Mathematical methods and its applications

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Lecture – 40
Properties of Z – transforms – I

Hello friends, welcome to my lecture on properties of Z transforms. The topic of properties of Z transforms will be covered in 2 lectures. This is first lecture on the properties of Z transforms. In the last lecture, we had obtained the values of z operating and u n equal to n and z operating on u n equal to n square. So, now, we will talk about z operating on cos n theta.

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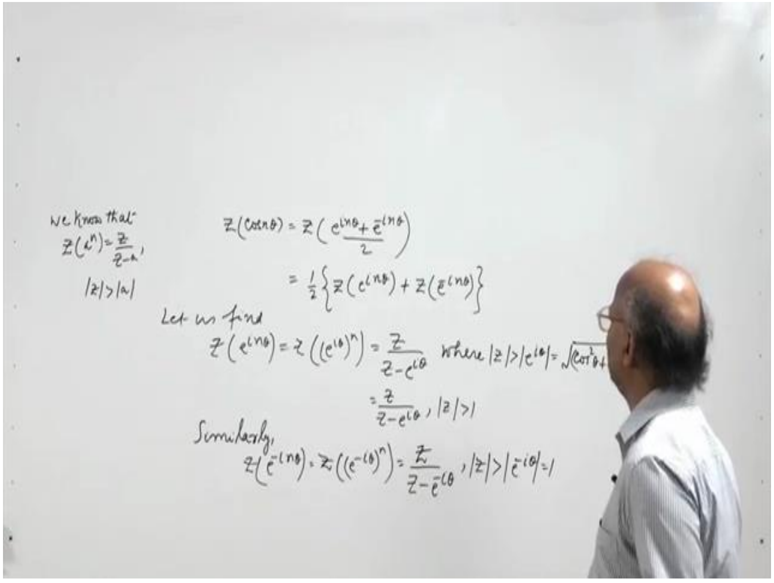

So, let suppose we have to find Z of cos n theta we know that cos n theta can be written as e to the power i n theta plus e to the power minus i n theta by 2. Now by applying the linearity property of z, we can write it as 1 by 2 z of e raise to the power of i n theta plus z of e to the power minus i n theta. Now let us recall the result; we can write, now let us find let us find z of e raise to the power i n theta e to the power i n theta we may write as z of e to the power i theta raise to the power n. Here we shall apply the formula for Z transform of a to the power n z we know that Z transform of a to the power n is z over z minus a provided mod of z is greater than mod of a. So, let us take a equal to e to the power i theta here.

So, then we shall have z over z minus e to the power i theta where mod of z is greater than mod of e i theta and the mod of e i theta is what e i theta is cos theta plus i sin theta. So, mod of e i theta is square root cos square theta plus sin square theta and you know that and it is equal to 1. So, this is z over z minus e i theta provided mod of z is greater than 1. Similarly Z transform of e to the power minus i n theta we can write as Z transform of e to the power minus i theta raise to the power n.

Now here we shall take a equal to e to the power minus i theta and then again apply he formula Z transform of a to the power n equal to z over z minus a now where mod of z is greater than mod of e to the power minus i theta mod of e to the power minus i theta is also equal to 1. So, this is equal to z over z minus e raise to the power minus i theta where mod of z is greater than mod of e raise to the power minus i theta which is also equal to 1 and the let us now put the values of Z transforms of e to the power i n theta e to the power minus i n theta. So, then we shall have.

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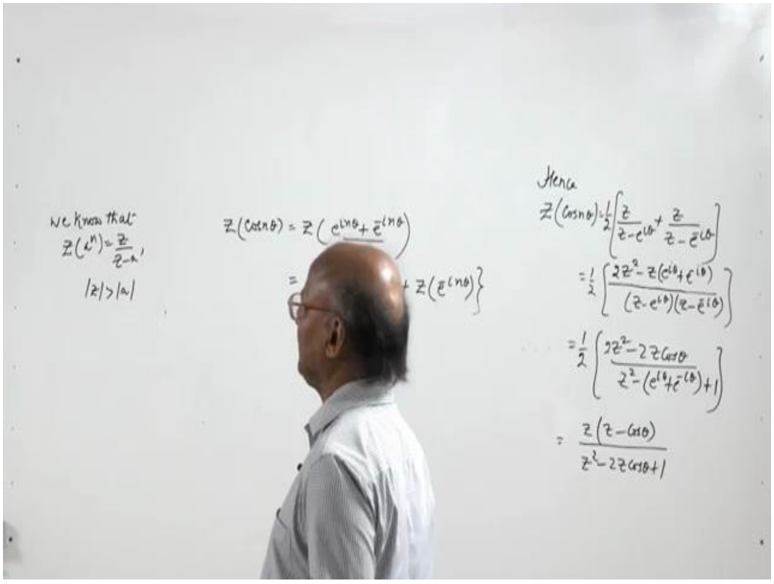

So, hence Z transform of cos n theta will be equal to 1 by 2 z over z minus e i theta plus z over z minus e to the power i theta and we can write this equal to; this is z minus e i theta into z minus e raise to the power minus i theta so then z square.

