Mathematical Methods and its applications
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Lecture - 41
Properties of Z - Transformations – II

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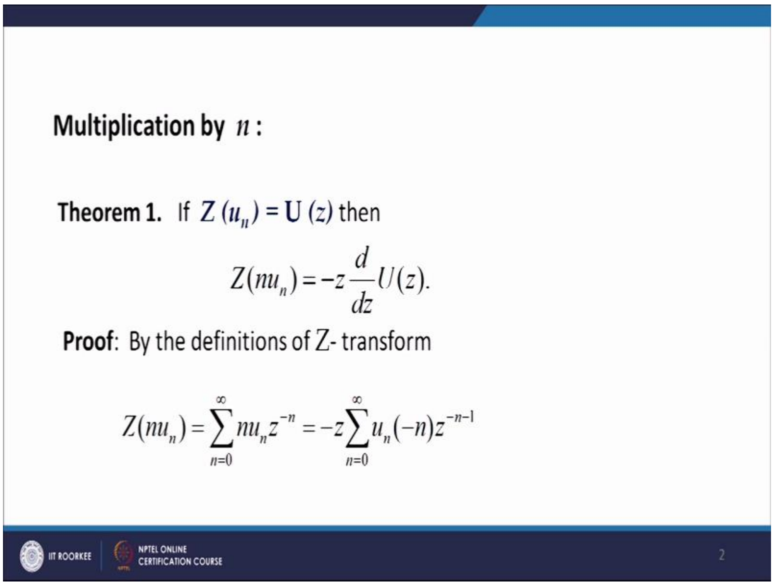

Hello friends, welcome to my second lecture on properties of Z transforms. So, first we will discuss what happens when we know the Z transform of a sequence u n say U z and then the Z transform of n into u n and to u n that is u n is multiplied by n. Then we see that Z transform of n u n is minus z into d over d z of U z. So, this result is also going to be very useful when we solve difference equations. So, by the definition of Z transform Z n u n, Z n u n is sigma n equal to 0 to infinity n u n into z to the power minus n and this we can write as minus z times sigma n equal to 0 to infinity u n into minus z and minus n times z to the power minus n minus 1. So, applying the definition of Z transform we will we are able to write Z transform of n u n in this form.

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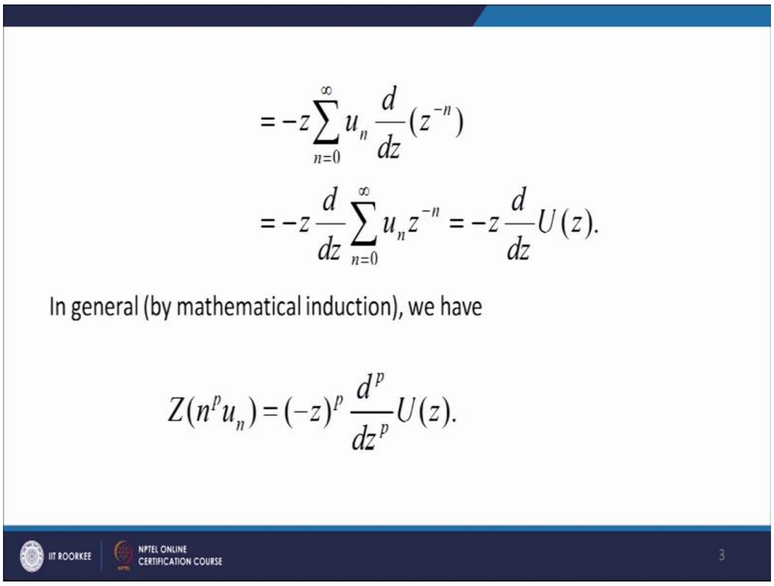

Now, but then we see that this can be written as minus z times sigma n equal to 0 to infinity u n d over d z of z to the power minus n because when you differentiate z to the power minus n with respect to z, what you get is minus n times z to the power minus n minus 1. So, we express in this form, but then this is equal to minus z d over d z of sigma n equal to 0 to infinity u n z to the power minus n. And therefore, it is equal to minus z times d over d z of u z. Now if you apply mathematical induction on p we can see that Z transform of n to the power p u n is equal to minus z to the power p d p over d z p U z where p is a positive integer.

Now say, suppose we have u n equal to n cos n theta, then n we want to find the Z transform of u n then, we can use the known result for Z transform of cos n theta and make use of the theorem which we have just proved that is the Z transform of n u n is minus z times d over d z of u z. So, let us see we, let us recall that Z transform of cos n theta is equal to.

