Mathematical methods and its applications
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Lecture – 04
Solution of second order homogenous
linear differential equation with constant coefficients II

Hello friends. Welcome to my second lecture on Solution of Second Order Homogenous Linear Differential Equation with Constant Coefficients.

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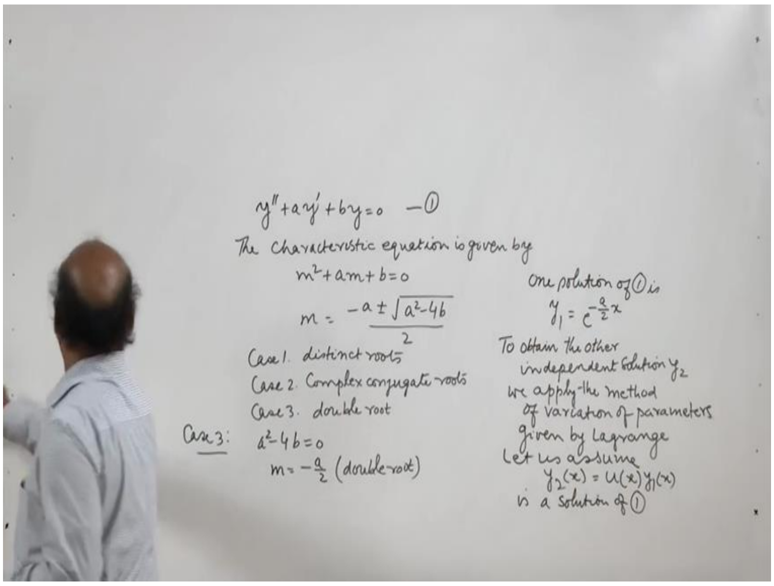

We are discussing the solution of second order homogenous linear differential equation with constant coefficients that is given by y double dash plus a y dash plus b y equal to 0. The characteristic equation here is given by m square plus a m plus b equal to 0. Let us find the roots of this quadratic equation m equal to minus a plus minus under root a square minus 4 b divided by 2. So, there are two values of m, we have three cases - case 1 distinct roots, case 2 complex conjugate roots, and then case 3 double root. So, the case 1, when the two roots of the characteristic equation are distinct, we have already covered in previous lecture; and the case 2, where the two roots are complex conjugate that also we have covered in the previous lecture.

So, in this lecture, we begin with the case 3. Let us say that the root of the quadratic the characteristic equation has double root that is in the case 3 the discriminant a square minus 4 b equal to 0, in this case the discriminant a square minus 4 b equal to 0. So, we have m equal to minus a by 2 which is a double root. Now this double root at first gives us one solution of the second order linear differential equation one. We can say the one solution is one solution of the given solution one, we can write as y 1 equal to e to the power minus a by 2 into x.

Now to obtain the other independent solution by 2, we apply the method of variation of parameters this method of variation of parameters was given by Lagrange for first order linear differential equation with first order linear differential equation. So, in this method, what we do is let us assume the other solution by 2 x to be equal to u x into y 1 x, we shall find out u x, so that y 2 is the other independent solution of the given differential equation. So, let us assume that y 2 x is equal to u x into y 1 x is the solution of equation 1

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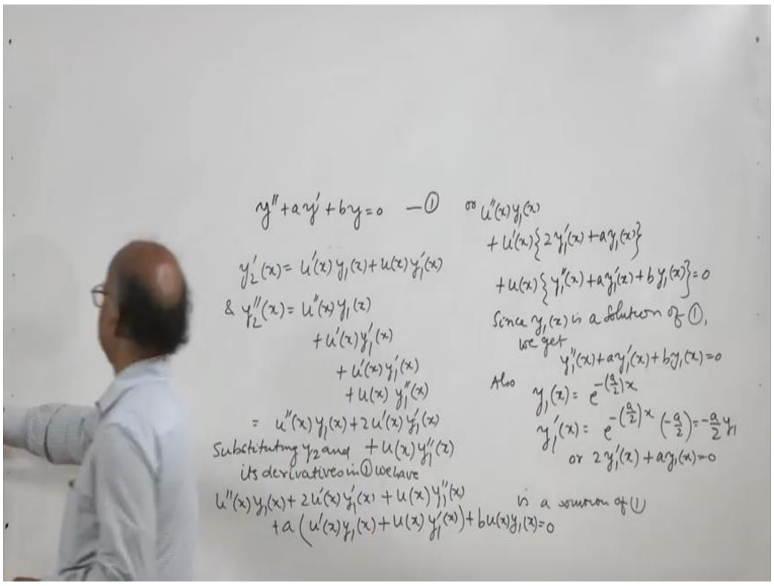

