Mathematical methods and its applications
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Lecture – 37
Application of Laplace Transforms – II

So, welcome to the series of lecture series on mathematical methods and its applications. So, on the last lecture, we have done some applications of Laplace transforms we have seen that how we can solve an ordinary equation; ordinary differential equation using Laplace transforms. We have also solved some circuit problem and solved them using Laplace transforms. Now again we will see some more application Laplace transform in this lecture.

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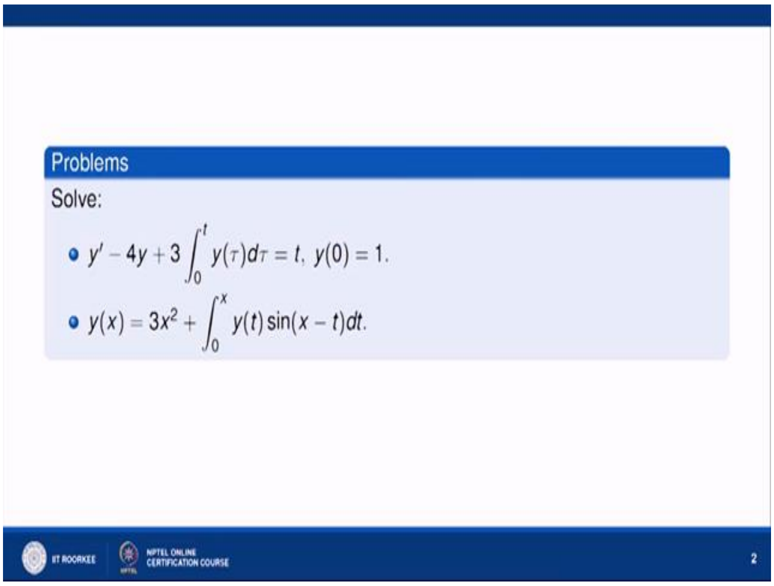

Now, suppose we have the first problem such problems are basically called integral equations where y is unknown function involved under sign of integral. So, in these problems y is unknown function and it involves under sign of integral. So, such equations are called integral equations. So, Laplace transform is also useful to solve integral equations. So, to illustrate this let us discuss these examples.

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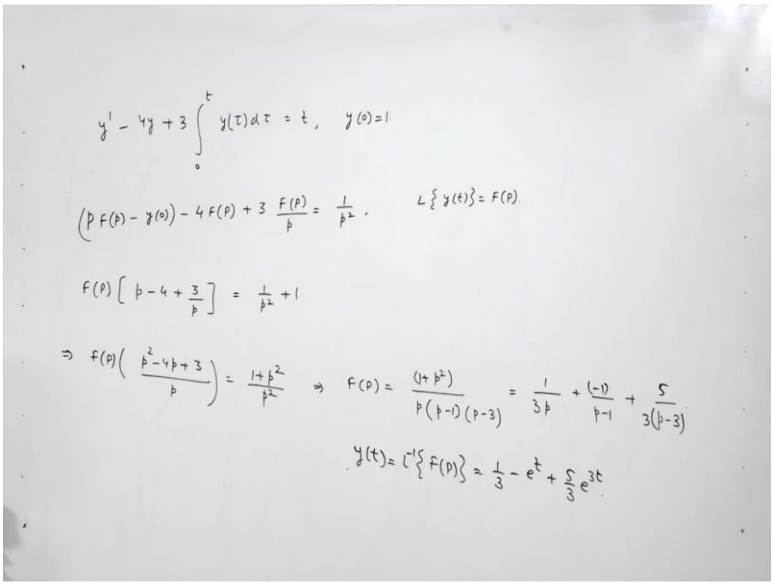

So, first problem; first problem is y dash minus 4 y plus 3 times integral 0 to t ,y tau d tau and equal to 1 equal to t and it is given to you that y 0 is 1. So, how can we solve this equation using Laplace transform? So, this is an integral equation because y is an unknown function involved in under sign of integral also. So, how can we solve this? Take Laplace transform both the sides. So, Laplace transform of y dash is p F p; p F p minus F 0 or y 0 minus 4 times F p plus 3 times. Now integral 0 to 2 y t y tau d tau; the Laplace of this is nothing but F p by p, we already know this. So, this is F p by p which is equal to the Laplace of p is nothing but 1 by p square. So, here I am calling Laplace of this y t as F p. Now it is F p; you can take common; F p coefficient F p will be p from here minus 4 from here and plus 3 upon p from here, y 0 is 1 given to us. So, you can substitute y 0 as 1 here. So, it will be equal 1 by p square plus 1. So, this implies F p into p square minus 4 p plus 3 upon p is equals to 1 plus p square upon p square. So, p P cancel out, this implies F p is nothing but 1 plus p square upon p into now this is p minus 1into p minus 3, the factor of this. So, this will be the F p. So, to find out y t we will make; we will take Laplace inverse both the sides.