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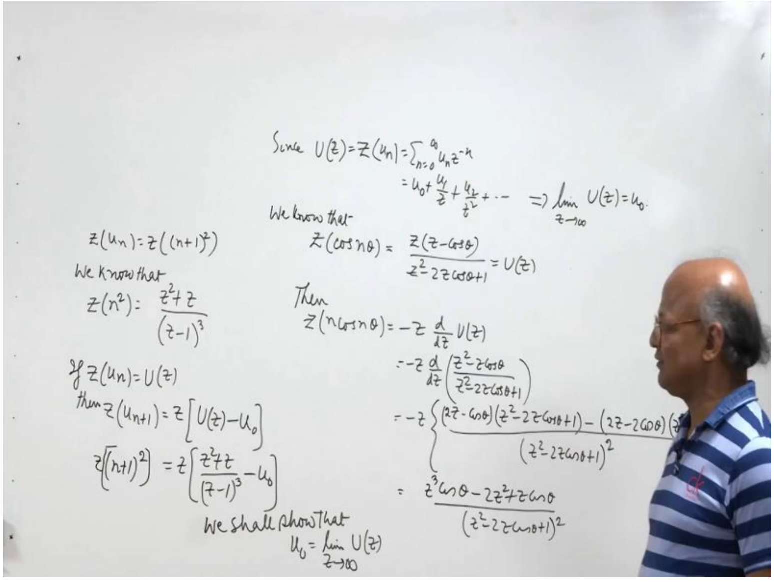

We know that Z transform of cos n theta this we have proved in the previous lecture that it is z into z minus cos theta divided by z square minus 2 z cos theta plus 1. So, now, let us apply the theorem multiplication by n. So, then Z transform of n cos n theta will be equal to minus z times d over d z of let us call it u z; so d over d z of u z.

So, this is minus z times d over d z of z square minus z cos theta divided by z square minus 2 z cos theta plus 1. So, this will be equal to minus z times the derivative of the numerator is 2 z minus cos theta multiplied by z square minus 2 z cos theta plus 1 minus 2 z minus 2 cos theta into z square minus z cos theta divided by z square minus 2 z cos theta plus 1 whole square. Now when we simplify the numerator, we will have this as minus z times z square sorry. So, this will be if you simplify the numerator you will get z cube cos theta minus 2 z square plus z cos theta divided by z square minus 2 z cos theta plus 1 whole square.

So, on simplifying we get the Z transform of u n, z cube cos theta minus 2 z square plus z cos theta over z square minus 2 z plus cos z cos theta plus 1 whole square. Now suppose we have u n equal to n plus 1 square. We want to find the Z transform of u n equal to n plus 1 square then, here we can use we know that Z transform of n square is z square plus z divided by z minus 1 raise to the power 3. So, and we know this shifting theorem if z of u n is equal to u z then, Z transform of u n plus 1 is z times u z minus u naught. So, here we will get Z transform of u n plus 1. So, here u n you can take as n square, if you take u n as n square then Z transform of n square is z square plus u z is z square plus z over z minus 1 whole cube and then Z transform of n plus 1 whole square. So, Z transform of n plus 1 whole square will be z times u z, u z is z square plus z by over z minus 1 whole cube. So, this is minus u naught now, here we need the value of u naught.

So, for that I need to know what is u naught and this u naught we shall require we shall do the initial value theorem there, we shall know that if we know Z transform of a sequence u n that is u z then by taking the limit of u z as z tends to infinity we get u naught. So, we know it, we shall show that u naught is equal to limit z tends to infinity u z and here u z is Z transform of n square, so z square plus z over z minus 1 whole cube. So, u naught is equal to limit z tends to infinity z square plus z divided by z minus 1 whole cube and this is a polynomial in z of degree 2, here we have a polynomial in z of degree 3. So, the limit is called; obviously, 0. So, u naught is equal to 0 and hence Z transform of n plus 1 whole square is equal to z cube plus z square divided by z minus 1 whole cube.

Now, you may wonder, how do you determine how we prove that limit of u z as z tends to infinity is u naught? So, this is very simple we know that since u z is equal to Z transform of u n is sigma n is equal to 0 to infinity, u n z to the power minus n which is u naught plus u 1 by z plus u 2 by z square and so on. So, when you take the limit of this z tends to infinity this gives you u naught. So, this implies, so u z equal to naught plus u 1 by z plus u 2 by z square and so on, this implies limit as z tends to infinity u z is equal to u naught. This is called initial value theorem which we will prove in the next lecture. So, we can use that initial value theorem to arrive at the Z transform of n plus 1 whole square; this is z cube plus z square over z minus 1 whole cube.