Then we shall have let us substitute y 2 x. So, let us y 2 x gives y 2 dash x equal to u dash x into y 1 x plus u x into y 1 dash x; and y 2 double dash x equal to u double dash x into y 1 x plus u dash x into y 1 dash x plus u dash x into y 1 dash x plus u x into y 1 double dash x. Or we can say this u double dash x into y 1 x plus 2 times u dash x into y 1 dash x plus u x into y 1 dash double dash x. Now, let us substitute the values of y 2, y 2 dash, y 2 double dash in equation 1. So, substituting y 2 and its derivatives in 1, we have u double dash x into y 1 x plus 2 u dash x into y 1 dash x plus u x y 1 double dash x plus a times y dash is y 2 dash. So, y 2 dash is equal to u dash x into y 1 x plus u x into y 1 dash x plus b times, y is replace by y 2, and y 2 is u into y 1.

Now, let us collect the coefficients of the derivatives of u. So, we may write this equation as u double dash, the coefficients of u dash x are 2 y 1 dash x plus a times y 1 x, the coefficient of u x is y 1 double dash x plus a y 1 dash x plus b y 1 x equal to 0. Now, since y 1 is a solution of equation 1, we get y 1 double dash x plus a y 1 dash x plus b y 1 x equal to 0; also y 1 x is equal to e to the power minus a by 2 into x. So, let us differentiate it with respect to x, it will give me which is equal to minus a by 2 y 1 or we can say 2 y 1 dash x plus a y 1 x equal to 0.

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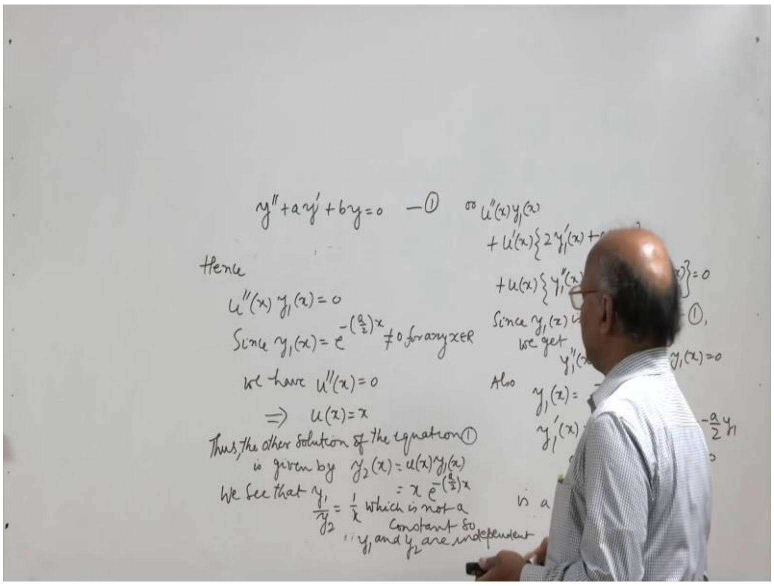

So, substituting these values here, it reduces we get u double dash x into y 1 x equal to 0; the coefficient of u dash x is 0 from here, the coefficient of u x is 0 from here. So, u double dash x into y 1 x is equal to 0. Since, y 1 x is e to the power minus a y 2 into x which is not equal to 0 for any x belonging to R, we have u double dash x equal to 0. So, when we integrate this twice, we get which implies u x equal to x, thus the other solution of the equation 1 is given by x into e to the power minus a by 2 into x. So, this is for one solution and then we to find the other solution we applied the method of variation parameters where we assumed y 2 x equal to u x into y 1 x.

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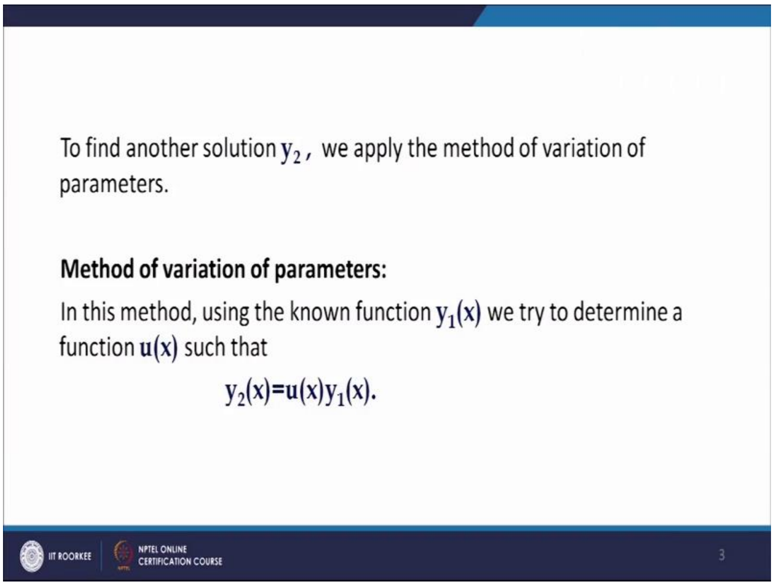

