Mathematical methods and its applications
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Lecture – 38
Application of Laplace Transforms – III

So, welcome to the lecture series on mathematical methods and its applications. So, this is the last lecture on Laplace transforms. We have seen so many things on Laplace transforms, so many applications. Now this is basically solution of partial differential equation using Laplace transform that how Laplace transform is useful for solving partial differential equations.

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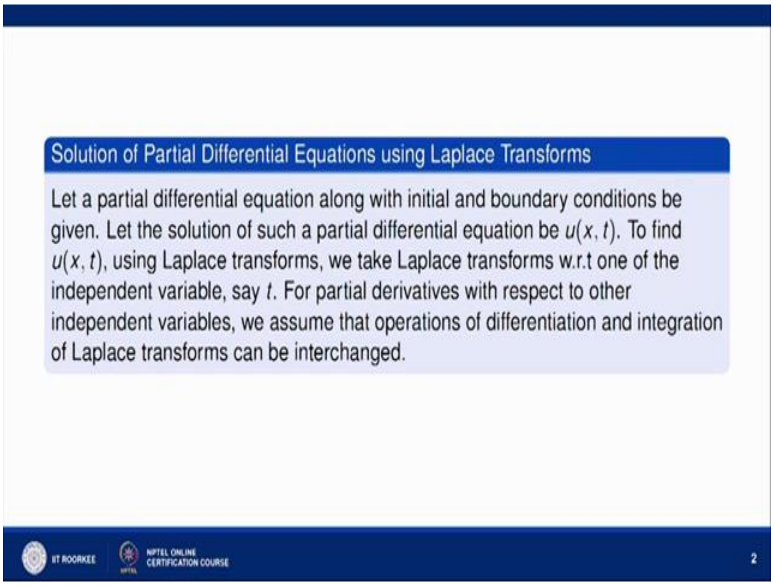

So, let a partial differential equation along with initial boundary condition is given to us. Let the solution of such a partial differential equation be u x t; u x t, 2 independent variables are involved x and t. To find u x t that is the solution of this partial differential equation using Laplace transforms, we take Laplace transforms with respect to one of the variable; one of the independent variable say t, for partial derivative with respect to the other independent variables we assume that the operations of differentiation, integration of Laplace transforms can be interchanged. So, this we assume that with respect to the other independent variable the operations of differentiation integrations can be interchanged.

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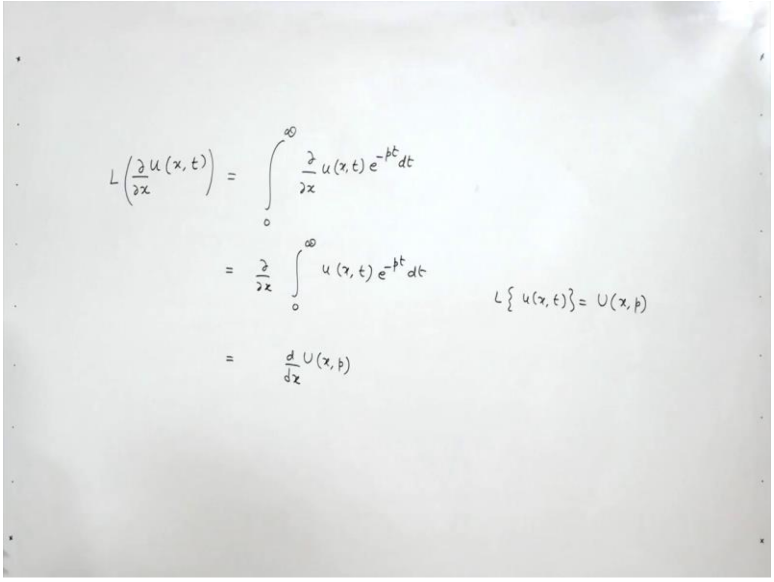

Now, let us see suppose you have a u x t which is a function of u x t. It is the function of 2 into independent variables x and t, now suppose you take partial derivatives with respect to 2 x and let us compute the Laplace transform of this. Laplace transform of del by del x of u x t. So, what it is by definition? 0 to infinity del by del x of u x t into e [FL] power minus p t d t because we are taking Laplace transform with respect to the independent variable t. So, that is why we are taking t here because we are taking Laplace transform with respect to the independent variable t.