So, and then, it is 2 z square minus z times e to the power i theta plus e to the power minus i theta.So, we can write it as 1 by 2; 2 z square minus 2 z cos theta divided by z square minus e to the power i theta plus e to the power minus i theta and then plus e to the power i theta into e to the power minus i theta is it power 0. So, we have 1. So, this is equal to z into z by minus cos theta divided by z square minus 2 z cos theta plus 1. So, Z transform of cos n theta is z into z minus cos theta divided by z square minus 2 z cos theta plus 1. You know similarly we can find Z transform of sin n theta.

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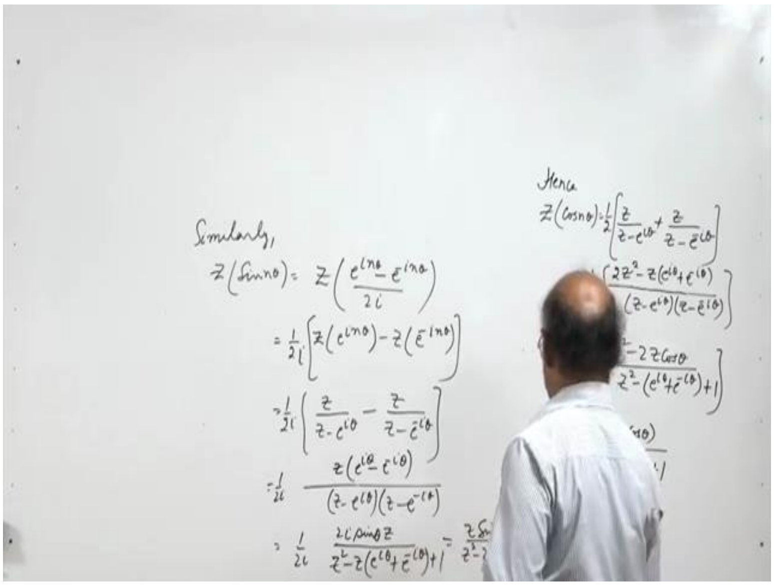

So, when we write Z transform of sin n theta we shall have similarly Z transform of sin n theta this will be equal to 1 by 2 i, let me write it like this, first we can write it as Z transform of e to the power i n theta minus e to the power minus i n theta divided by 2 i and by using linearity property, this is 1 over 2 i Z transform of e to the power i n theta minus Z transform of e to the power minus i n theta which is 1 by 2 i Z transform of e to the power i n theta is z over z minus e i theta and this is z over z minus e to the power minus i theta this is equal to 1 by 2 i z minus e i theta into z minus e raise to the power minus i theta and then z square times z square minus z e to the power minus i theta minus z square z square will cancel and will get minus z times e raise to the power.

Let me write it like this. So, this is equal to z minus z e to the power minus i theta and here we will have plus z e to the power i theta. So, z times e to the power i theta minus e to the power minus i theta and e to the power i theta minus e to the power minus i theta is 2 i sin theta. So, we shall have 1 by 2 i 2 i sin theta into z divided by z square minus z times e to the power i theta plus e to the power minus i theta plus 1. So, this is 2 i; 2 i cancel and we get z sin theta divided by z square minus 2 z cos theta plus 1 where mod of z is greater than 1.

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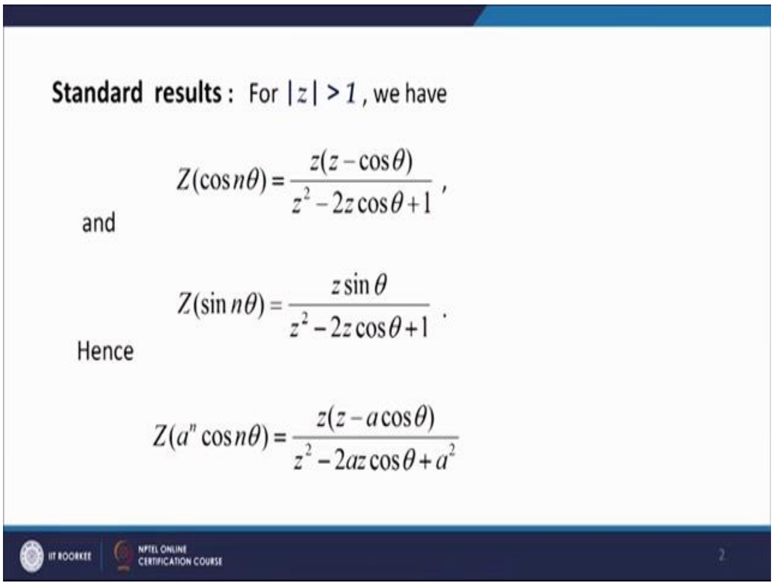

