Mathematical methods and its applications
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Lecture – 39
Z – Transform and inverse Z – Transform of elementary functions

Hello friends, this is my first lecture on Z transforms. So, in this lecture, we shall discuss the Z transform and inverse Z transform of some elementary functions.

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The Z transforms are useful in solving difference equations which arise in the discrete systems the development of the communication branch of engineering is actually based on the theory of Z transforms. The Z transforms has the same role in discrete systems as the Laplace transform in continuous systems. Many properties of the Z transforms are similar to the properties of the Laplace transforms Z transform operates on a sequence of u n of discrete integer valued arguments n equal to 0 plus minus 1 plus minus 2 and so on and like the Laplace transform which operates on continuous functions. Z transform is we can say is the discrete analog of the Laplace transform for every operational rule and application of Laplace transform there is a there is an operational rule and application of Z transforms.

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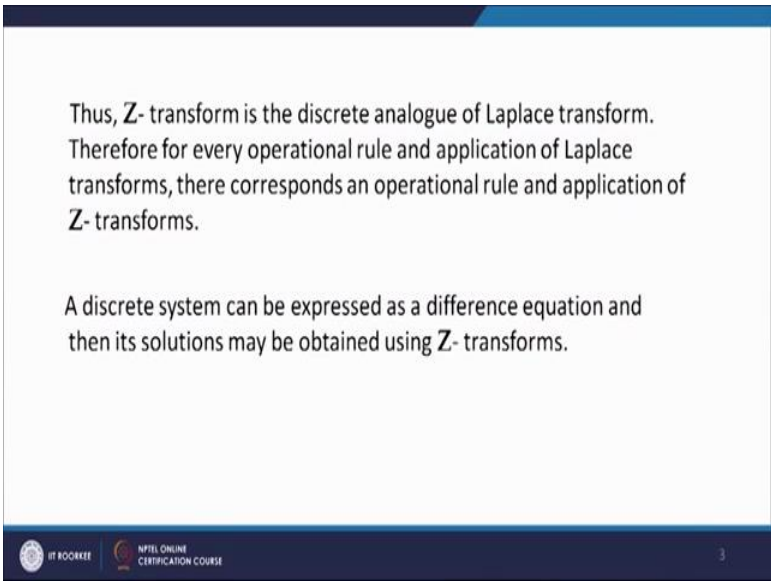

A discrete system can be expressed as a difference equation and then its solution can be obtained by using Z transforms.

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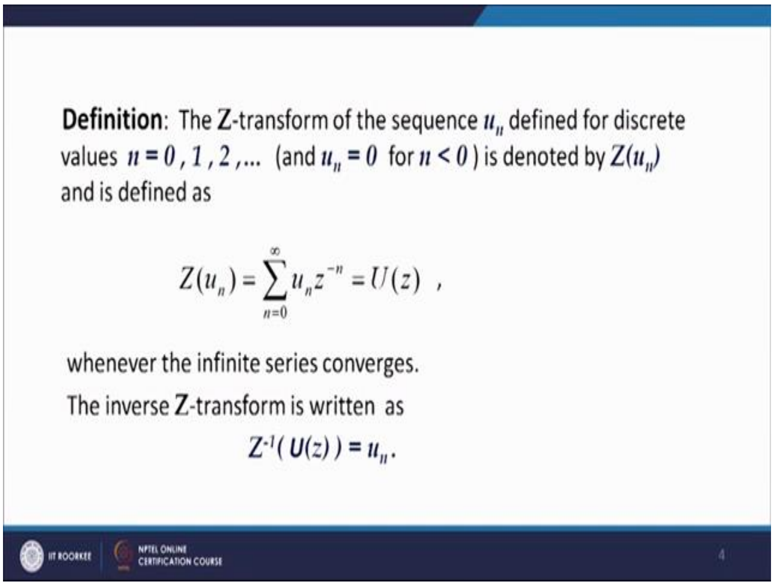

So, let us see how we define the Z transforms the Z transform of the sequence u n defined for discrete values n equal to 0 1 2 3 and so on and e 1 equal to 0 for N less than 0 is denoted by Z u n and is defined as Z u n equal to sigma n equal to 0 to infinity u n z to the power minus n which is a function of z. So, we write it as U z provided the infinite series convergence. So, whenever this infinite series converges it will give us a function of z. So, it is U z the inverse Z transform is written as Z inverse of U z, Z inverse of U z equal to u n. So, whenever we know the Z transform of a sequence, its inverse Z transform will give us back the sequence even like in the case of Laplace transform we have we define the inverse Laplace transform here again the inverse Z transform is defined in a similar manner. So, Z inverse U z equal to u n.