So, this we can write like this, it is equal to basically this can be taken out. So, it is del by del x of integration 0 to infinity u x t because this integration is with respect to t. So, del by del x can be taken out. So, e [FL] power minus p t d t and this expression is nothing but Laplace transform of u x t. So, suppose Laplace transform of u x t is suppose u x p, it is again a function of 2 variable x is there and we are taking Laplace transform with respect to t, place of t we have we will be having p. So, this will be nothing but this is nothing but u x p, Laplace transform of u x t small u x t and it is del by del x we can write d by d x because p is the parameter. So, this can be written as Laplace transform of del by del x of u x t can be written as d by d x of capital U x p where capital U is nothing but Laplace transform of small u with respect to t.

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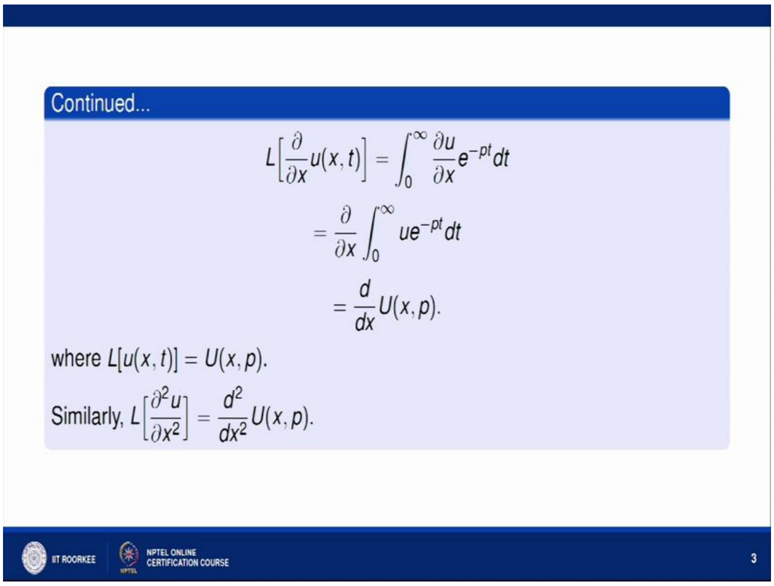

Now, similarly if we find second partial derivative of u and take the Laplace transform of this. So, this is nothing but d square by d x square of U x p, this can be derived on the same lines.

Now, let us compute Laplace of partial derivative of u with respect to t, what is this?

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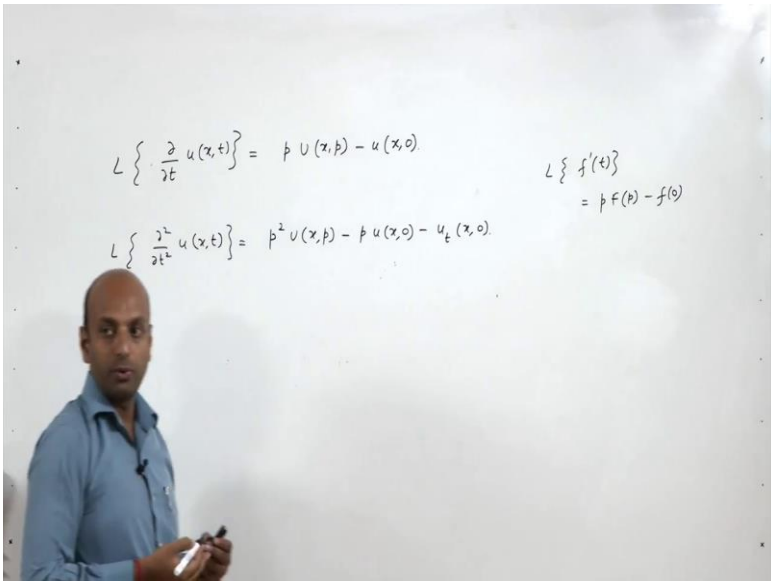

So, this we can easily see because so Laplace transform of del by del t of u x t. So, we know that Laplace transform of f dash t is nothing but p f p minus f 0 where this f p is nothing but Laplace transform of f t and this f 0. So, in the same way, this can be written as we treat x as constant term. So, this can be written as p into Laplace transform of u x t which we are taking as U x p minus u x 0. This can be written as this way because x we are taking as constant, we are treating we are taking Laplace with respect to t.