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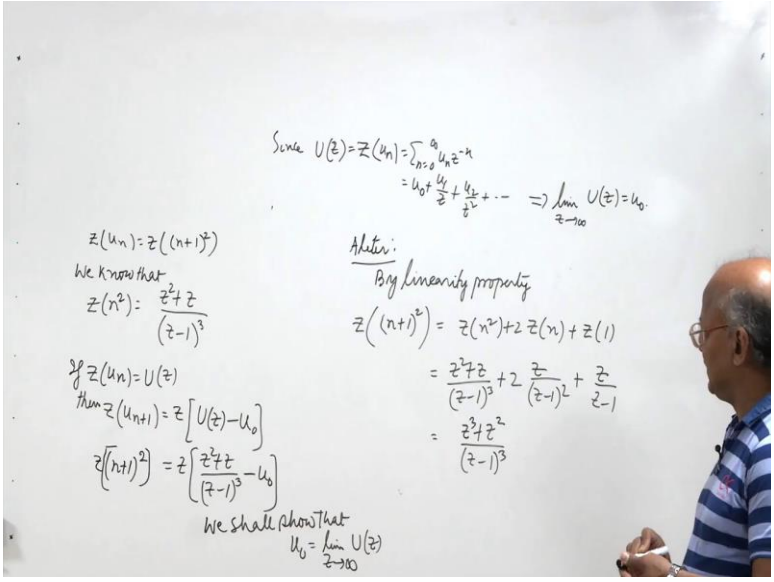

So, this result uses Z transform of the value of u naught which we get by initial value theorem, but if we do not want to use the initial value theorem then, we can write by linearity property. So, we can do this result by applying by linearity property; Z transform of n plus 1 whole square is equal to Z transform of n square plus 2 times Z transform of n plus Z transform of 1 and Z transform of n square is z square plus z divided by z minus 1 whole cube plus 2 times z transform of n, which is z over z minus 1 whole square and this is Z transform of 1 is z over z minus 1.

So, simplifying this we shall have z cube plus z square divided by z minus 1. So, by using the initial value property, by using the linearity property we can obtain the Z transform of n plus 1 Whole Square. And if you do like this that is you use the shifting property then, you need to know the value of u naught which we will get by using the definition of u z as sigma n equal to 0 to infinity u n z to the power minus n, taking the limit as z tends to infinity to get the value of u naught.

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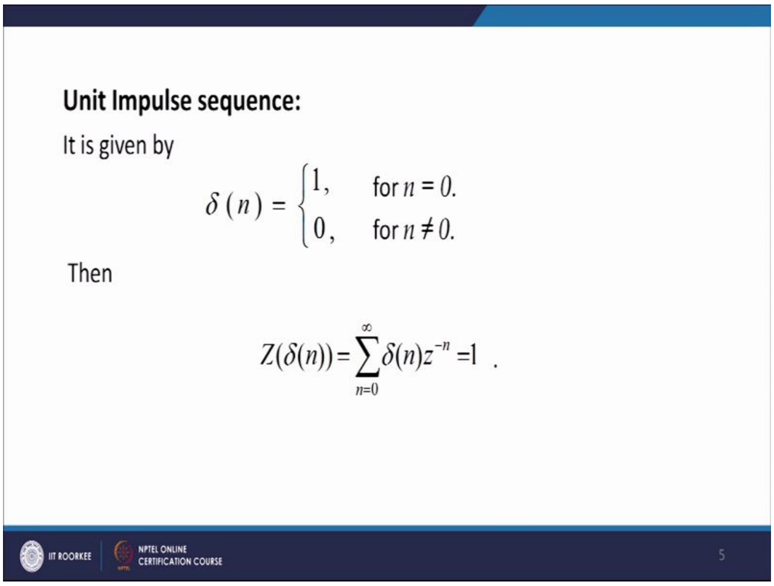

Now, let us go to unit impulse sequence, it is given by we define a unit impulse sequence by delta n equal to 1 when, n is 0 n 0 for all n different from 0. So, if you find the Z transform of delta n, let us apply the definition. The definition, sigma n equal to 0 to infinity delta n z to the power minus n when n is equal to 0, we are given delta n to be equal to 1 otherwise it is 0. So, all other terms are 0 except the term for n equal to 0 where delta n is 1. So, we get 1 into z to the power 0 which is 1. So, Z transform of delta n is equal to 1. So, whenever we need the inverse Z transform of 1, we will write there delta n.