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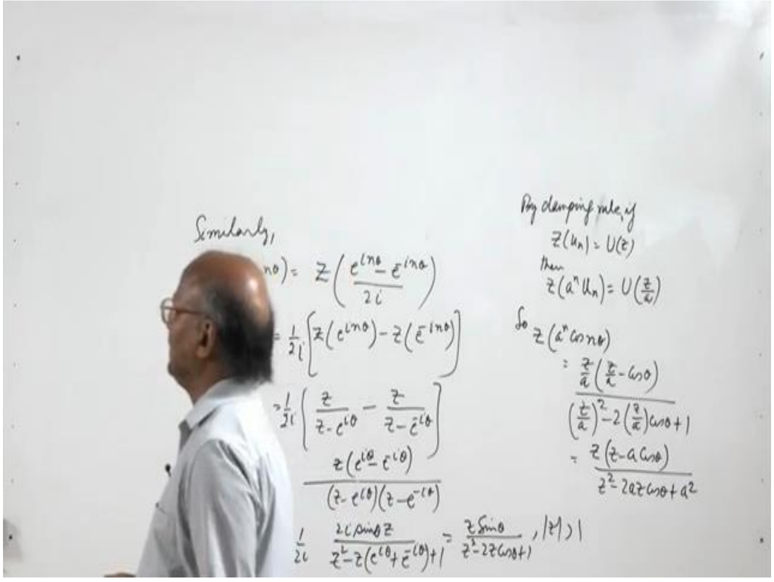

So, that is how we find the Z transform of sin n theta and then Z transform of a to the power n cos theta. We can obtain by dumping rule, let us recall the dumping rule, we know that by dumping rule, if Z transform of sequence u n is U z then Z transform of a to the power n u n; u n is equal to U z by a. So, Z transform of a to the power n cos theta. So, Z transform of a to the power n cos theta will be equal to in the Z transform of cos n theta in the Z transform of cos n theta which is z into z minus cos theta divided by z square minus 2 z cos theta plus 1. Let us replace z by z by a. So, we get and this will give you z minus z into z minus a cos theta divided by z square minus 2 a z cos theta plus a square. So, we get Z transform of a to the power n cos theta by using the dumping rule.

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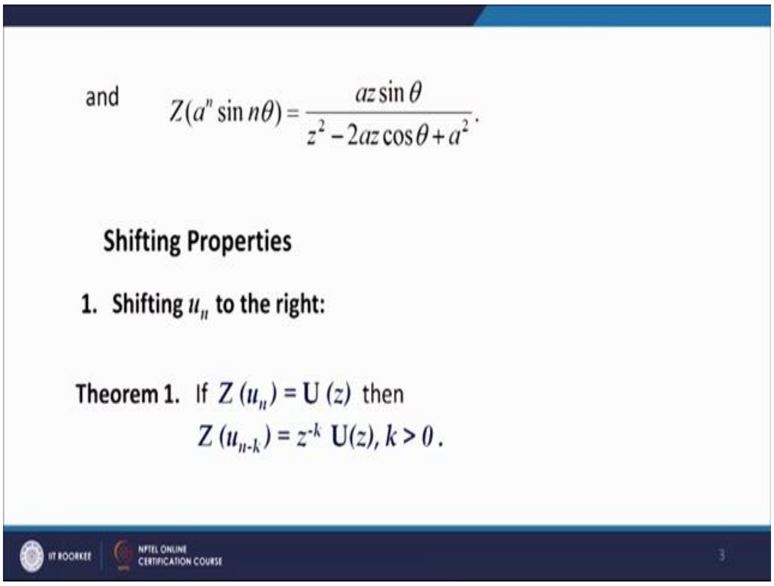

And similarly by using the dumping rule we can obtain the Z transform of a to the power n sin theta sin n theta in the Z transform of sin n theta which is z sin theta over z square minus 2 z cos theta plus 1. We replace z by z by a to arrive at a z sin theta over z square minus 2 a z cos theta plus a square. Now let us discuss the shifting properties like we have the shifting theorems in the Laplace transform here also we have the shifting theorems, now the first theorem is giving us a result when there is a shifting of the sequence u n to the right, suppose Z transform of a sequence u n is given by U z then Z transform of u n minus k is equal to z to the power minus U z where k is a positive integer.

