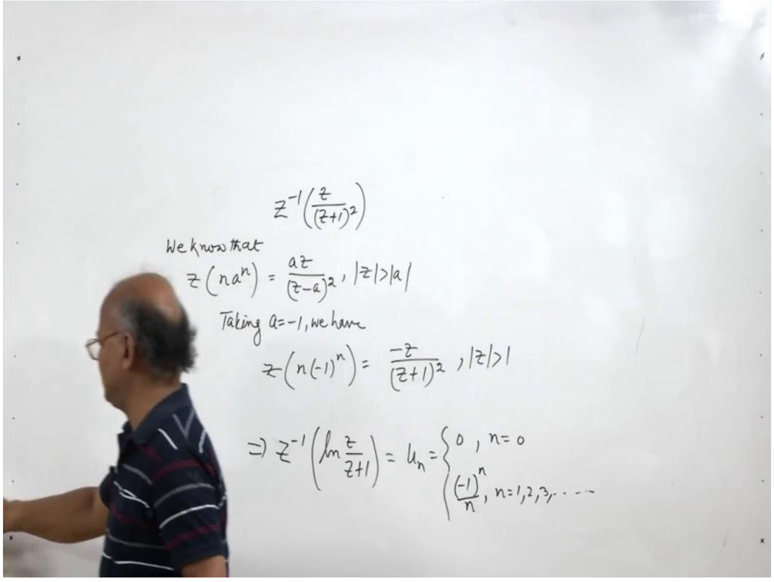

This is the example 2, let us now go to the next slide and see how we find the z inverse; Z transform of z over z plus 1 whole square, we know that Z transform of n into a to the power n is a z over z minus a whole square provided mod of z is greater than mod of a. So, when mod of a is greater than mod of a, we know that Z transform of n a to the power n is equal to a z over z minus whole square. So, taking a equal to minus 1, we have Z transform of Z transform of n times minus 1 to the power n equal to minus z divided by z plus 1 whole square provided mod of z is greater than mod of minus 1 which is 1.

And this gives us inverse transform of, using the linearity property of the Z transform, what we get is minus Z transform of n times minus 1 to the power n minus 1 is equal to z over z plus 1 whole square when mod of z is greater than 1 or we can say that inverse Z transform of z over z plus 1 whole square. This is equal to minus 1 to the power n minus 1 into n.

Now, this is the method where we have used the known result to get the inverse transform there is a method which we call long division method we can find the inverse transform by that using the using that long division method also. So, let us see how we that method works.

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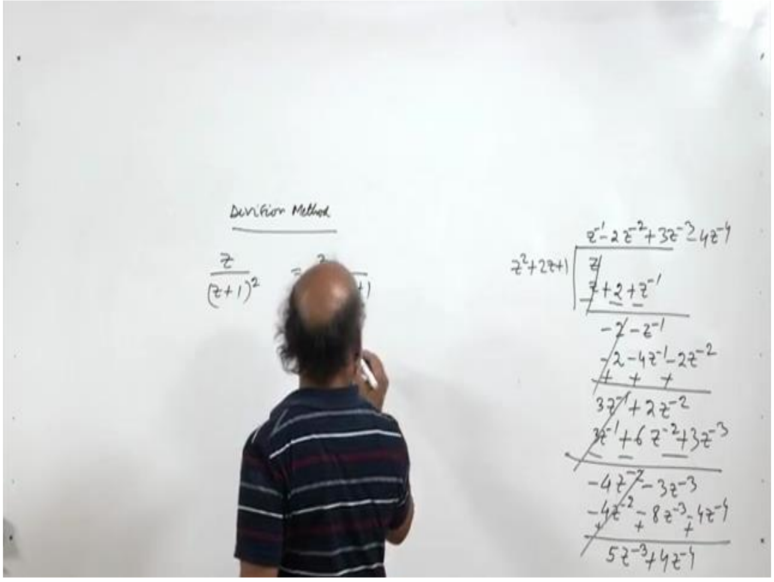

So, we shall apply the long division method, what we will do here is that we have z over z plus 1 whole square, z over z plus 1 whole square is z over z square plus 2 z plus 1. So, let us divide z by z square plus 2 z plus 1, we have to divide z by z square plus 2 z plus 1 in such a way that it will lead us to a finite series in powers of 1 over z.