Again if suppose you want to find out second order derivative of this term of u x t. So, again by the same concept we can write, it is p square U x p minus p into you x 0 minus u dash. So, dash derivative with respect to the variable for which Laplace transform is to be is we have taken. So, here we have taken Laplace transform with respect to t. So, derivative with respect to t only so that is u t x t x 0 because when we take Laplace transform of second derivative of f it is nothing but p square f p minus p f 0 minus f dash 0 and derivative is with respect to t so here also the partial derivative is respect to t only.

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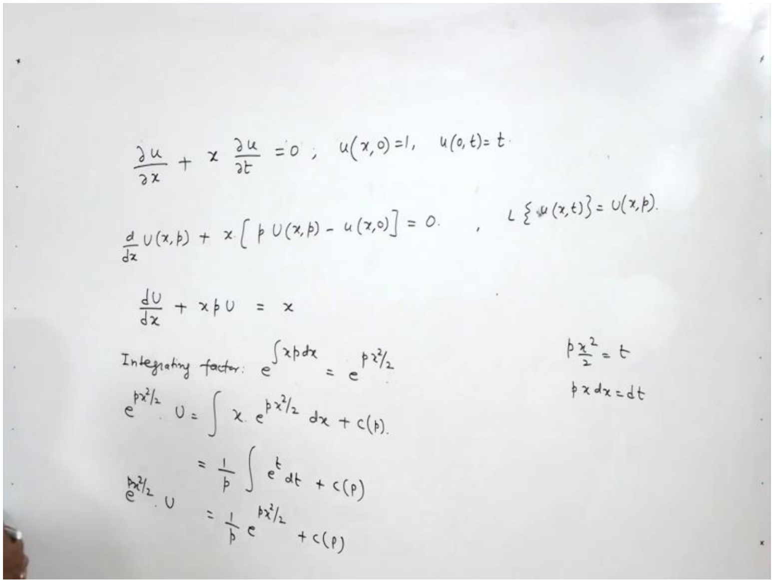

Now, let us solve this problem using Laplace transforms with simple problem. Now what this problem is it is del u by del t plus x del u by it is del u by del x sorry, del u by del x plus x del u by del t is equals to 0, u at x comma 0 is one these are the initial conditions and u 0 t is equals to 0 this is given to us. So, this is some first order partial differential equation and we want to solve it. So, let us solve this problem using Laplace transforms. So, you take Laplace transforms both the sides. So, now, we are taking Laplace transform with respect to t independent variable t.

So, this can be written as d by d x of Laplace transform of this which we are taking as U x p plus x t treat as constant and del u del t is nothing but p into U x p minus u x 0 is equals to Laplace transform 0 is 0. So, here Laplace transform of u x t u x t, I am taking as U x p. Now this is nothing but it is d u by d x, this capital U is this, I am taking like. So, plus x into p capital U and u x 0 is 1. So, when you substitute 1 here and this will go to right hand side this is nothing but equal to x.

So, this is simple first order linear differential equation. If the first one linear basically when you take Laplace transform in any p d this will give a ordinary differential equation you use your usual techniques to solve this equation and then take Laplace inverse to find out the solution of the given p d. Now how to find the solution of this first you find integrating factor? So, what is the integrating factor e [FL] power integral x p d x? So, this is nothing but e [FL] power p x square by 2 and the solution will be nothing but e [FL] power p x square by 2 into u is equals to integral right hand side into integrating factor d x plus integrating constant c.

Now, of course, this is basically a function of p because here 2 variables are involved p and x p is an arbitrary variable; arbitrary constant. We can say and x is a variable, it will be a; we are integrating with respect to x. So, it will be some function of p now how will integrate this we take this as say t. So, p x square by 2 we can take as t. So, it is p x d x will be nothing but d t. So, this is equal to 1 by p integral e [FL] power t d t plus c p and it is 1 by p integral of e [FL] power t is e [FL] power t itself and t is nothing but p x square by 2 so e [FL] power p x square upon 2 plus c p e [FL] power x square by 2 into p into u in the left hand side.