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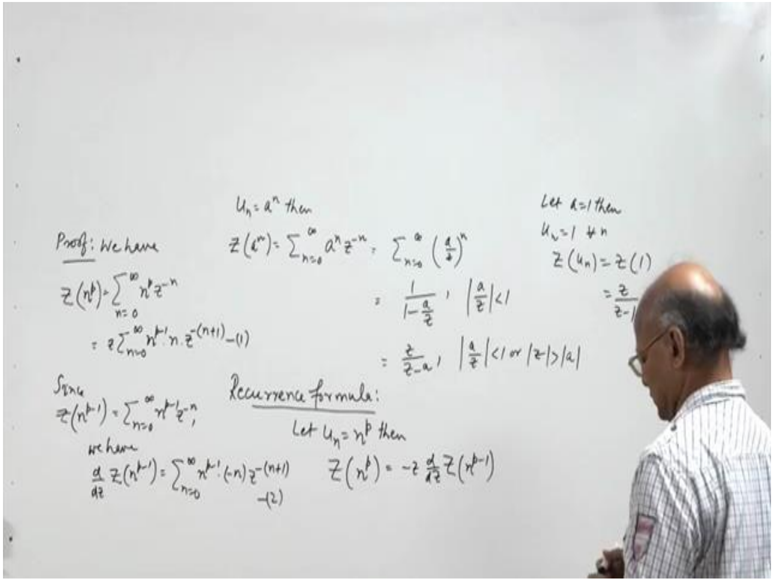

Now, let us discuss Z transforms of some a standard sequences. So, let us begin with u n equal a to the power n. So, when u n is a to the power n then where a is a complex number then z of a to the power n is sigma n equal to 0 to infinity. So, u n here is a to the n, so n equal to 0 to in between u n z to the power minus n. So, we have a to the power n and z to the power minus n this capital Z, this is small z.

Now this is summation n equal to 0 infinity a by z raise to the power n. Now we know that here z is a complex number a is also a, a is a complex constant. So, a by z is a complex number. So, the series of complex numbers sigma n equal to 0 to infinity a by z to the power n converges provided mod of z a by z is less than 1 and we have this is equal to 1 over 1 minus a by z provided mod of a by z is less than 1. So, this gives you z over z minus a provided mod of a by z is less than 1 or mod of z is greater than a. So, the condition for the convergence of the series here is that mod of z must be greater than mod of a.

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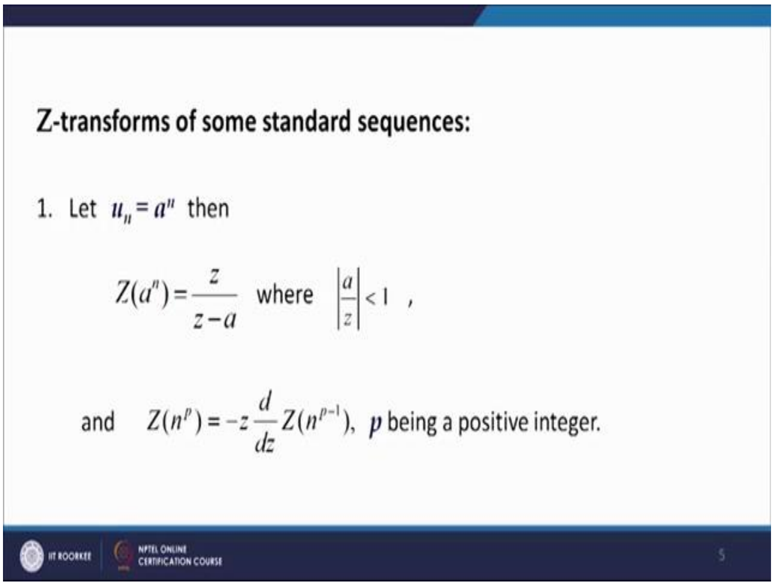

And now let us look at the formula which is known as the recurrence formula if you take u n to be to the power p, let us say u n equal to n to the power p where p is a positive integer then we have a z of n to the power p this formula is very useful in determining the Z transforms of other sequences. So, minus z d over d z of z and to the power p minus 1 so here what do we let us prove this formula?