So, the now let us see that y 1 and y 2 we see that y 1 over y 2 is equal to 1 over x which is not a constant. So, the y 1 and y 2 are independent of two each other.

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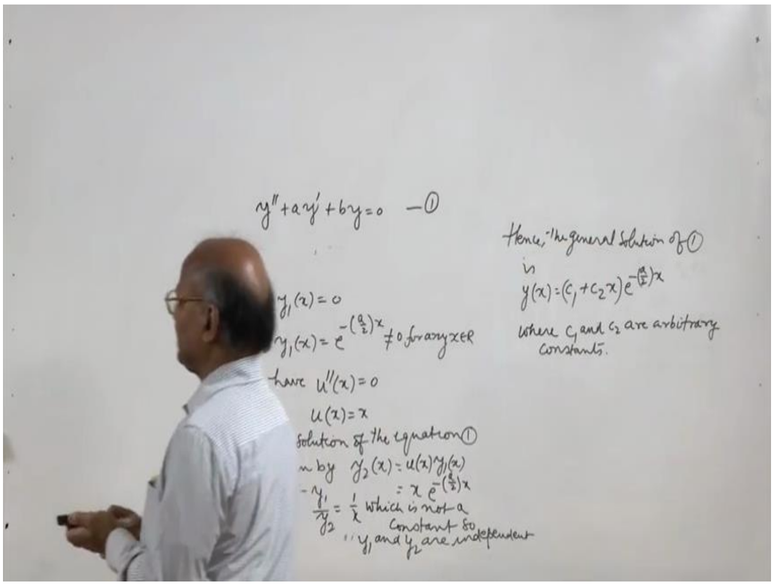

So, we got two independent solutions of the equation 1, and therefore, we can write the general solution as y x equal to c 1 plus c 2 x into e to the power minus a by 2 into x, where c 1 and c 2 are arbitrary constants.

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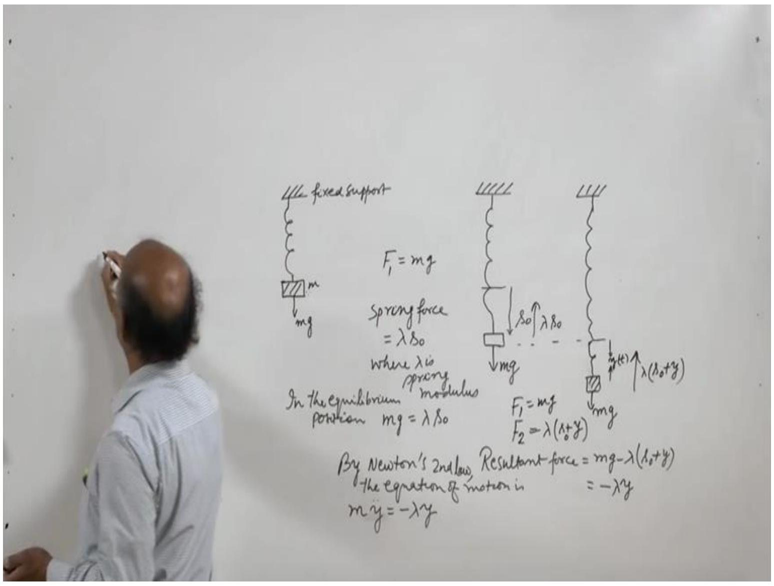

Now, let us take an example on this. First due to gravity which will at downward will be mg. So, let us take f 1 equal to mg. Now, due to the mass m that is due to the gate m g there will be a extension in this spring, before it comes to the equilibrium position. So, let us say in the equilibrium position, the force due to gravity that is f 1 equal to mg will have downward, while the spring force the spring force will be lambda s naught where this spring force is, when lambda is spring modulus, and s naught is the extension or stretch in the spring.

So, let us say this is s naught. This is spring force will at in the direction of opposite to mg. So, in the equilibrium position mg and lambda is not will balance each other, and we shall have mg equal lambda s naught. Here, we are using the hooks law, which says that the restoring force of the spring or we can also call it as the spring force is proportional to the stretch or extension of the spring.