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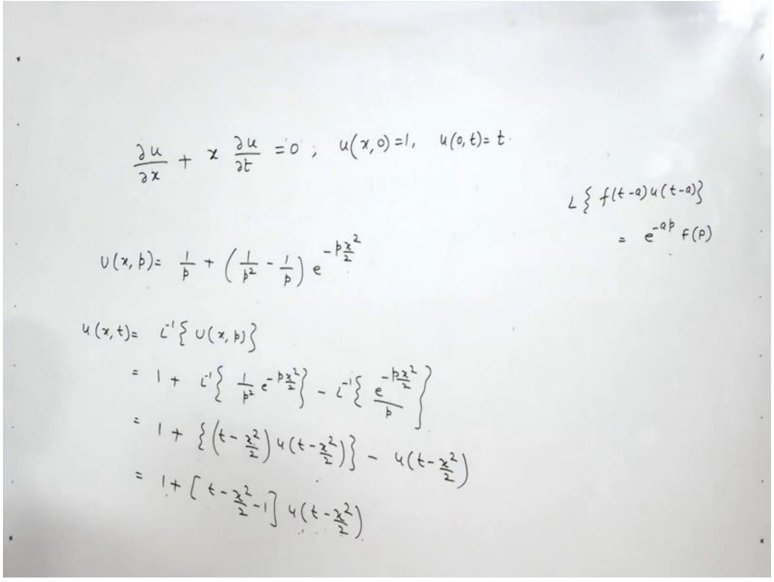

So, what we obtain? U is equal to 1 by p plus c p into e [FL] power minus p x square by 2.

Now, to calculate c p, we will use the second condition u x u 0 t is equals to t. Now suppose it is 1, what is U x p? U x p is nothing but Laplace transform of u x t which is nothing but 0 to infinity u x t into e [FL] power minus p t d t.

Now, when you take x equal to 0 so here x equal to 0 and here also x equal to 0, so, u 0 p will nothing but a Laplace transform of u 0 t and that is nothing but Laplace transform of t because u 0 t is nothing but t. So, this is nothing but 1 by p square. Now when you apply this condition here, it is when you take x equal to 0 put x equal to 0. So, u x 0 is u 0 p is nothing but 1 by p square. So, 1 by p square equal to 1 by p plus c p into 1. So, this implies c p is nothing but 1 by p square minus 1 by p. So, therefore, U x p is nothing but therefore, U x p is nothing but 1 by p plus 1 by p square minus 1 by p into e [FL] power minus p x square upon 2. Now u x t will be nothing but Laplace inverse of u x p. So, it is Laplace inverse of this expression. So, Laplace inverse of 1 by p is 1 plus now we have to find out Laplace inverse of 1 by p square e [FL] power minus p s square by upon 2 and minus Laplace inverse of e [FL] power p s square upon 2 by p.

Now, this x is here is not a variable because we are taking Laplace with respect to t. So, this x is not a variable, we are taking it as a constant to find out Laplace inverse only p is a parameter. So, we know this we know our result that Laplace transform of f t minus a into u t minus a where u is the unit step function at t equal to 2 a is nothing but e [FL] power minus a p into f p, where f p is nothing but Laplace transform of f p. Now this one will remain as it is plus now the Laplace inverse of this is t 1 by p square Laplace inverse is t e [FL] power minus a s a is x square by 2 if you compare with this. So, a is nothing but x square by 2. So, and Laplace inverse of this is t. So, f t is t. So, the Laplace inverse of this entire expression by this ex formula is nothing but t minus x square by 2 because a is x square by 2 and u of t minus x square upon 2 plus minus.

Now, here Laplace inverse of 1 by p is 1. So, f t is 1 and a is x square by 2. So, again this expression will give u t minus x square upon 2. So, therefore, the final answer is 1 plus t minus x square upon 2 minus 1 u of t minus x square upon 2. So, this will be the solution of this differential equation, we can use this property to simplify this expression. So, the variable under which we have taken the Laplace transform take only that variable as a I mean other variables take as a constant take other variables as constant. So, this is how we can simplify this P d e.