So, to prove this formula, we have z of n to the power p z of n to the power p by definition is sigma n equal to 0 to infinity n to the power p z to the power minus n. So, we have this we have given here the proof this can be written as z times summation a equal to 0 to infinity n to the power p minus 1 into n into z to the power minus n plus 1. Now since by definition z of n to the power p minus 1 is equal to summation n equal to 0 to infinity n to the power p minus 1 z to the power minus n we have d over d z of d over d z of z n to the power p minus 1 equal to summation n equal to 0 to infinity n to the power p minus 1 into minus n z to the power minus n plus 1. Now let us combine this equation 1 and 2.

So, when you combine 1 and 2, what you find is sigma n equal to 0 to infinity n to the power p minus 1 into n into z to the power minus n plus 1 that is z of n to the power p. So, we get z of n to the power p equal to minus z into d over d z of n to the power p minus 1. So, combining 1 and 2 we have the required recurrence relation from here. So, z of n to the power p minus z d over d z z of n to the power p minus 1, now this result is going to be very useful in finding the Z transforms of some other sequences.

So, let us first look at the special cases of the result z of a to the power n equal to z over z minus a here if you take a equal to 1 we will have see if you take let us say we take let a b equal to 1 then u n is equal to 1 for all n and z of u n is equal to z of 1 will be equal to z over z minus 1 from this result. So, this is valid 1 mod of z is greater than 1.

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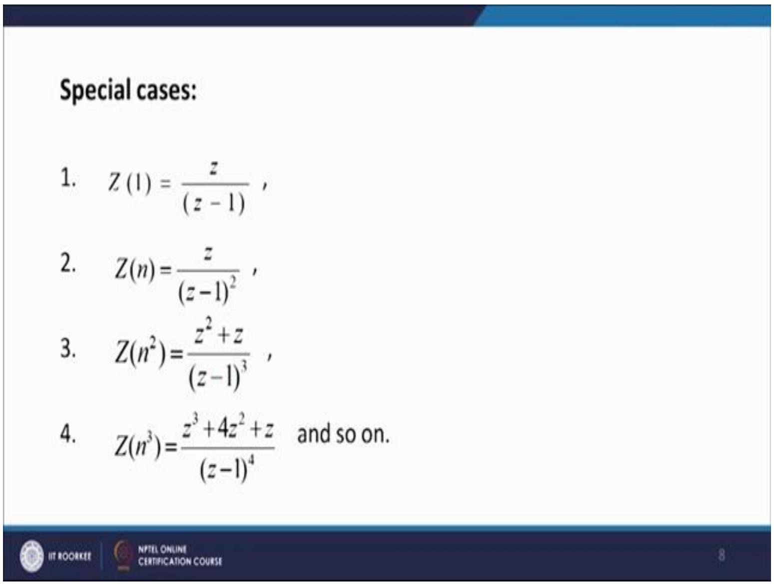

So, z of 1 can be found z of a to the power n equal to z over z minus a by taking a equal to 1. So, z of let us see, how we find z of n equal to z over z minus 1 whole square. So, let us use this result.

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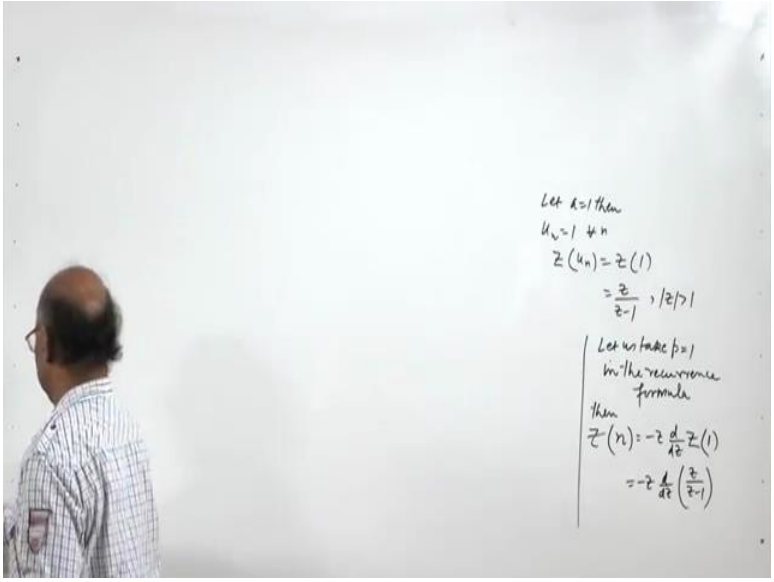