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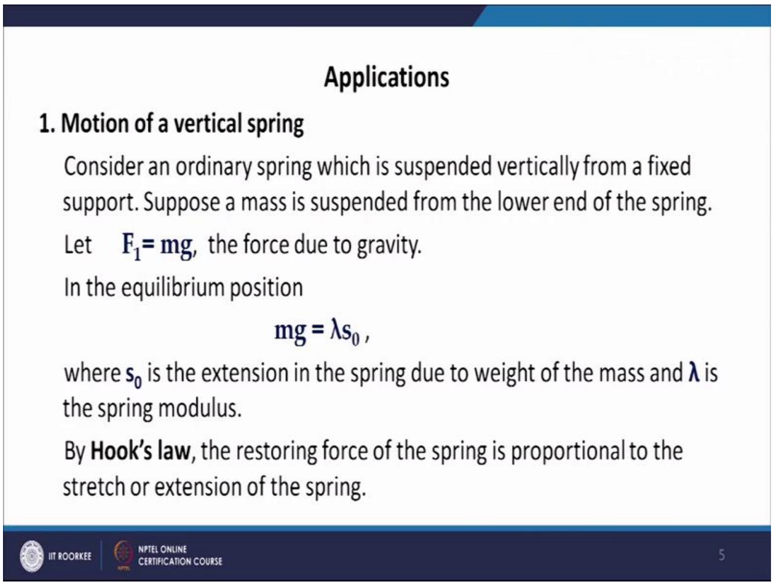

So, here there is a stretch or extension s naught in this spring. So, the spring force are restoring force is proportional to s naught and therefore, we write it as equal to lambda s naught by lambda is constant or proportionality we call it as a spring modulus. Now, let us pull the mass m downward and then release it. So, the mass m is pulled down and then released, then there will be a vertical motion in the spring the forces that will be obtained let say at some time t during the motion of the spring y may be displacement from the equilibrium position at time t. So, then the forces that will be acting on the mass m will be the force due to gravity mg downwards f 1 equal to mg, f 2 equal to this spring force which will be lambda times s plus s naught plus y because s naught plus y is the total extension in the a spring.

So, this lambda s naught plus y will act in the direction opposite to mg. The resultant force therefore is since mg is equal to lambda s naught, we have resultant force as minus lambda y. Now, let us apply the Newton's second law, by Newton's second law, resultant force is equal to m times d square y over dt square.

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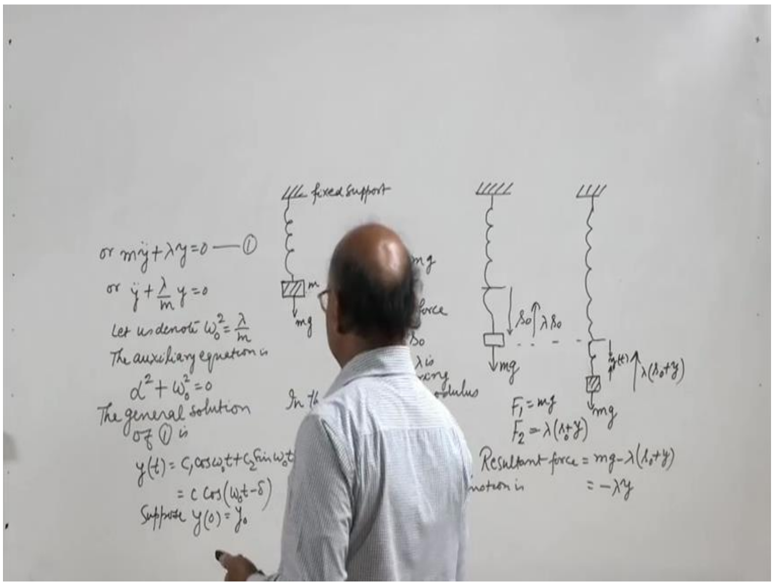

So, we have the equation of motion as m by double dot equal to minus lambda by or where y double dot means d square y over dt square. Now, we can also write it as. So, if we denote omega naught, let us denote by omega naught square lambda by m then we have the auxiliary equation is let me not write m. So, alpha square plus omega naught square equal to 0.