Now, let us solve a second order P d e. So, these are the applications of Laplace transforms. It cannot not a not only solve o d or integral equations, it may solve P d e also as we have seen in these examples.

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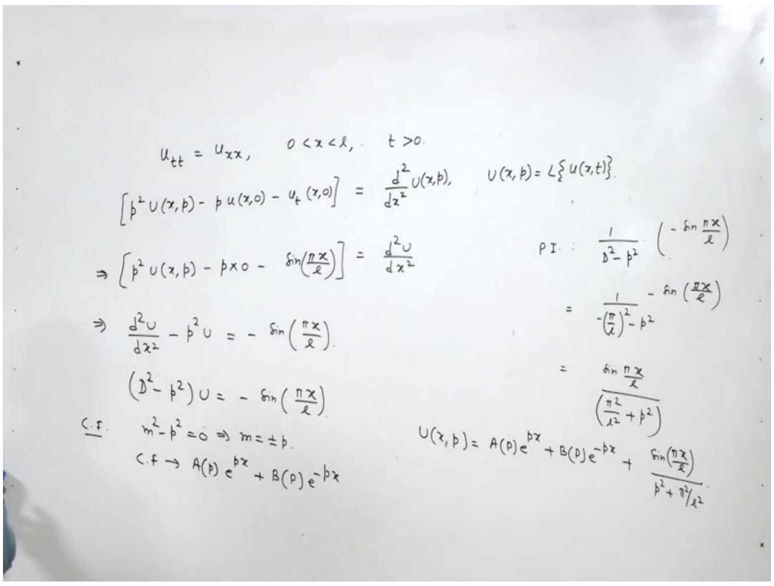

So, what is the second order equation u t t is equals to u x x and 0 is less than x less than l and t greater than 0. Conditions are given to you, what are the conditions we will use when we apply Laplace transform both the sides? So, take Laplace transform both the sides. So, what will be the Laplace transform of del square u by del t square it will be nothing but p square into U x p minus p u x 0 minus u t x 0 we are applying Laplace transform with respect to t. So, this is equal to this is nothing but d square upon d x square of u or U x p where U x p is nothing but Laplace transform of u x t.

Now, this is p square U x p minus now u x 0 is 0 given to us. So, p into 0 minus and u t 0 is nothing but sin pi x by l. So, it is sin pi x by l given tools and this is nothing but d 2 u upon d x 2. So, this implies d 2 u upon d x 2 minus p square u is nothing but minus sin pi x by l. So, it is a second order linear differential equation. So, we can find it find the solution of this equation first we find complimentary function or complimentary solution then we find particular integral and the sum of these 2 will give the complete solution of this differential equation.

So, what is the complementary equation here? So, it is nothing but d square minus p square time u is equals to minus sign pi x by l. So, complementary function will be nothing but complementary function will be nothing but it is m square plus p square equal to 0. So, m will be nothing but plus minus p. So, complementary function will be nothing but a now it will it will be a; a and b which are the arbitrary constants are nothing but function of p here because we have we have 2 because u contains 2 variables x and p, a p e [FL] power p x plus b p e [FL] power minus p x.

Now, for a particular integral particular integral it is nothing but 1 upon d square minus p square of minus sign pi x by l. So, we simply replace d square by minus a square and a is nothing but pi by l. So, this is equal to 1 upon minus pi by l whole square minus p square and it is minus sin pi x by l. So, this is nothing but sin pi x by l upon pi square upon l square plus p square. So, this will be p i. So, what will be the complete solution that is capital U? So, capital U x p will be nothing but c x plus p i. So, c f is nothing but a p e [FL] power p x plus b p e to the minus p x plus p i which is sin pi x upon l upon p square plus pi square upon l square. So, that will be the complete solution of this differential equation.

Now, we use the other conditions given to us to calculate a p and b p to calculate these functions we will use the other conditions now what is what are the other conditions now u 0 t is equal to 0.