So, to find out the Laplace inverse of this F p, we will make use of partial fractions. So, this is nothing but upon p, A upon p plus B upon p minus 1 plus C upon p minus 3. So, to calculate A, we will substitute p equal to 0 in this expression. So, when we take p equal to 0 it is nothing but 1 upon 1 upon 3. So, 1 upon 3 will be the value of A for B you can substitute p equal to 1, p is 1, it is 2 upon it is 2 upon 1 into minus 2 that is minus 1 and when p is 3 it is 9 plus 1; 10 upon 3 into 2. 2 5s are 10, it is 5 by 3. So, these are the values of A, B and C. So, what will be y t? Y t will be nothing but Laplace inverse of F p and which is nothing but Laplace inverse of this expression. So, Laplace inverse of this expression is nothing but 1 by 3 minus e [FL] power t plus 5 by 3 e [FL] power 3 t. So, that will be the Laplace inverse of this expression.

So, you can check when y 0 equal to 1, when t equal to 0; when t equal to 0 this value is 1 by 3 minus 1 that is minus 2 by 3; minus 2 by 3 plus 5 by 3 is 1. So, this condition is also satisfied. So, this is the solution of this integral equation. Now suppose you want to solve the second problem, it is again a integral equation. So, how can we solve this problem? Let us see again using Laplace transforms, this problem is y x is equal to 3 x square plus integral 0 to x y t sin s minus t d t.

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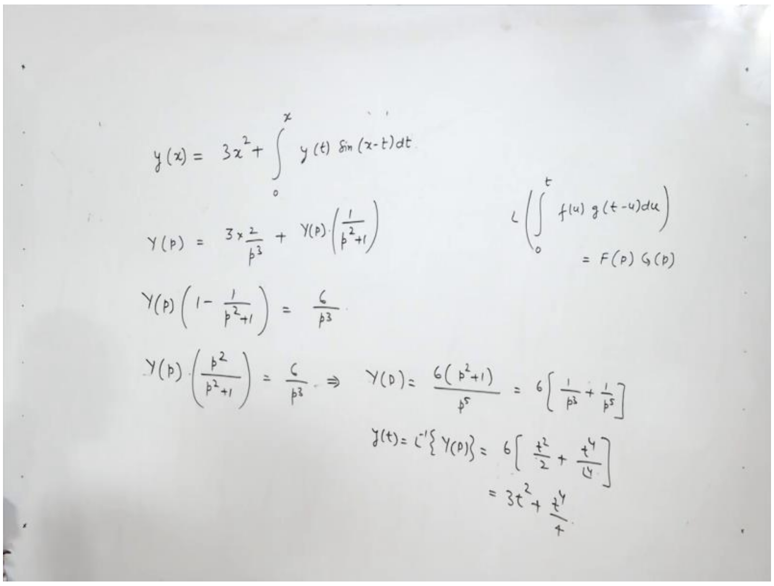

So, this is the problem. Now how can we solve this integral equation using Laplace? So, again take Laplace both the sides. So, Laplace of y x suppose it is y p; suppose Laplace y is y p; 3 x square. So, Laplace x square or t square is nothing but 2 upon p cube plus.

Now, to find out Laplace of this expression it is nothing but the convolution of y with sin; the convolution of y with sin function is this expression. So, by the convolution theorem, we already know that convolution theorem of Laplace transform it is 0 to t f u g t minus u d u is nothing but Laplace of this is nothing but F p into G p where F p is the

Laplace of function F and G p is a Laplace of function g; small g. So, this is by the convolution theorem of Laplace transforms. So, we again apply a convolution theorem over here in this expression. So, we will get Laplace transform of y t we have already assumed that it is y p into Laplace transform of sin function is nothing but 1 upon p square plus 1. This is by the convolution theorem of Laplace transforms, now y p you can collect the terms containing y p.