And now since this equation has two complex roots. So, we have the general solution of equation 1 as by t equal to c 1 cos omega naught t plus c 2 sin omega naught into t. Now, it can also be expressed as c times cos omega naught t minus delta, where c is equal to under root c 1 square plus c 2 square and tan delta is equal to c 2 by c 1 it can be easily checked. Now, if y naught is equal to if we impose the initial conditions on this motion as suppose y 0 is equal to y naught that means, when we release the mass m at that time t equal to 0, the displacement of the mass m from the equilibrium position is y naught and the velocity at that instant is given by y 1, then we can see from here, then put t equal to 0 in this we get y 0 equal to y naught, y naught is equal to c cos delta.

And from y t equal to c cos omega naught t minus delta, if you use y dash 0 equal to y 1, you get y 1 equal to c omega naught sin delta.

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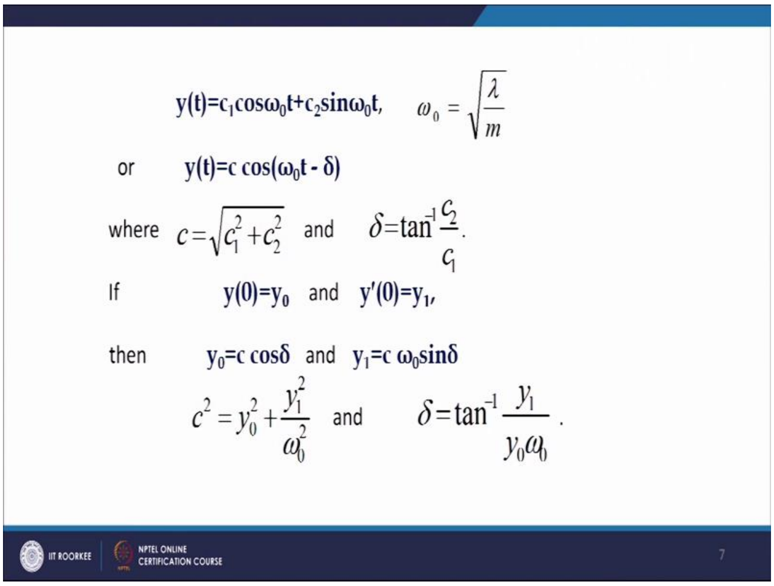

So, from these two equations, y naught equal to c cos delta and y 1 equal to c omega naught sin delta, we can find the value of c, c square is equal to y naught square plus y 1 square over omega naught square and delta equal to tan inverse y 1 over y naught omega naught. So, the two constants c and delta can be found from the initial conditions y 0 equal to y naught, and y dash 0 equal to y 1. Or we can also put c as 1 over omega naught under root y 1 square plus omega naught square y naught square until delta equal to this.

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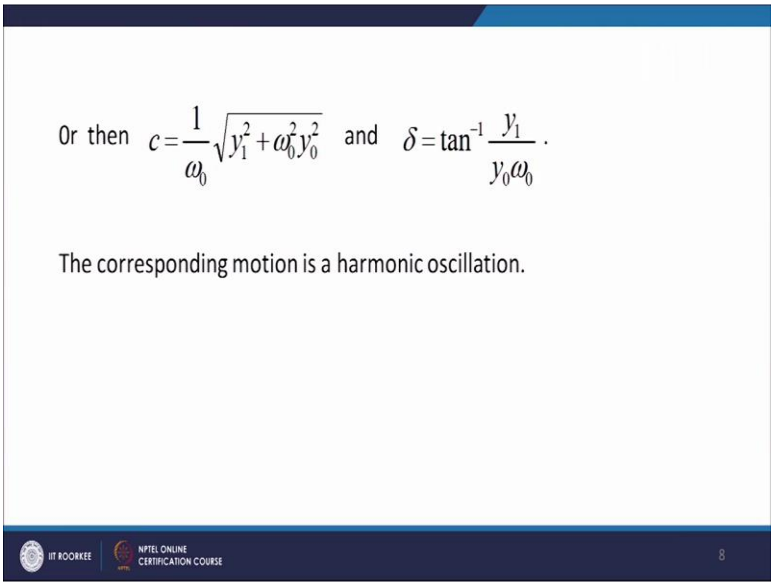

Now, we can see here that since y t is equal to c cos omega naught t minus delta, the motion is a harmonic oscillation.