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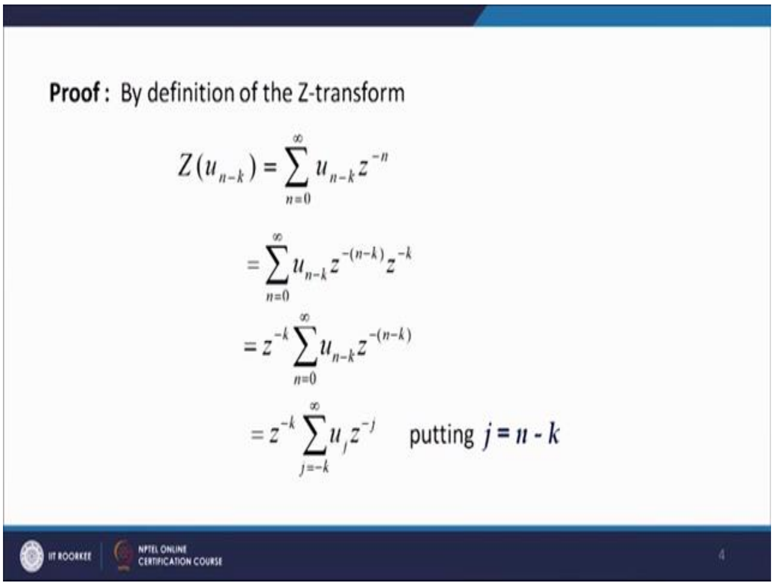

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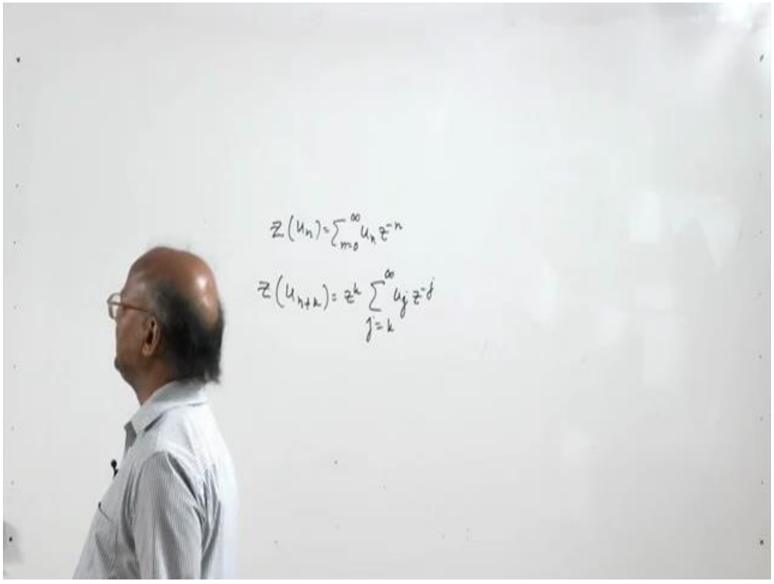

Let us see how we prove it, z we know that Z transform of u n Z transform of u n is sigma n is equal to 0 to infinity u n z to the power minus n. So, apply this definition to the sequence u n minus k. So, Z transform of u n minus k is equal to sigma n equal to 0 to infinity u n minus k z to the power minus n now z to the power minus n we can write as z to the power minus k into z to the power minus k this z to the power minus k is independent of n. So, we can write it outside the summation, so z to the power minus k sigma n equal to 0 to infinity u n minus k z to the power minus n minus k. Now let us define the index j equal to n minus k. So, when index j is n minus k since n begins with 0 n is equal to when n is equal to 0 j is equal to minus k.

So, we have this summation we of over j starting with minus k and when n is infinity j is equal to infinity. So, we have infinity here. So, we what we get is Z transform of u n minus k is equal to and z to the power minus k sigma j equal to minus k 2 infinity u j z to the power minus k now k is a positive integer. So, u j wherever j is negative then we know that when we define the Z transform where we have assumed he sequence u n to be equal to 0 when n is less than 0. So, we will be make use of that property of the sequence u n.