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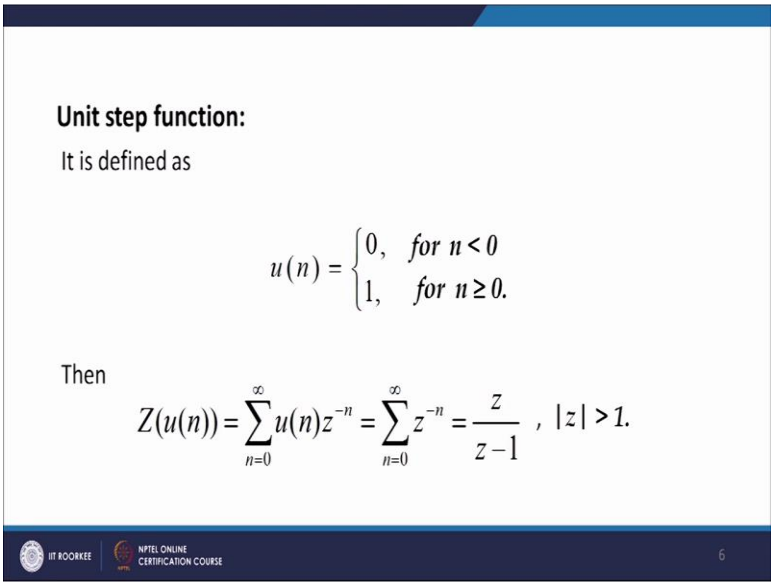

Then we need the Z transform of unit step function. So, unit step function we defined as u n equal to 0 when, n is less than 0 and 1 for all n greater than or equal to 0; that is for all non negative integers we define u n to be equal to 1 and for all negative integers we define it as 0. So, Z transform of the unit step function u n is then again by definition sigma n equal to 0 to infinity, u n 0 to the power minus n which is sigma n equal to 0 to infinity z to the power minus n because u n is equal to 1 for all n bigger than or equal to 0.

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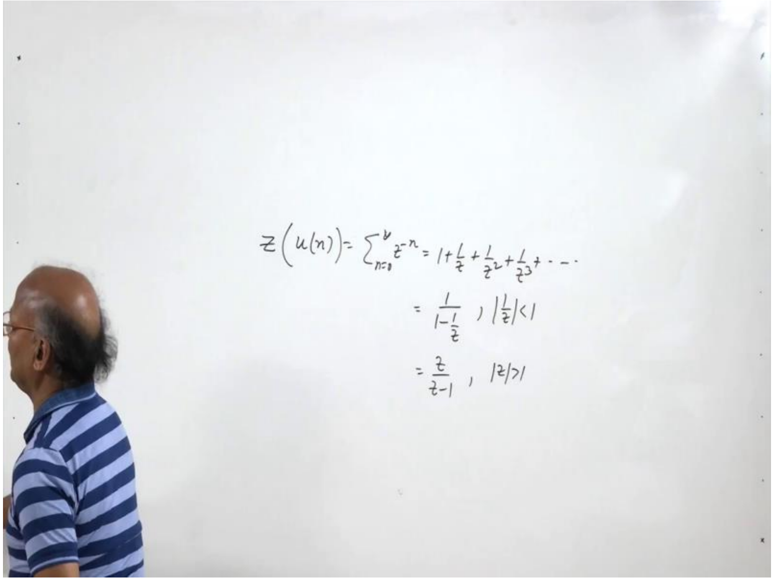

Now, this is z to the power. So, this is 1 plus z plus z square and so on. Unit step function; this is sigma n equal to 0 to infinity z to the power minus n.

So, this is 1 plus 1 by z plus 1 by z square 1 by z cube and so on. So, this series converges absolutely when mod of z is greater than 1 and the sum of the series is 1 over 1 minus 1 by z. So, this is true when, this is greater than this is less than 1 or this is z over z minus 1 when mod of z is greater than 1. So, the region of convergence of this series is mod of z greater than 1. The Z transform of unit step function is z upon z minus 1, now let us consider an example u of a rational function in z and see how we determine the inverse Z transform. So, here we know U z.

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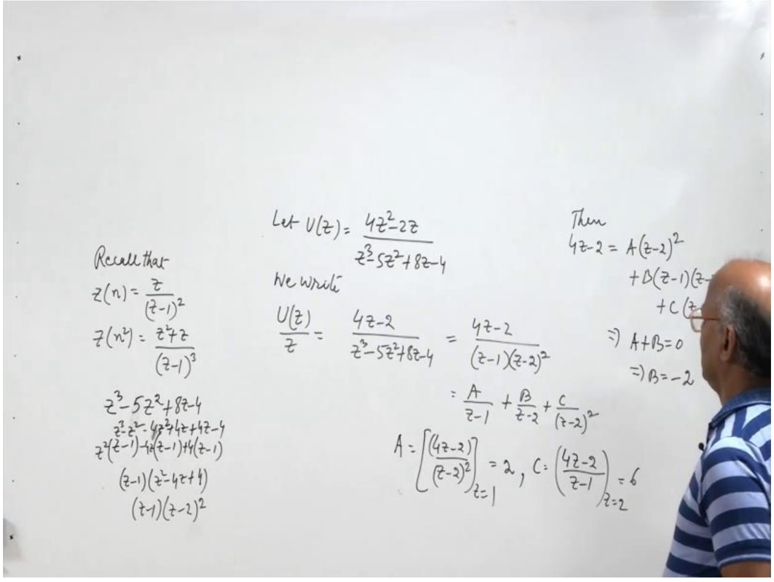