So, let us take p equal to 1 in the recurrence result, in the recurrence formula. So, then z of n to the power p that we are taking p equal to ones z of n equal to minus z d over d z of z to the power n to the power 0 which is 1 and z of 1 we have found just now which is minus z d over d z of, so d over d z of z over z minus 1.

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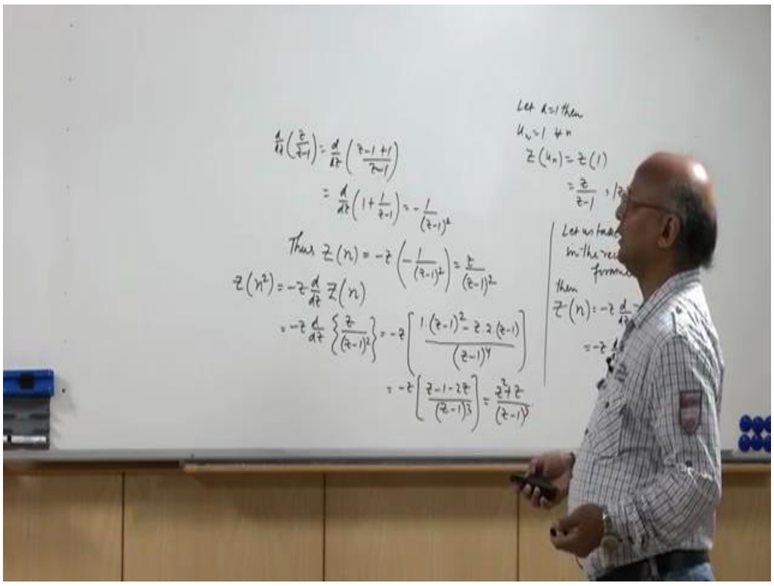

Now, let us differentiate z over z minus 1 with respect to z d over d z of z over z minus 1, we can write it as d over d z of z minus 1 plus 1 divided by z minus 1 which is equal to d over d z of 1 plus 1 upon z minus 1. So, when we differentiate this we get minus 1 upon z minus 1 whole square and thus z of the sequence u n equal n is minus z into minus 1 upon z square which gives you z over z minus 1 whole square. So, that is how we find z of the sequence u n equal to n then next you can find z of u n equal to n square in a similar manner.

So, z of n square now in the recurrence formula we take p equal to 2. So, we shall have in the recurrence formula take p equal to 2. So, at z of n square is minus z d over d z of z of n and z of n we have already found. So, minus z d over d z of z n is z over z minus 1 whole square. So, while differentiating this result by differentiating d over z over z minus 1 whole square we shall have one derivative of z is 1 into z minus 1 whole square minus z into 2 times z minus 1 divided by z minus 1 to the power 4.

So, we can cancel z minus 1 and then what do we get minus z times z minus 1 minus 2 z upon z minus to the power 3. So, what do we get this is minus z minus 1. So, z square plus z divided by z minus 1 whole to the power 3, we get the formula for n square which is z square plus z over z minus 1 whole cube now we can find we can find the value of z n cube in a singular manner by taking p equal to 3 in the difference formula we will need the value of z n square which we have already found s by differentiating the value of z n square and multiplying by minus z we shall arrive at the formula for z n cube which is z cube plus 4 z square plus z over z minus 1 to the power 4 and so on.

So, this is how we find the Z transforms of these sequences u n equal to 1 u n equal to n u n equal to n square u n equal to n cube and then we can see that if we want to find the inverse the inverse Z transform then inverse Z transform of z over z minus 1 Z inverse z inverse of z over z minus 1 will be equal to 1.