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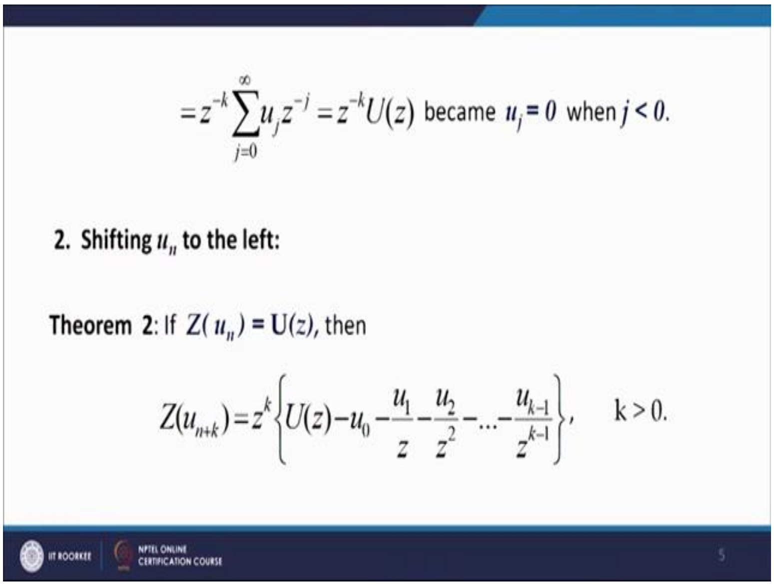

So, z to the power minus k and u j you can see here u j is 0 because whenever z j is 0 u j 0. So, j which was starting with minus k here, all these values of u j where j is negative will be equal to 0 and so the summation will start with j equal to 0 and go to infinity. So, we have z to the power minus k sigma j equal to z to infinity j z to the power minus j. Now sigma j equal to 0 to infinity u to the power u; u j is z to the power minus j is U z. So, what we get is z to the power minus k U z.

So, when there is a shifting by k on derived side the; see the Z transform of the sequence u n minus k s given by z to the power minus k into Z transform of u n. Now suppose there is shifting of u n to the left. So, if Z transform of u n is U z then we shall see that Z transform of u n plus k the sequence u n plus k where k is again a positive integer is equal to z to the power k into U z which is z, Z transform of u n minus u naught minus u 1 by z minus u 2 by z square minus u k minus 1 over z to the power minus 1.

So, this can also be said that here in the inside he bracket we have the Z transform of u n sequence and from that we subtract the terms of the series sigma u n z to the power minus n from n equal to 0 to n equal to k minus 1. These are the terms of the series sigma u n z to the power minus n where n runs over 0 and goes up to k minus 1. So, let us see the proof of this again recall the definition of Z transform of u n Z transform of u n is sigma a equal to 0 to infinity u n z to the power minus n. So, Z transform of u n plus k Z transform of u n plus k will be sigma n equal to 0 to infinity u n plus k z to the power minus n again we apply a use their use the same procedure here apply the same procedure z to the power minus n can be written as z to the power minus n plus k into z to the power k and z to the power can be taken outside the summation because it is independent of n. So, z to the power k sigma n equal to 0 to infinity u n plus k z to the power minus and plus k now let us define the index j s equal to n plus k.

So, here n was beginning with 0 and ending at infinity. So, n is equal to 0. So, j equal to k. So, summation over j will begin from k and go up to infinity because when n is infinity j is infinity. So, z to the power k sigma j equal to 0 to k equal to sigma j equal to k to infinity u j z to the power minus j now. So, what we have is z u n plus k equal to z to the power k and then we have summation over j from k to infinity and we have u j z to the power minus j what we will do is now we will we can do summation can be expressed as sigma j equal to 0 to infinity u j z to the power minus j and from that we subtract the terms sigma where sigma runs from 0 to k minus 1.

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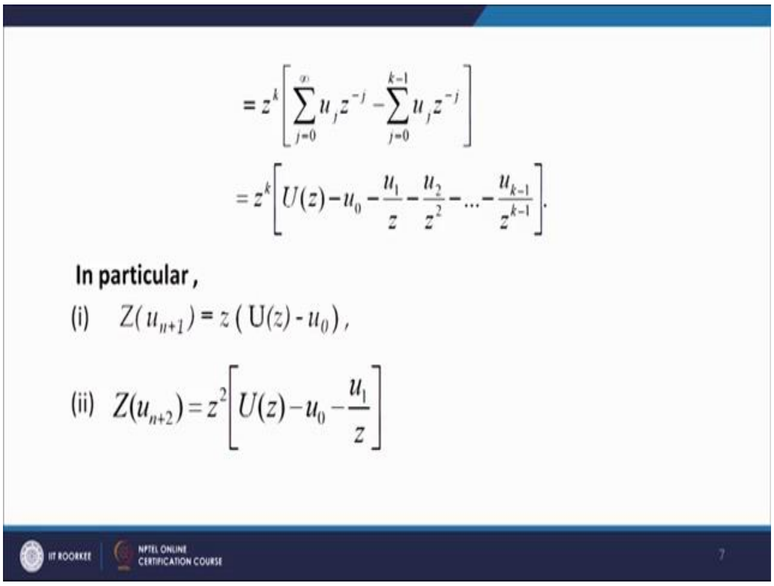