So, z square plus z plus 1 by; we multiply this by z to the power minus 1 so that here when we multiply by z to the power minus 1, we get z here so that z will cancel with this z here. So, z to the power minus 1, we take then we multiply z to the power minus 1, we take here. So, we get z plus 2 times z in to z minus 1 is 2 plus z to the power minus 1 and what we get is when we subtract the sign changes, so minus 2 minus z to the power minus 1. Now our aim will be to eliminate minus 2. So, we write here minus 2 times z to the power minus 2. So, when you multiply by minus 2 times z to the power minus 2 to z square, you get minus 2 and then we get minus 4 z to the power minus 1 and then minus 2 times z to the power minus 2 again we subtract. So, the sign changes and this cancels and we get 4 z to the power 1 minus 1 minus z to the power minus 1. So, we get 3 z to the power minus 1 plus 2 times z to the power minus 2.

Now, we multiply by 3 times z to the power minus 3 so that when we multiply it to z square, it will give us 3 times z to the power minus 1. So, 3 times z to the power minus 1 and then we get 6 times z to the power minus 2 and we will get 3 times z to the power minus 3. So, like this we continue. So, we will get here 4 times z to the power minus 2 be the negative sign minus 3 times z to the power minus 3.

So, now our aim will be to multiply z square by such by an expression which will give us minus 4 z to the power minus 2, so minus 4 z to the power minus 4, so minus 4 z to the power minus 2 and then we get minus 8 z to the power minus 3 and then we get minus 4 z to the power minus 4. So, this is plus this cancels and we get this minus this is 5 times z to the power minus 3 plus 4 times z to the power minus 4.

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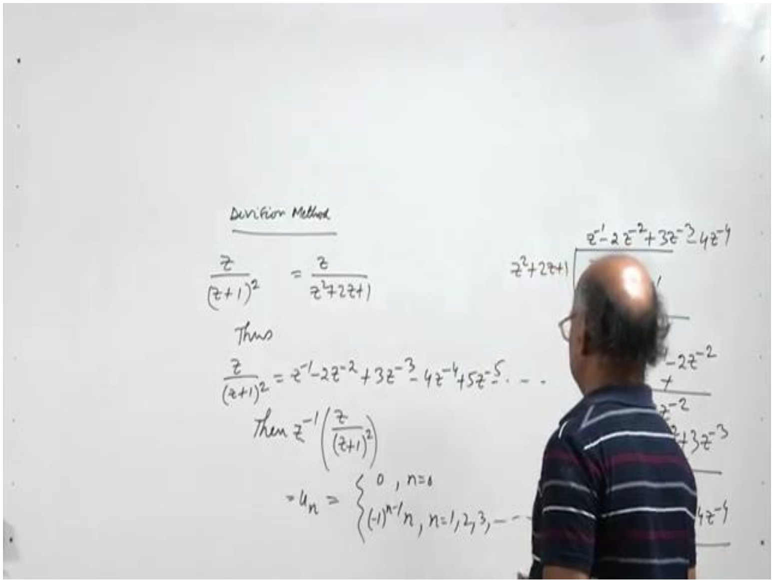

So, continuing this process. So, thus z over z plus 1 whole square, it will come out to be z to the power minus 1 then minus 2 times z to the power minus 2 and then 3 times z to the power minus 3 minus 4 times z to the power minus 4 and then next 1 will be plus 5 times z to the power minus 5 and so on and when we compare it with that series sigma u 1 z to the power minus n. So, this gives us then Z inverse of z over z plus 1 whole square. This equal to u n equal to 0 because there is no term contained which is free from z. So, 0 when n is equal to 0 n minus 1 to the power n minus 1 when minus 1 to the power n into n when n is equal to when n is equal to 1, we will write here minus 1.

So, this how we get the inverse transform of z over z plus 1 whole square, with that I would like to conclude my lecture.

Thank you very much for your attention.