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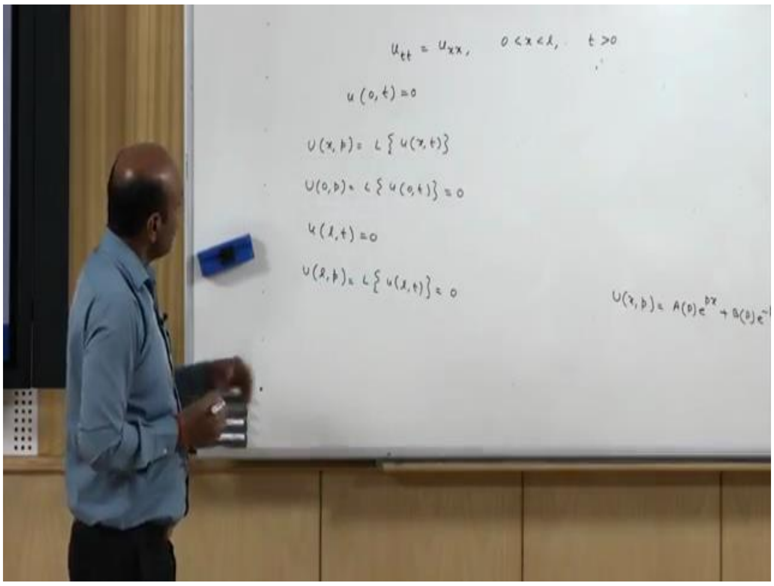

Now, U x p is nothing but Laplace transform of u x t. So, u 0 p will be nothing but Laplace transform of u 0 t which is nothing but 0 because this is 0 and other condition is u l t equal to u l t equal to 0 Laplace. So, you substitute x equal to l here first you substitute x equal to 0 when you substituted x equal to 0. So, u 0 t is 0. So, Laplace transform of 0 is 0.

Now, we substitute x equal to l here. So, U l p will be nothing but Laplace transform of u l t and u l t is again given as 0. So, it is again 0 now apply both these conditions both these conditions and this condition in this equation to calculate a and b.

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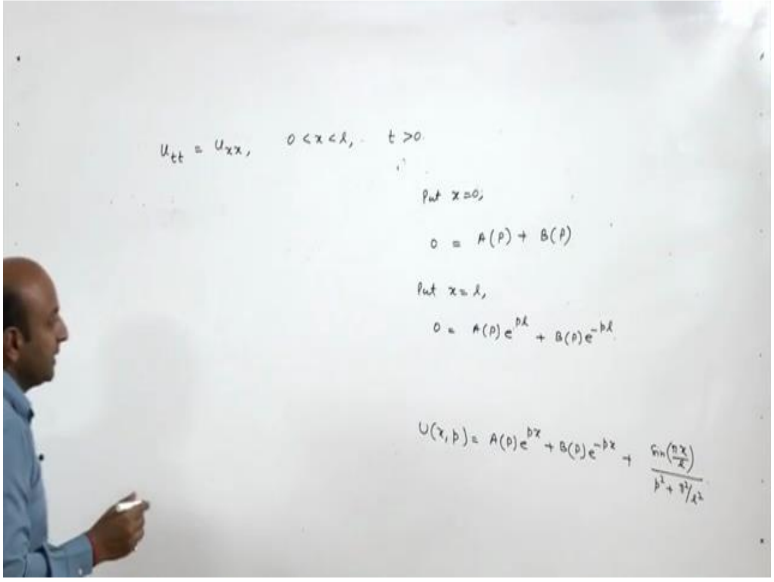

Now you substitute x as in this equation you substitute first x as 0, put x equal to 0 when you put x equal to 0 here. So, the left hand side is nothing but 0; 0 is equal to e [FL] power 0 is 1 is a p plus this is nothing but plus b p and sin 0 is 0. So, it will be 0. So, this is 1 condition.

Now, put x equal to n here when you put x equal to l. So, u capital U l p is again 0. So, when you put it x equal to l, it is 0 is equals to a p e [FL] power p l plus b p e [FL] power minus p and at x equal to l sin pi which is 0.

Now, these are the 2 equations in 2 unknowns A and B; capital A and capital B, if you take the determinant.