So, j equal to 0 to k minus 1 u j z to the power minus k. So, we subtract this sum and add this sum. So, when we add this sum to the existing sum that is sigma j equal to 0 k sigma j equal to k to infinity u j z to the power minus j then we will have this sigma j equal to 0 to infinity u j z to the power minus j now this is nothing, but U z, Z transform of the sequence u n. So, z to the power k z this is U z and minus; you can write the terms of the sum. So, this is k j equal to 0 means you will not j equal to 1 means u 1 by z and j equal to 2 means u 2 by z square and so on minus u k minus 1 over z to the power minus 1.

So, this is shifting of this sequence going to the left, now in particular suppose we want to get the value of z of u n plus 1. So, take k equal to 1 in this exp expression. So, z of u n plus 1 will be z times u z minus u naught and when you take k equal to 2 z of u n plus 2 will be equal to z square u z minus u naught minus u 1 by z.

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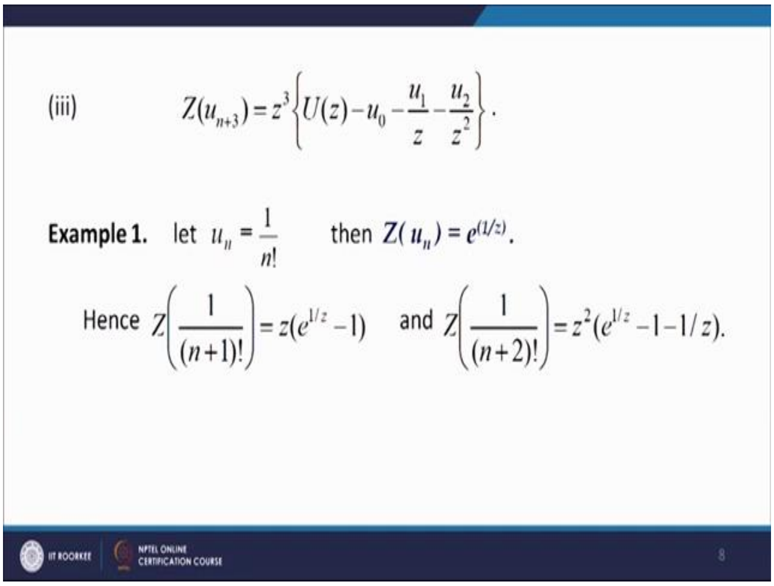

And in a similar manner when we take k equal to 3 we get z of u n plus 3 equal to z q U z minus naught minus q 1 by z minus u 2 by z square.

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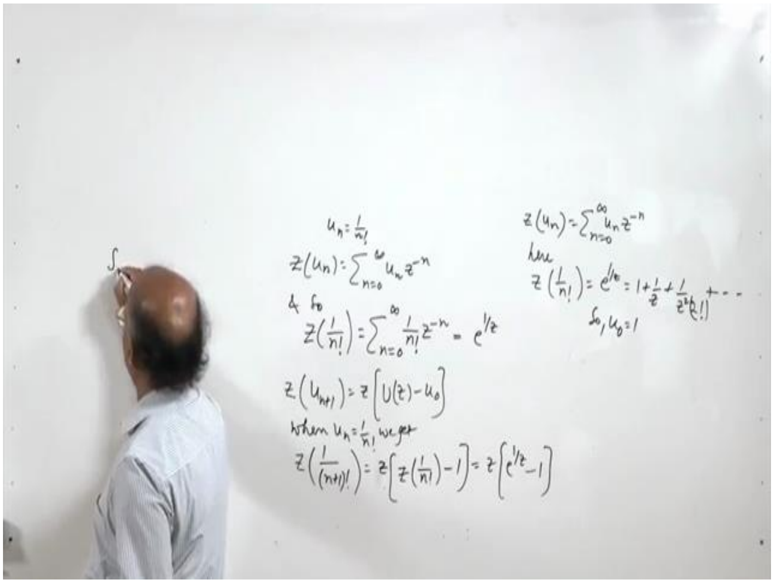