So, it is y p into it is p square upon p square plus 1 which is equals to 6 upon p cube. Now this implies y p is equal to 6 p square plus 1 upon p [FL] power 5. So, this would be equal to 6 1 by p cube plus 1 by p [FL] power 5. So, to calculate to calculate y t is nothing but Laplace inverse of y p and it is nothing but Laplace inverse of this expression. So, Laplace inverse of this expression is nothing but 6 into Laplace inverse of 1 by p cube is t square upon 2 and Laplace inverse of 1 by t [FL] power 5 is nothing but t [FL] power 4 upon factorial 4. So, this is nothing but 3 t square plus t [FL] power 4 upon 4. So, this is t the solution of this integral equation.

Now, let us solve next problem, again it is a integral equation based on the same concept.

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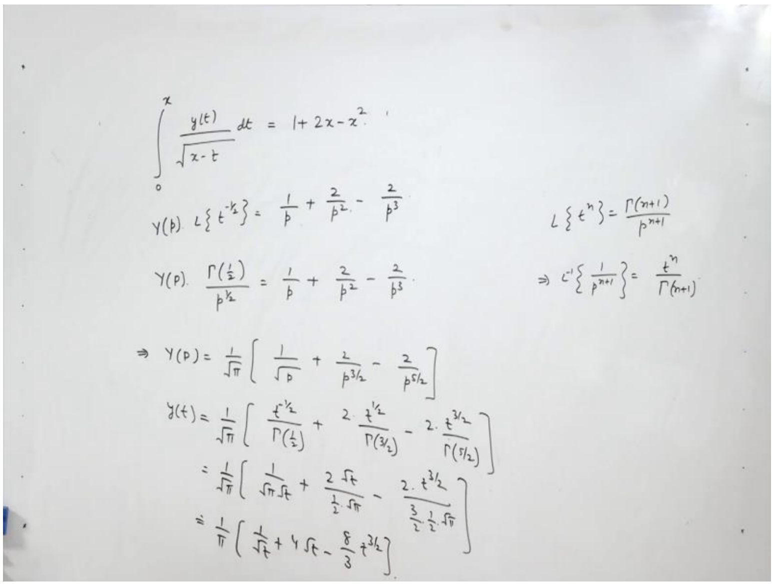

So, in the next problem is that is 0 to x y t upon under root x minus t d t and it is equal to 1 plus 2 x minus x square. So, we have to find that y which satisfies this expression. Basically we have to find that y which satisfies this expression. So, this is an integral equation basically. So, how can we solve this again take Laplace transforms both the sides now to find out the Laplace transforms of this expression we will again use convolution theorem for Laplace transforms because it is a convolution of under root x with under root t with y t because a because it involve 2 functions basically 0 to x y t into x minus t [FL] power minus half.

So, it is a convolution of this function with function convolution of y t and t. So, we will take the Laplace transform both the sides. So, Laplace transforms of this by the convolution theorem is Laplace transform y t which are which suppose is y p and Laplace transform of t [FL] power minus half is equal to or F [FL] power minus half both are same you can take t k power minus half or s k power minus half.

Now, Laplace transform of 1 is 1 by p 2 by p square minus it is Laplace transform of t square t Laplace transform of t square is 2 by p cube. So, it is y p Laplace transform of t [FL] power minus half is gamma half upon p [FL] power half which is 1 by p plus 2 by p square minus minus 2 by p cube gamma half is under root pi. So, y p is nothing but 1 by under root p time 1 by under root pi times 1 by under root pi plus 2 upon p [FL] power 3 by 2 minus 2 upon p [FL] power 5 by 2.

Now, y t is nothing but Laplace inverse of this expression. So, take Laplace inverse both the sides. So, Laplace inverse of this will be nothing but it is 1 by under root pi now Laplace inverse of 1 by under root p again recall the formula of t [FL] power Laplace of t [FL] power n is nothing but gamma n plus 1 upon p [FL] power n plus 1. So, from here Laplace inverse of 1 upon p [FL] power n plus 1 is nothing but gamma is nothing but gamma n plus 1 upon oh it is nothing but t [FL] power n upon gamma n plus 1.

So, Laplace inverse of this, you substitute n s minus half to get under root p. So, it will be t [FL] power minus half upon gamma half plus 2; 2 into 1 upon p [FL] power 3 by 2. So, you will substitute n s half it is nothing but t [FL] power half upon gamma 3 by 2 minus again 2 into it will be t [FL] power 3 by 2 upon gamma 5 by 2. So, you can we can simplify this very easily. So, this will be nothing but this will be nothing but 1 by under root pi the first expression is 1 upon under root pi into under root t plus 2 under root t and it is 1 by 2 under root pi minus 2 into 3 t [FL] power 3 by 2 upon 3 by 2 1 by 2 under root pi.