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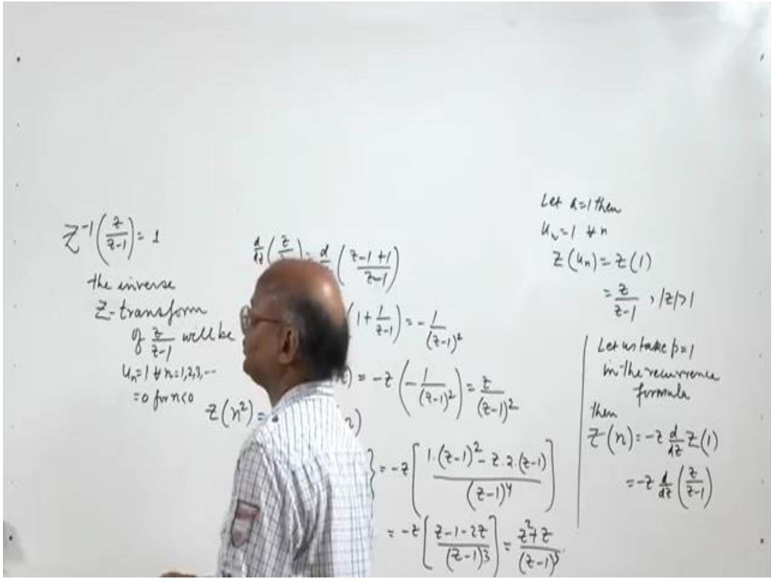

So, the inverse Z transform inverse Z transform inverse Z transform of z over z minus 1 will be u n equal to 1 for all n equal to 1, 2, 3 and so on and 0 for and less than 0.

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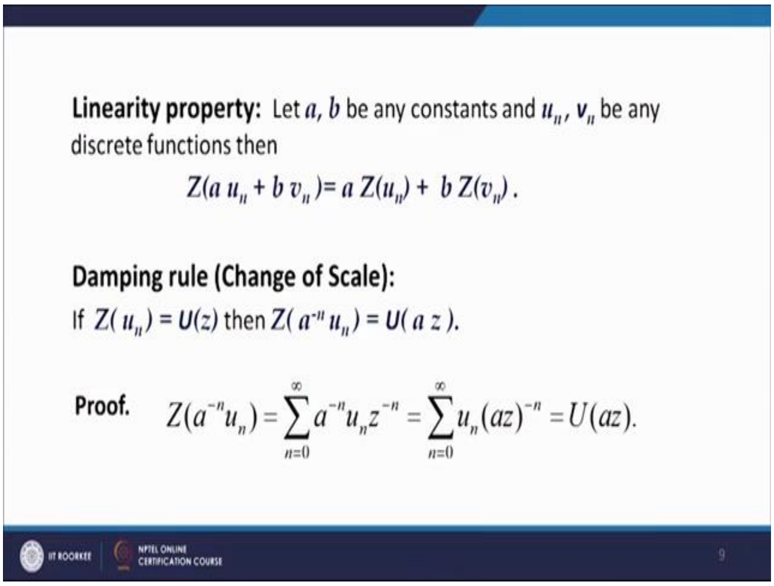

Now, let us go to properties of the Z transform if we know that in the Laplace transform, we have the linearity property. So, here also we have the linearity property if n be are any complex constants real or complex constants and u n and b n are 2 sequences defined on the discrete arguments then z of a u n plus v v n equal to a Z u n plus b z b v n this.