Let us take for an example the sequence u n to be equal to 1 by n factorial. So, let say when u n is 1 by factorial Z transform of u n is equal to sigma n equal to 0 to infinity u n z to the power minus n and so Z transform of 1 by n factorial equal to sigma n is equal to 0 to infinity 1 by n factorial z to the power minus n and we know that this is nothing, but the Maclaurin series of the function e to the power 1 by z. So, we get the Z transform of the sequence u n equal to e to the power 1 by z and hence, but z; Z transform of 1 over n plus 1 factorial we know the Z transform. So, Z transform of u n plus 1 we have seen this is equal to z times u z minus u naught and U z is a transform of un. So, if u n is 1 by n factorial when u n is 1 by n factorial, we get Z transform of 1 over n plus 1 factorial equal to Z transform z times z times Z transform of 1 by n factorial minus u naught; u naught is what? U naught is the term occurring for n equal to 0 in the summation u n z to the power minus n and running from c and running from 0 to infinity see we have Z transform of u n Z transform of u n is sigma n equal to 0 to infinity u n z to the power minus n. So, u naught is the term in the sum which is where there is power of z is 0. So, here Z transform of 1 by n factorial we have seen t is equal to e to the power 1 by z and is equal to 1 plus 1 by z plus 1 by z square into 2 factorial and so on.

So, the first term here is 1. So, u naught is equal to 1 k. So, this is minus 1 and this is z times e to the power 1 by z we have found minus 1. So, Z transform of 1 over n plus 1 factorial is z times e to the power 1 by z minus 1 and similarly, Z transform of 1 over n plus 2 factorial.

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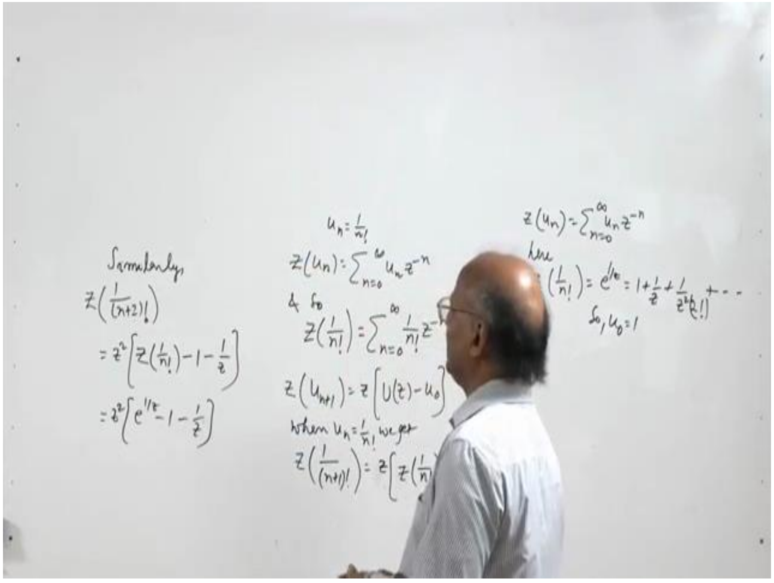

So, if u n is 1 by n factorial u n plus 2 is 1 over n plus 2 factorial. So, this is z square times U z; z is Z transform of u n. So, 1 by n factorial minus u naught u naught we have seen u naught is equal to 1 minus u 1; u 1 is the coefficient of 1 by z which is also 1 here. So, 1 by z because Z transform of n plus 2 is z square times Z transform of u n minus u naught minus 1 by z. So, this is z square times e to the power 1 by z minus 1 by z.

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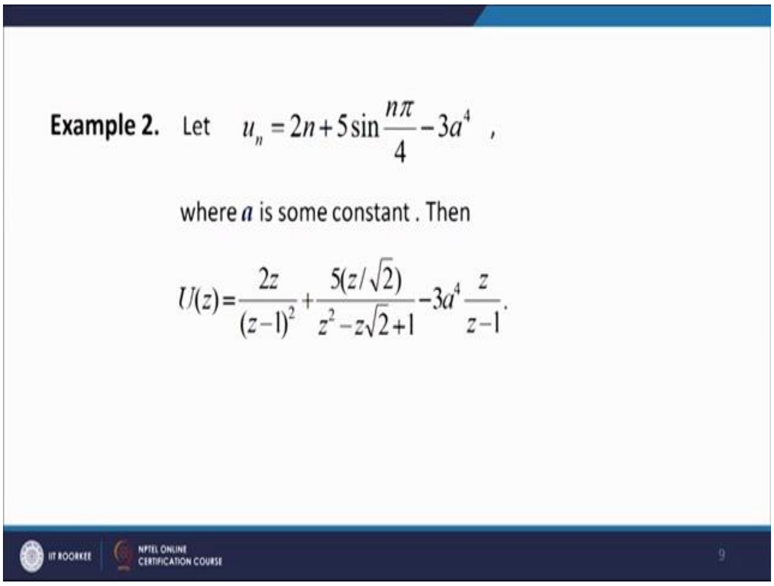