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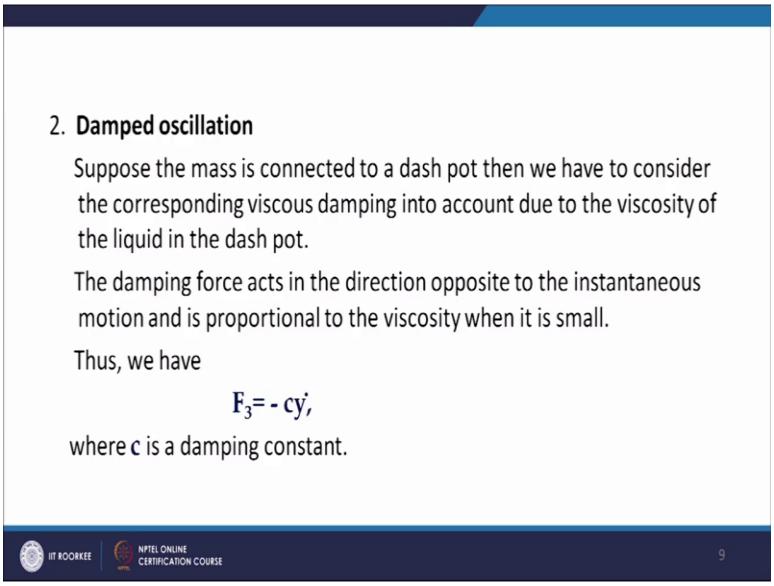

Now, let us go to damped oscillation.

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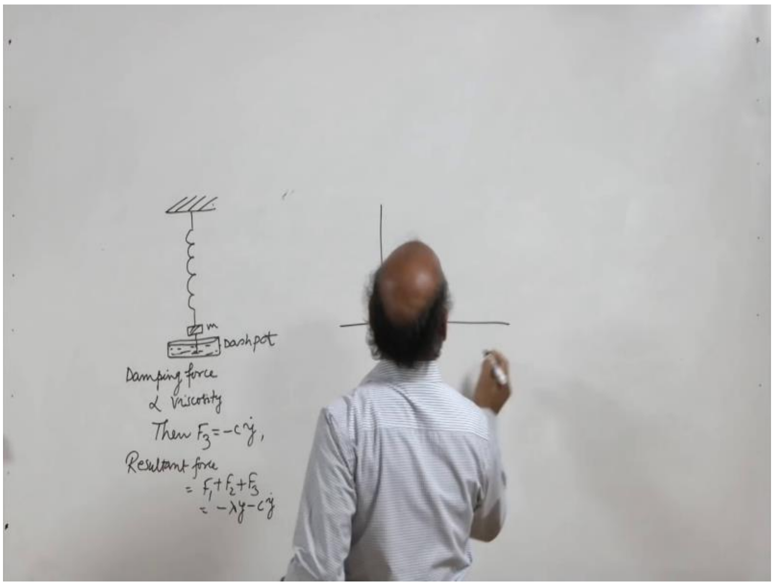

So, in the case of damped oscillation suppose the mass is connected to a dashpot, this is dashpot. So, suppose the mass is connected to a dashpot, then we have to consider the viscous force due to the viscosity of the liquid in the dashpot. Now, the damping forces due to the viscous liquid x in the direction opposite to the instantaneous motion and is proportional to the viscosity. So, the damping force is proportional to the viscosity, when it is small. So, then we have the third force which is F we have taking as F 3, F 3 equal to minus c y dot, y dot is the velocity at instant t.

So, the force due to the viscosity of the liquid, which is there in the dashpot, there will be a force which is called damping force it x in the direction opposite to the instantaneous motion. And this c is a dumping constant; this c is a damping constant.

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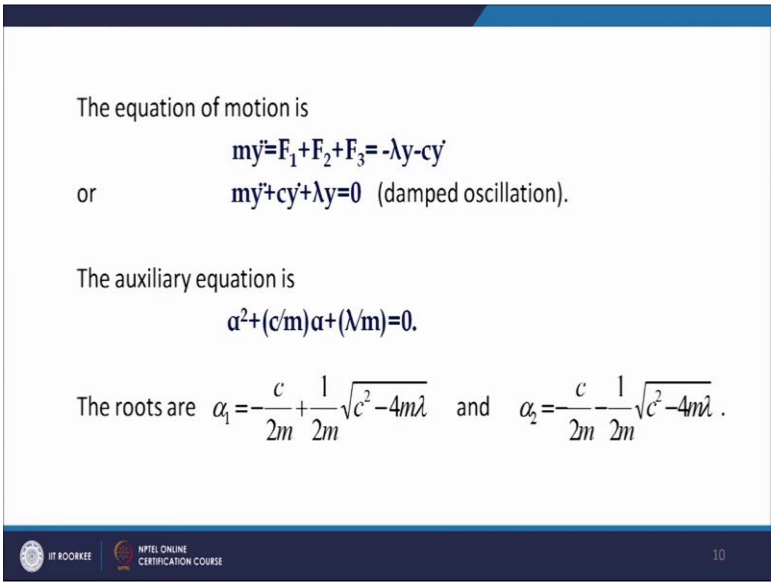