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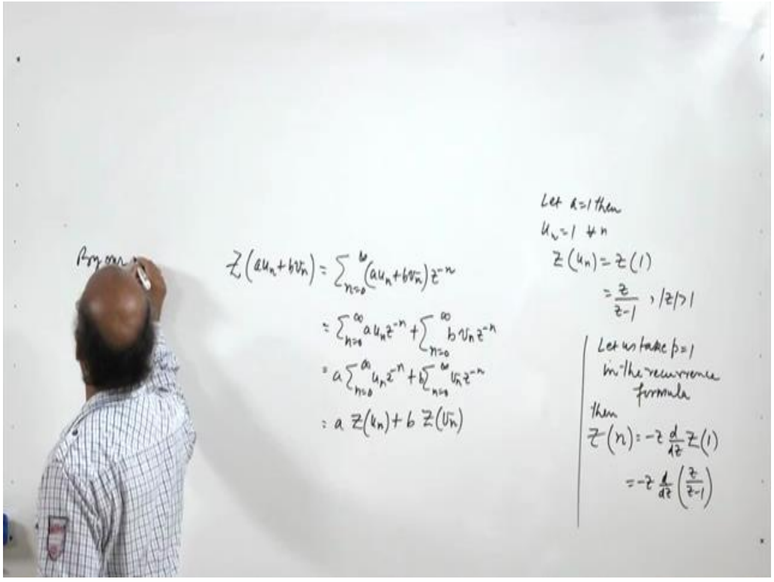

We can easily prove by the definition of the Z transform z of a u n plus b v n will be equal to summation n equal to 0 to infinity a u n plus b v n into z to the power minus n which can be broken as summation n equal to 0 to infinity a u n z to the power minus n plus summation n equal to 0 to infinity b v n into z to the power minus n and which is equal to a times summation n equal to 0 to infinity u n z to the power minus n plus summation b times summation n equal to 0 to infinity b n z to the power minus and this is a times Z u n plus b times z v n. So, Z transforms satisfies the linearity property and then we have the change of scale property if you remember we have the change of the scale of the in the Laplace transforms. So, here we also we have the change of scale property if z of the sequence u n is given by U z then z of a to the power minus n u n is equal to u a z this can be proved very easily.

Let us write z of a to the power minus n u n by definition this will be sigma n equal to 0 to infinity a to the power minus n u n into z to the power minus n and this can be then written further as sigma n equal to 0 to infinity u n into a z to the power minus n now by our definition by our definition z of u n is summation n equal to 0 to infinity u n z to the power minus n which we have denoted by U z which we have denoted by u z.

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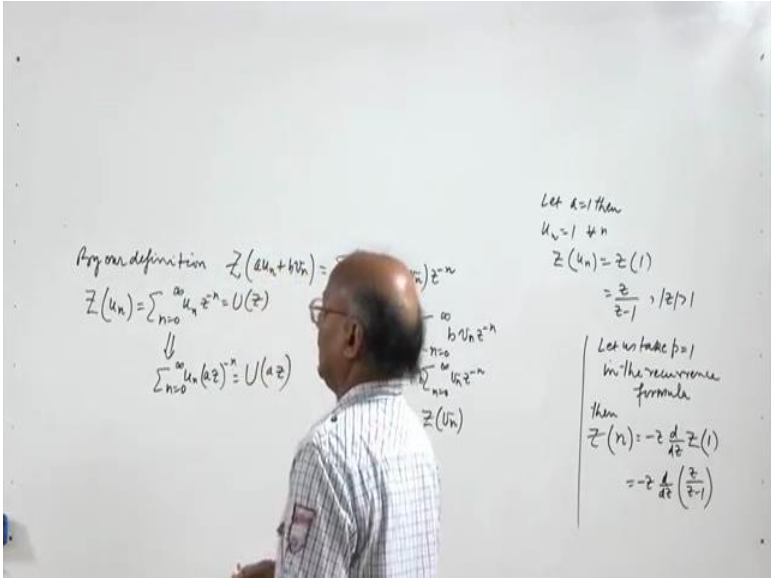

So, here we have sigma n equal to 0 to infinity u n a z to the power minus in place of z we have a z. So, this implies sigma n equal 0 to infinity u n a z to the power minus n equal u a sorry, u a z. So, that is how we get the change of scale property when whenever we multiply u n sequence by a to the power minus n there is a change of a scale in place of z in U z we get a z U z becomes u a z.