So, this is how we find the Z transform of 1 over n plus 2 factorials. Now let us see an example where we shall find the Z transform of sequence u n consisting of the terms 2 n 5 sin n pi by 4 and minus 3 a 4.

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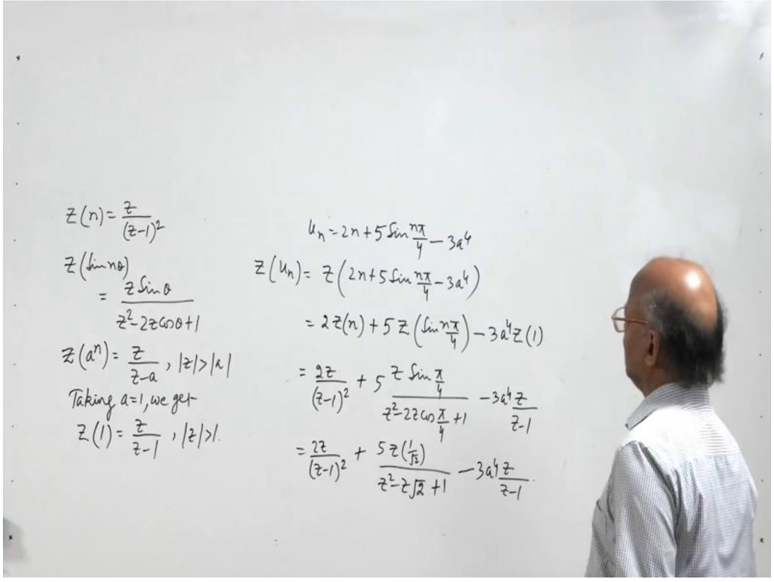

So, let us find the Z transform of u n un is 2 n plus 5 sin n pi by 4 and then minus 3 a to the power 4 a is given to be a constant. So, Z transform of u n is equal to Z transform of 2 n plus 5 sign n pi by 4 minus 3 a 4. Now let us apply the linearity property. So, by the linearity property, this is 2 times Z transform of n plus 5 transform 5 times Z transform of sin n pi by 4 and then minus 3 a 4 Z transform of 1. Now let us recall that Z transform of n is z over z minus 1 whole square we have earlier found the Z transform of n it came out to be z over z minus 1 whole square let us also recall the Z transform of sin n theta it came out to be z sin theta divided by z square minus 2 z cos theta plus 1 and then let us recall the Z transform of 1.

So, z, let us recall that Z transform of a to the power n is equal to z over z minus a provided mod of z is greater than mod of a. So, here let us take a equal to 1. So, taking a equal to 1, we get 1 to the power n that is 1. So, Z transform of 1 is equal to z over z minus 1 provided mod of z is greater than 1. So, now, let us apply let us put these values here. So, 2 times z over z minus 1 whole square and then 5 times z sine theta is pi by 4 here divided by z square minus 2 z cos pi by 4 plus 1 and then minus 3 a 4 z over z minus 1 now let us recall that sin pi by 4 and cos pi by 4 both are equal and equal to 1 by root 2. So, we will have 2 z divided by z minus 1 whole square and then we will have 5 times z into 1 by root 2 and then we shall have this is 1 by root 2 root 2; 2 over root 2 will be root

2. So, z square minus z is square root 2 plus 1 and minus 3 a 4 z over z minus 1.

So, this how we found the find the Z transform of this sequence even by using the linearity property and the standard formulae for Z transform of n Z transform of sin n theta Z transform of a to the power n. So, as I said we have to remember the Z transform of some special sequences here which we shall need to find the Z transform of other sequences and these will also help when we solve the difference equations.

So, in the next lecture we shall discuss some more properties of the Z transforms like initial value theorem, final value theorem and one more theorem is that which is multiplication by n. So, we will discuss some more properties in our next lecture on properties of Z transforms. With that I would like to conclude my lecture.

Thank you very much for your attention.