So, let U z be equal to 4 z square minus 2 z divided by z cube minus 5 z square plus 8 z minus 4.

So, we will now let us recall that, Z transform of n is z over z minus 1 whole square, Z transform of n square is z square plus z divided by z minus 1 whole cube. Now these tendered technique to determine the inverse Z transform is that when, U z is a rational function of z we write first U z by z. So, U z by z is equal to 4 z minus 2 divided by z cube minus 5 z square minus 8 z minus 4. It will be clear in when we do the solution why we are taking U z by z? So, let us break it U z by z into partial fractions, we see that when you put z equal to 1 here this denominator becomes 0. So, z minus 1 is the factor of this. So, we have to find factors of this polynomial, z cube minus 5 z square plus 8 z minus 4. So, let us find the factors of this, z minus 1 is the factor and it is a polynomial. So, what we do is, we write it as z minus 1, z minus 1; we multiply by z square we get z cube minus z square then, minus 4 z minus 4 z square then plus 4 z plus 4.

So, we get z minus 1 into z square minus 4 z plus 4, which is z minus 1 into z minus 2 whole square. So, we get here, 4 z minus 2 divided by z minus 1 into z minus 2 whole square and the factors then, the partial fractions we can write as A over z minus 1 plus B over z minus 2 plus C over z minus 2 whole square corresponding to z minus 1 and z minus 2 whole square. Now the values of A and C can be found directly. So, A is equal to we remove z minus 1 from here. So, 4 z minus 2 over z minus 2 whole square at z equal to 1. So, this will be 4 minus 2 that is 2 when minus 2 is minus 1 whole square that is 1. So, we get the value as 2 and then C can also be obtained directly, C is equal to z minus 2 whole square, we remove from here. So, 4 z minus 2 divided by z minus 1 and when we put z equal to 2.

So, this will be 4 into 2; 8 minus 2, 6. So, C is equal to 6, we have to determine now the value of B; for that we write the LCM. So, LCM is z minus 1, z minus 2 whole square and. So, then 4 z minus 2 is equal to A times z minus 2 whole square plus B times z minus 1, z minus 2 and C times z minus 1. Now on the right side, we get terms in z square, but on the left side there is no term in z square. So, let us collecting the terms in z square here, the coefficient of z square is A here, here the coefficient of z square is B. So, A plus B is equal to 0. So, we get A plus B equal to 0 and A, we have already found. So, this implies B is equal to minus 2. So, thus we get the value of U z by z. So, hence U z by z is equal to A; A is 2, 2 over z minus 1 and then B over z minus 2. So, minus 2 over z minus 2 and then C is 6, so 6 over z minus 2 whole square.

Now, we multiply by z. So, we get U z equal to 2 z upon z minus 1 then minus 2 z upon z minus 2 and then 6 z upon z minus 2 whole square. Now let us see why we see, we bring we consider U z by z because then the partial fraction will be some constant divided by z minus 1 and constant divided by z minus 2 square and what we need when we find the inverse Z transform of U z, we need here the forms, the standard forms z over z minus 1 whole square or z minus z over because we know that Z transform of a to the power n is equal to z over z minus a provided mod of z greater than mod of a. So, here we need z over z minus there, if we do not consider U z over there and instead consider U z equal to U z 4 z square minus 2 z upon this and then we bracket into partial fractions. We shall not get the standard forms where z should come in the numerator.

So, we consider U z by z, now this is 2 z over z minus 1. So, we can take the inverse Z transform now and let us see when we take the inverse Z transform.