So, this we can easily simplify. So, this after simplification we get 1 upon pi under root pi can be taken out common. So, it is 1 under root t minus plus 4 times under root t minus it is 4 8 upon 3 times 8 upon 3 times at t [FL] power t [FL] power 3 by 2. So, this will be the final answer for this problem.

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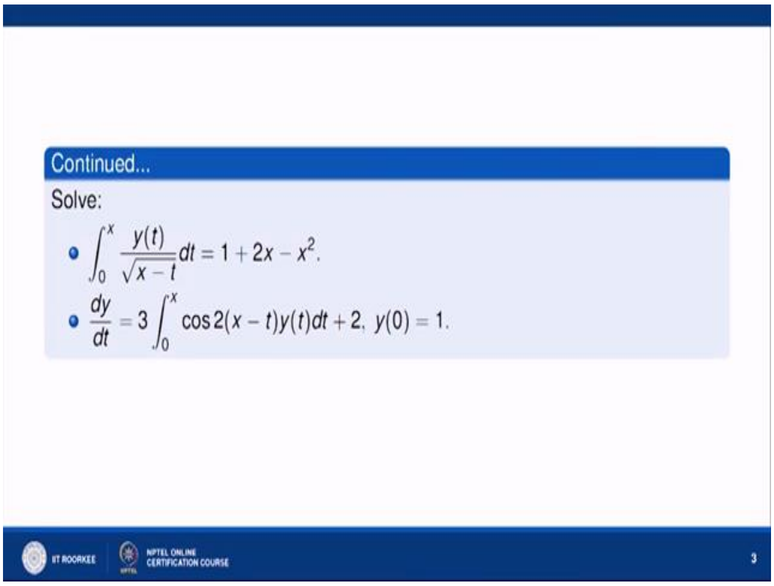

Now, similarly we can solve the last problem of this slide again we will take Laplace transform both the sides in the left hand side we have d y by d t. So, the Laplace transform of d y by d t will be p F p minus y 0 y 0 given as 1 and the when the right hand side we will apply convolution theorem for Laplace transforms. So, this is that is how we can solve integration equations using Laplace transforms.

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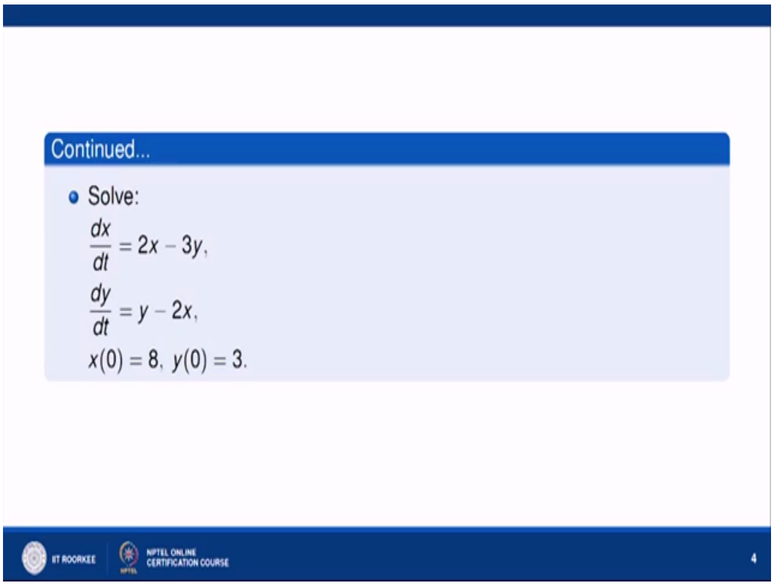

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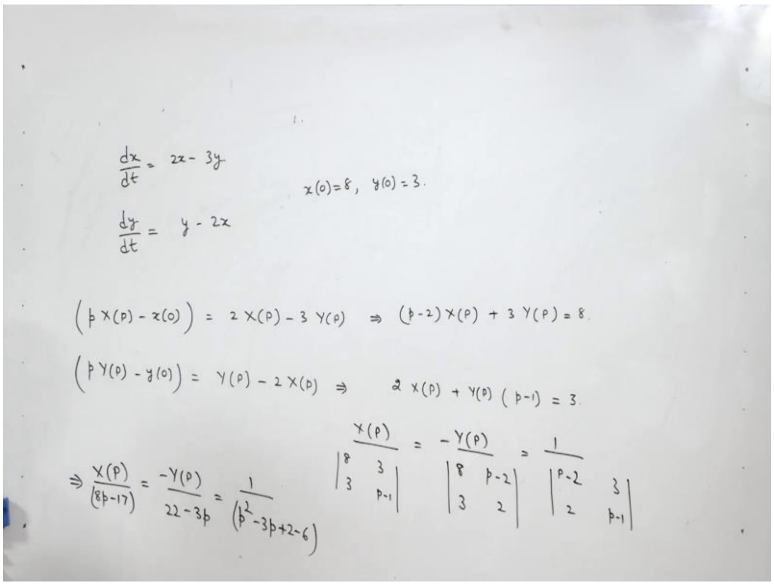