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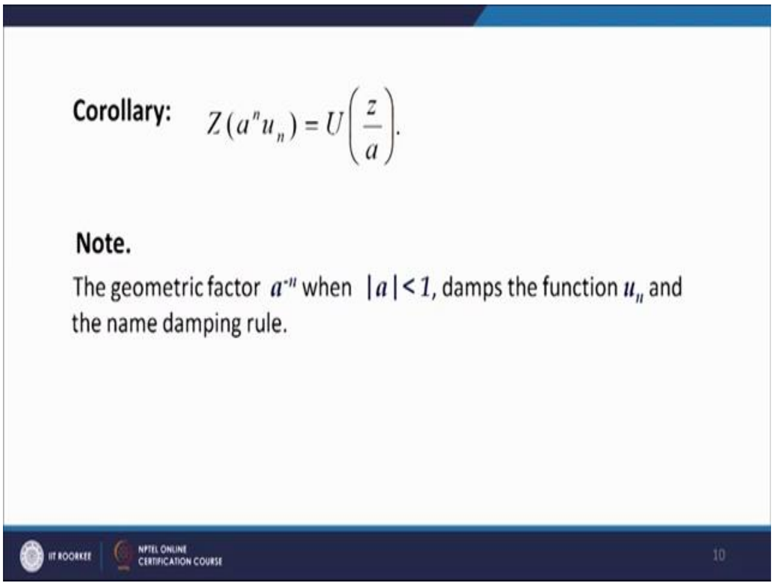

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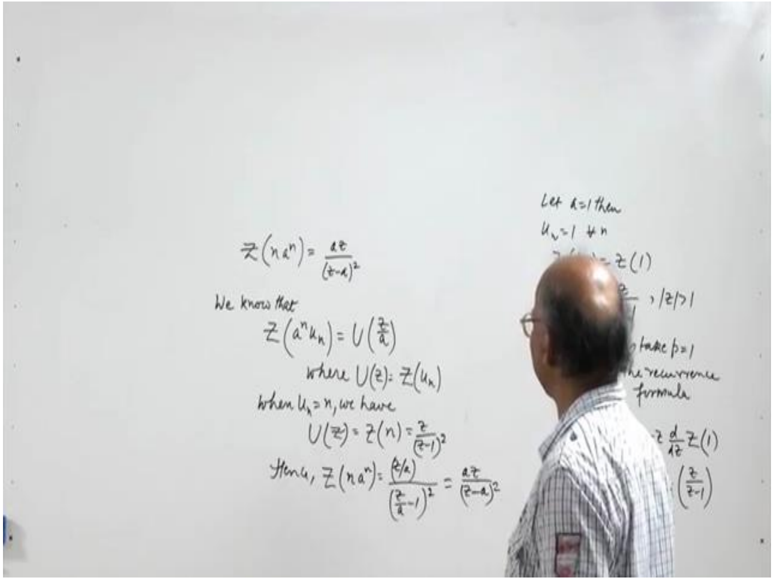

So, now the corollary of this result is z of a to the power n u n equal to U z by a. So, let us see a; we have z of in the dumping rule we have z of a to the power minus n u n equal to u u a z. So, this if we replacing a by 1 by a, we have z of a to the power n u n equal to U z upon a on replacing a by 1 by a in the dumping rule. So, we have z of a to the power n u n equal to U z by a the geometric sector let us see the geometric sector u a to the power minus n which occurs here dumps the function u n when mod of z is less than 1 and that is why we call it as the dumping rule. So, now, using this dumping rule z of u n equal to if z of u n equal to U z we have z of a to the power minus n u n equal to u a z using this dumping rule we can find the Z transforms of several other sequences which will be important when we will solve the difference equation we find the general solution of differential equation.

So, let us see what how we can use this dumping rule from the dumping rule it follows that z of a to the; n a to the power n z of n a to the power n is equal to a z upon z minus a whole square let us let us prove this we know that we know that from the dumping rule z of a to the power n z of a to the power n u n is equal to U z by a where U z is the Z transform of the sequence u n. So, what we do is a in order to prove z of n a to the power n equal to a z over n minus a whole square let us take u n equal to n. So, n when u n equal to n we have a z over z minus a whole square. So, this is how we find the Z transform of the sequence n into a to the power n by using the dumping rule.