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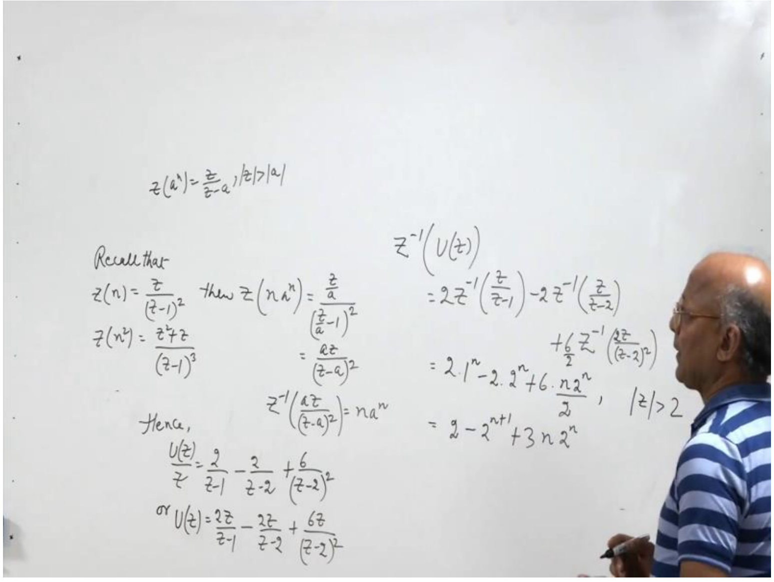

So, Z inverse of U z is equal to Z inverse 2 times Z inverse of z over z minus 1 then, minus 2 times Z inverse of z over z minus 2 and then plus 6 times Z inverse of z over z minus 2 whole square. Z transform of a to the power n is z over z minus a. So, when mod of z is greater than 1 this transform will be put a equal to 1. So, 2 times 1 to the power n for this and then minus 2 times Z transform of n is z over z minus 1 whole square. We need the Z transform, Z inverse Z transform of z over z minus 2. So, if we take Z transform of n is z. So, this implies, so then Z transform of n into a to the power n. If we want this then we multiply, then we divide using the dumping rule we will get z over a divided by z over a minus 1 whole square and so alright. So, this will be, this is a z divided by z minus a whole square.

Now, here we can get it directly. So, minus 2 times z over z minus taking mod of z greater than 2, mod of z get 2 to the power n. So, here mod of z greater than we want, here we want mod of z greater than 2 and then plus 6 times and then Z inverse of. So, Z inverse of a z divided by z minus a whole square. This is equal to n into a to the power n. So, we can here we have in place of a we have 2 here. So, this will be n into a to the power n divided by a, so 6 into n into 2 to the power n divided by 2. We can multiply; n divide by 2 here; 6 by 2 and then we can bring 2 here like this. So, this will be. So, this is this first inverse Z transform is valid when mod of z is greater than 1, second transform is valid when mod of z is greater than 2. So, the common region we have to take, so mod of z greater than 1 and mod of z greater than 2. So, mod of z greater than 1 if we have mod of z greater than 2, then both will be valid.

So, we have 2 minus 2 to the power n plus 1 plus 3 times n into 2 to the power n. So, this is valid for mod of z greater than the 2. We have taken the common region of convergence; mod of z greater than 1 and mod of z greater than 2. So, in this is how you find the inverse Z transform u n. Now we take up example 2; where suppose u n is equal to 1 over n plus 1.

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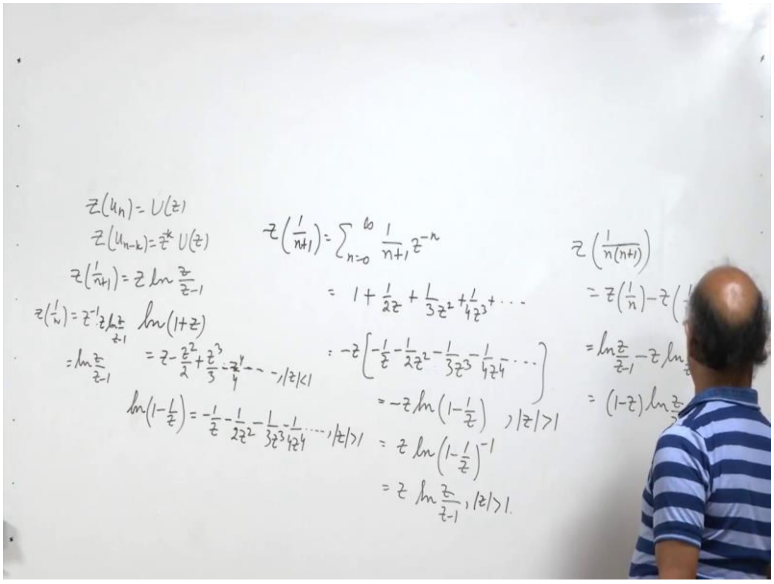

Then Z transform of 1 over n plus 1, we have to find. So, we know that Z transform of 1 by n, if we find the Z transform of 1 by n, the Z transform of 1 by n is given by sigma n equal to 0 to infinity, 1 by n z to the power minus n and Z transform of 1 by n plus 1 for this, we can use the shifting property. If you take u n equal to 1 by n then u n plus 1 can be considered as 1 by n plus 1.