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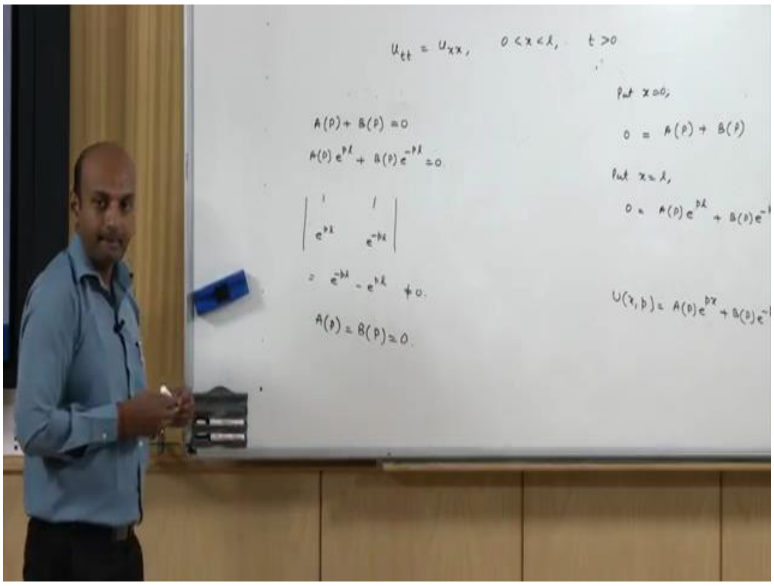

So, these are 2 equations A p plus B p is equal to 0 and A p e [FL] power p l plus B p e [FL] power minus e p l is equals to 0, if it is a determinant of 1 e [FL] power p l and e [FL] power minus p l. So, it is equals to e [FL] power minus p l and minus e [FL] power p l which is not equal 0 for all p. So, this implies this will be having a unique solution and the unique solution in homogenous equations where right hand side is 0 is nothing but the trivial solution that is a p is equals to b p is equals to 0 because data coefficient determinant coefficient matrix the determinant of that is equal to 0 is not equal to 0. So, not equal to 0 means unique solution and the only unique solution at trivial solution that is a p equal to b p equal to 0.

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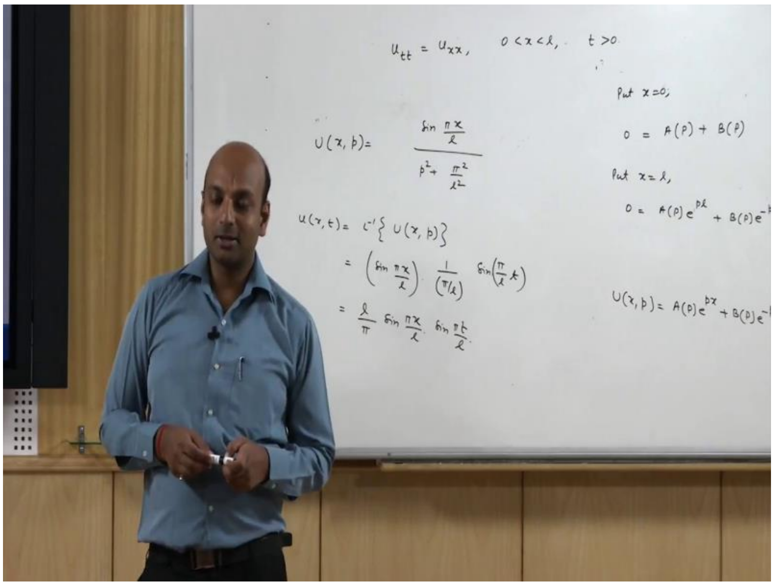

So, when you substitute a p equal to b p equal to 0 what we will obtain. So, what will be U x p now U x p will be nothing but a p and b p are 0. So, it is nothing but sin pi x y l upon p square plus pi square upon l square and u x t is nothing but Laplace inverse of U x p which is equals to now i, now you want to take Laplace inverse of this equation this expression. So, sin pi x by l will come outside because it is free from t it is free from p and t it is free from p sorry and this for this Laplace inverse 1 upon this you divide and multiply by pi by l, 1 upon pi by l into Laplace inverse of this sin pi by l pi upon l into t.

So, the final answer is l upon pi sin pi x by l into sin pi t by l. So, this will be the final solution of this equation. So, similarly the questions based on p d e can be solved using Laplace transforms.

Thank you very much.