Now, let us see a; so in the now the resultant force will be F 1 plus F 2 plus F three which is equal to f one plus f two we have seen is minus lambda y, so minus lambda y plus minus c y naught dot. Thus we get the equation motion as n y double dot plus c y dot plus lambda y equal to 0. The auxiliary equation will be given by alpha square plus c by m into alpha plus lambda by m equal to 0. The two roots of this auxiliary equation are given by alpha 1 and alpha 2. The alpha 1 is minus c by 2 m plus 1 by 2 m under root c square minus 4 m lambda; and alpha 2 is minus c by 2 m minus 1 by 2 m under root c square minus 4 m lambda.

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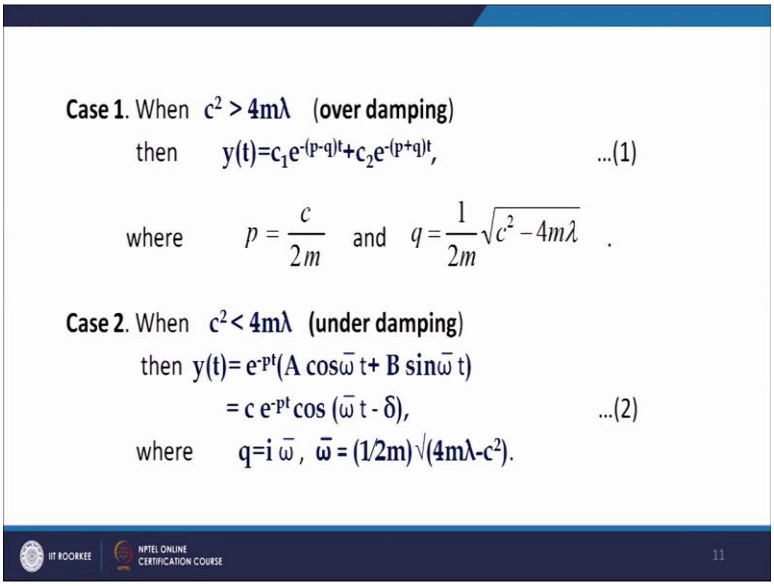

Now, due to the discriminant c square minus 4 m lambda, there arise three cases the case 1 is c square is greater than 4 m lambda; in that case, the two roots alpha one and alpha two of the auxiliary equation will be distinct. So, now we have three cases, in the case 1 when c square is greater 4 lambda, we have the case of over dumping because the dumping constant c is high here.

So, in this case, the general solution is given by y t equal to c 1 e to the power minus p minus q into t plus c 2 e to the power minus p plus q into t. Now, where p is equal to c by 2 m and q equal to 1 over 2 m under root c square minus 4 m lambda. Now from here we can see that p is strictly greater than q. So, when t goes to infinity, e to the power minus p minus q into t goes to 0 as well as e to the power minus p plus q into t goes to 0. So, in the equation 1, when y t goes to 0, s t goes to infinity that is the system comes to rest after a sufficiently long time.

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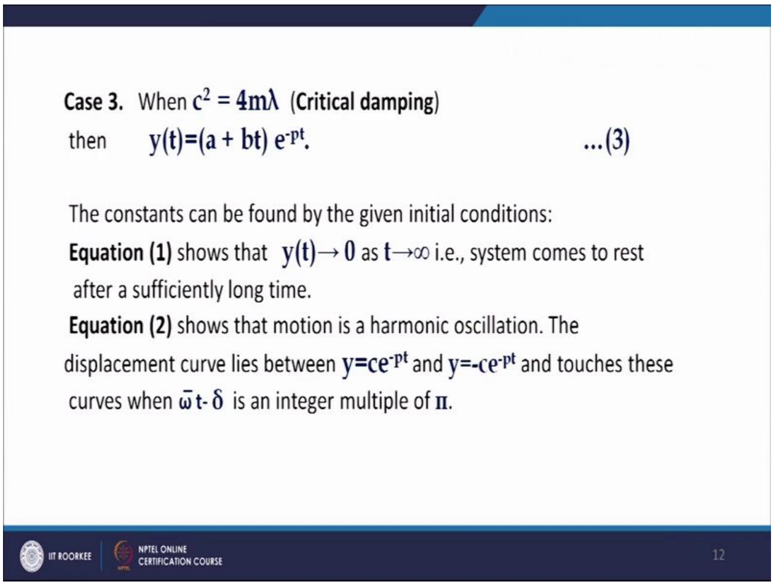

In equation 2, we see that y t is given by c e to the power minus p t cos omega bar t, and therefore the motion is a harmonic oscillation. The displacement curve y t lies between y e equal to c times e to the power minus p t and y equal to minus c times e to the power minus p t. So, this is let us say t axis, this is y t.