So, this is z times U z minus u naught, u naught is the term in z 1 by n which does not contain your z does which does not contain and your z term. So, here we will consider n to begin with I think I should have taken z directly 1 by n plus 1. Let me write it again, Z transform of 1 by n plus 1 is sigma n equal to 0 to infinity 1 by n plus 1 z to the power minus n and if you expand this, what we will get; n equal to 0. So, 1 minus n equal to 1 by 2 over z then n equal to 2, so 1 by 3, here also plus 1 by 3 z to the power 2, 1 by 4 z to the power 3 and so on.

Now, what we will do is, we can write it as see let us recall that log l n 1 plus z. Let us recall the expansion of l n 1 plus z, this is z minus z square by 2 plus z cube by 3 and so on minus z cube z 4 by 4 and so on. So, l n 1 minus 1 by z let us. So, this is mod of z less than 1 and l n 1 minus 1 by z, if we write this then what we have minus 1 by z, minus 1 by 2 z square, minus 1 by 3 z cube, minus 1 by 4 z 4 and so on. So, this I can write as minus z times, if you identify with this minus 1 upon z, minus 1 by 2 z square, minus 1 by 3 z cube, we can write like this. So, here mod z less than 1 here mod of 1 by z will be less than 1, this means mod of z must be greater than 1. So, mod of z must be greater than 1. So, this is minus z l n 1 minus 1 by z provided mod of z is greater than 1 and this is raised to the power l n z minus. So, this is z times l n 1 minus 1 by z raised to the power minus 1 or z times l n z over z minus 1. So, this is the Z transform of 1 over n plus 1 provided mod of z is greater than 1. And then, we can find the Z transform of 1 over n into n plus 1, Z transform by using linearity property.

So, this is equal to Z transform of 1 by n minus Z transform of 1 by n plus 1. If we know the Z transform of u n, we can also find the Z transform of 1 by n then we will need that n equal to 0 term we cannot take, z equal to Z transform of, we know that if Z transform of u n is equal to U z then, Z transform of u n minus k is z to the power k into U z; z to the power minus k into U z. So, Z transform of we know Z transform of 1 over n plus 1. So, Z transform of 1 over n plus 1, we know this is z of l n z over z minus 1. So, from here if we write Z transform of 1 by n using this result we will have z to the power minus 1 into U z, that is z in to l n z over z minus 1. So, we have l n z over z minus 1.

So, this will be l n z over z minus 1 minus z times l n z over z minus 1, by using the shifting theorem. So, this is l n 1 minus z times l n z over z minus 1 provided mod z is greater than 1. So, by using shifting theorem we can find is, now let us consider this problem example 3.

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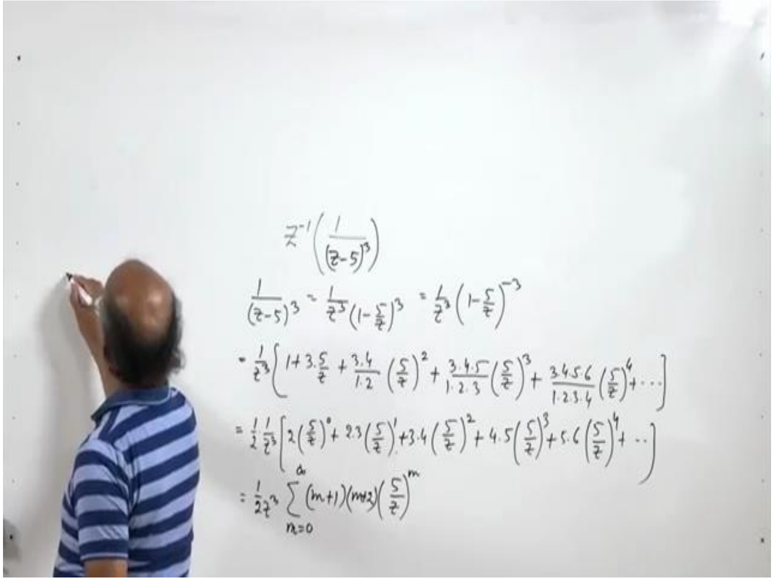

So, example 3, we have to find inverse Z transform of 1 over z minus 5 whole to the power 3, inverse Z transform of 1 over z minus 5 to the power 3. Let us write 1 over z minus 5 to the power 3 as 1 over z cube times 1 minus 5 by z to the power 3, so then 1 by z cube 1 minus 5 by z to the power 3 minus 3.