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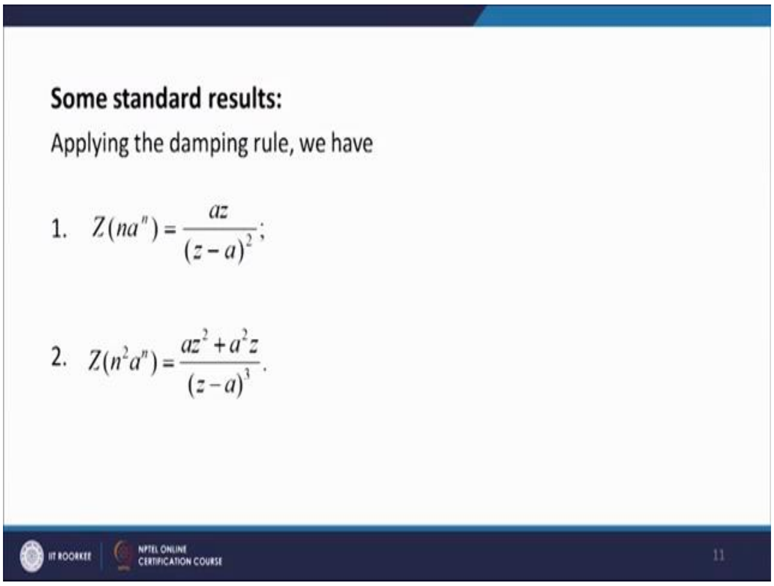

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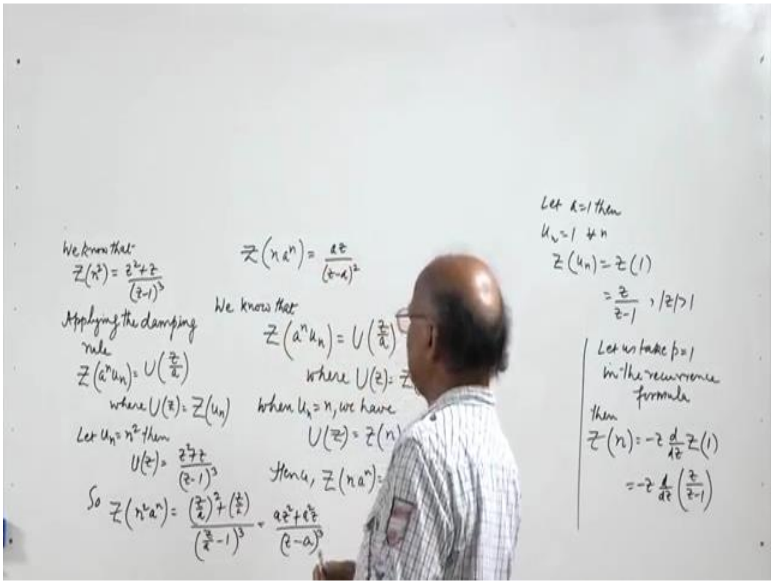

Similarly, we can find the Z transform of n square a to the power n. So, we know that Z transform of n square Z transform of n square we had obtained earlier it is z square plus z over n minus z minus 1 whole cube. So, we apply the dumping rule again apply the we applying the dumping rule z of a to the power n into u n equal to U z by a where U z is sigma U z is Z transform of u n sequence.

Let u n be equal to n square then Z transform of Z transform of n square then U z is equal to U z is equal to z square plus z divided by z minus 1 whole cube. So, Z transform of n square u n is n square n square a to the power n will be replacing z by z over a. So, z by a whole square plus z by a divided by z by a minus 1 whole cube this will give you a z square plus a square z over this is a square here. So, a square when we multiply a cube we are multiplying a cube we are multiplying. So, we get a z square over plus a square z over minus a whole cube.

So, that is how we find the Z transform of n square into a to the power n, similarly you can find the Z transform of n cube into a to the power n because we have found the Z transform of n cube. So, once Z transform of n cube is known Z transform of n cube into a to the power n will be replaced z by z minus a in the Z transform of n cube and you get Z transform of n cube a to the power n. So, these are some of the standard results that we shall need in the sol in when we solve the difference equations. So, one should memorize. In fact, these results because when we solve the difference equation there we will have to inverse Z transform. So, for taking the inverse Z transform we need to memorize some of these important results like when you take the lap inverse Laplace transform you use the known results of Laplace transforms Laplace transforms of some extended functions similarly here. So, we need to memorize some of these extended results to take the inverse Z transform while solving the difference equations. So, with that I would like to conclude my lecture.

Thank you very much for your attention.