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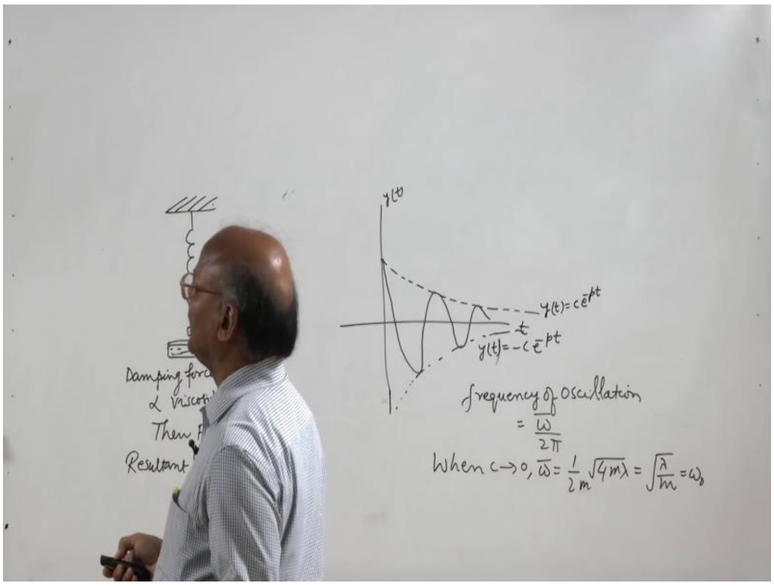

These are the curves y t equal to c e to the power minus p t, and y t equal to minus the displacement curve lies between these two curves, and touches these curves when omega bar t minus delta is an interior multiple of pi. Moreover, the frequency here is frequency of oscillation is equal to omega bar over 2 pi.

So, when omega bar is more frequency is more and when let us see when omega bar will be more, omega bar will be more when c is smaller. When c is smaller here and it will be greatest when c tends to 0. So, when c tends to 0, omega bar goes to 1 over 2 m under root 4 m lambda, which is equal to under root lambda by m which is equal to omega naught. The omega naught which be found in the case of the simple harmonic oscillation this omega naught y t equal to c cos omega naught, here omega naught you can see omega naught is root lambda by m. Now, let us discuss the last case, in the equation 3, you can see, we have y t equal to a plus b t into e to the power minus p t. Since, e to the power minus p t is not equal to 0 y t can never be 0 except possibly once when a plus b t is equal to 0 that is t is equal to minus a by b.

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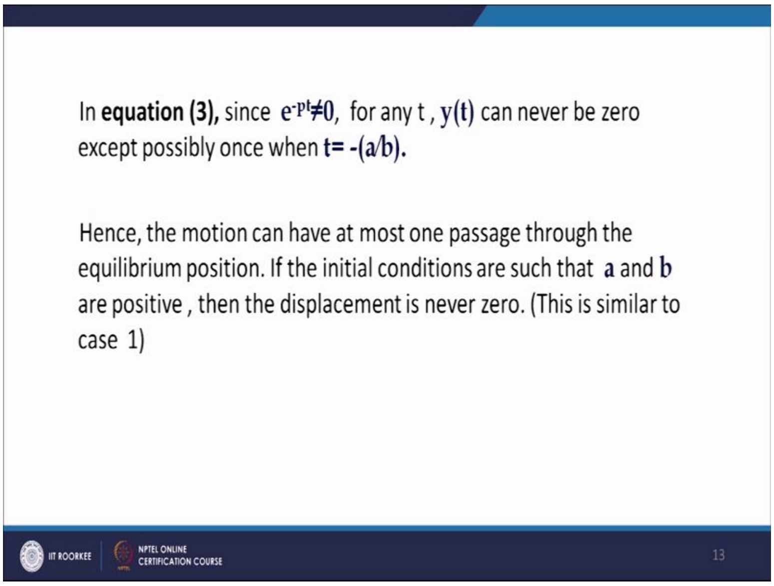

Hence the motion can have at most one passage through the equilibrium position now if the initial conditions are such that a and b are positive then you can see y t can never be 0, because if a and b are positive then y t can never be 0. So, the displacement will never be 0. And this case is similar to the case 1; in the case 1, you can see y t tends to 0 as t goes to infinity. So, the system comes to rest after sufficiently long time.

Thanks.