Now, we are given that mod of z is greater than 5. So, mod of 5 by z is less than 1. So, we can apply the (Refer Time: 32:54) theorem and write it as 1 over z cube and then 1 plus 3 times 5 by z then, 3 into 4 divided by 1 into 2 factorial 5 by z whole square then, 3 into 4 into 5 divided by 5 into 2 into 3, 5 by z whole cube and so on. 3, 4, 5, 6 divided by 1, 2, 3, 4, 5 by z whole cube and so on. I can write it as 1 by 2 into 1 by z cube then 2 into 5 by z to the power 0 then, 2 into 3, 5 by z to the power 1 then, 3 into 4, 5 by z to the power 2. Here we shall have 4 into 5, 5 by z to the power 3 and so on, 5 into 6, 5 by z to the power.

Here, 5 by z to the power 0, 5 by z to the power 1, 5 by z to the power 2, 5 by z to the power 3, this 5 by z to the power 4, so 5 by z to the power 4 and so on. So, this is, this can be written as 1 by 2 z cube sigma, we have 2, 2 into 3. So, we have m plus 2, 5 by z to the power, let us say m here, m plus 3, m here, m equal to. If I write it for say from 0 to infinity then, what do we get here; m equal to 5 m plus 2, m plus 3, we get here n is equal to m plus 3, so m plus 3 means m plus 2, m plus 1. Let me see m is equal to 0, if I put here I get here 2 into 1, m equal 1 I put here, 2 into 3 m equal to 2 I put here. So, 3 into 4; yeah, so m plus I can write here m plus 1 and here m plus 2 and then 5 by z to the power m. Now I think it is, m equal to 0, so 2 into 5 by z to the power 0 then, 2 into 3, 5 by z to the power 1 then, 3 into 4, 5 by z to the power 2 and so on.

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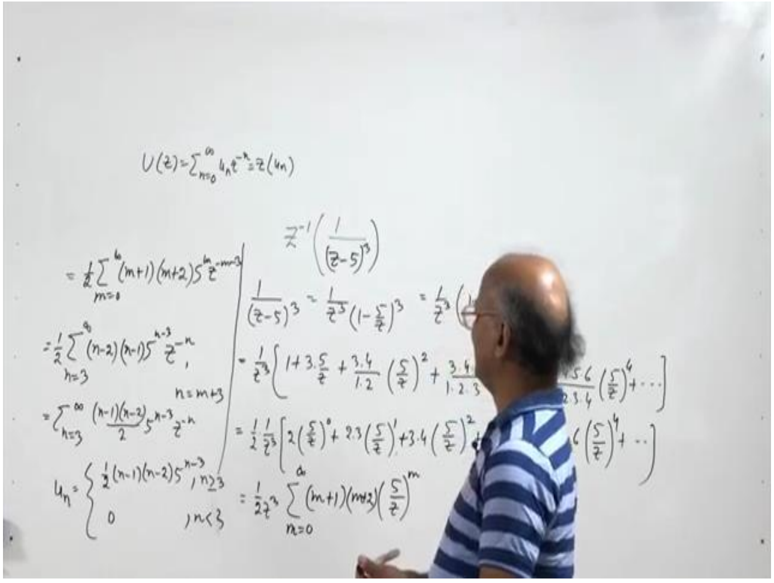

So, we can write like this and then this will be equal to 1 by 2, summation m equal to 0 to infinity. So, m plus 1 into m plus 2 into 5 to the power m, z to the power minus m minus 3, z to the power 3, we have in the denominator.

So, z to the power minus m minus 3 we get. Now let us put n equal to m plus 3, let us put n equal to m plus 3. So, then this will be 1 by 2 times summation n is equal to 3 to infinity and m is equal to n minus 3. So, we will get n minus 2 here and here we will get n minus 3 plus 2. So, n minus 1 and then we get 4 to the power n minus 3 into z to the power minus n where, we are putting n equal to m plus 3.

So, from here we can see that now this is nothing, but this summation n equal to 3 to infinity n minus 1, n minus 2 divided by 2, 5 to the power n minus 3 into z to the power minus n. So, when n is greater than or equal to (Refer Time: 37:54) of this, now let us compare it with these U z is equal to sigma n is equal to 0 to infinity u, n z to the power minus n. This is Z transform of u n sequence. So, if we compare with this then, we can say that here u n is equal to 1 by 2, n minus 1 and n minus 2, 5 to the power n minus 3 when n is greater than or equal to 3 and this is equal to 0 when, n is less than 3. So, this is the inverse Z transform of 1 over z minus 5, whole cube. With that I would like to conclude my lecture.

Thank you for your attention.