Now, next is we can also solve simultaneous ordinary equations using Laplace transforms. So, let us see some problems based on it. So, the first problem is simultaneous differential equation is d x upon d t is equals to 2 x minus 3 y and d y by d t as given as d y d t is y minus 2 x and given as x x at 0 is 8 and y at 0 is 3 y at 0 is 3.

So, let us solve this problem again, we will take Laplace transform of both the expressions. So, Laplace when we take Laplace transform of first expression. So, it will be nothing but it is d by d s by d t. So, it will be nothing but p into x p minus x 0 is equals to 2 x p minus 3 y p. So, that will be Laplace of first expression what is x 0 x 0 is 8. So, what this expression implies this implies p minus 2 times x p plus 3 y p is equal to 8. So, that will be the first equation which is obtained here.

Now, apply Laplace transform in the second equation when you apply Laplace transform both the sides here we will be having p into y p. So, this y p is nothing but Laplace transform of y t now minus y 0 which is equals to y p minus 2 x p. So, again this implies you compare take the coefficients. So, it is 2 into x p 2 times x p and the coefficient of y p is coefficient of y p is p minus 1 which is equals to y 0 y 0 is 3. So, these are the 2 equations which we obtain p minus 1 times y p.

. So, these are 2 equations which we obtain over here now we will find the; these are the 2 linear equations in x p and y p 2 linear equations x p and y p. So, we will solve them for x p and y p and then take the Laplace inverse to find out x t and y t that is a solution of this simultaneous differential equation. So, how can we solve this? So, to find out the solution it is nothing but I mean simultaneous linear equations to into unknowns. So, we can solve it like this it is x p upon minus y p upon and it is 1 upon. So, it is the determinant of 8 3 and 3 p minus 1 this is by the Cramer rule, we are solving it then minus y p y p coefficient will be 8 3 C 1 C 2 and it is p minus 2 and 2 and here the right hand side will be nothing but p minus 1 p minus 2 it is p minus 2 and 3 2 and p minus 1.

So, this implies x p upon it is 8 p minus 8 minus 9 is minus 8 minus 9 is minus 17 minus y p upon it is 16; 16 plus 6; 16 plus 6 is 22 minus 3 p, it is 1 upon it is nothing but p square p square minus 3 p plus 2 minus 6 . So, this is x p and y p. So, what are x p and y p.

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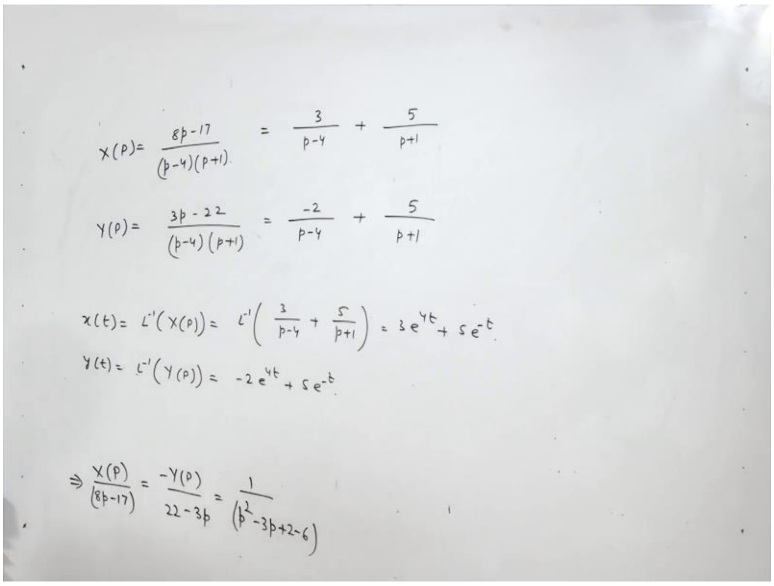

Now, so, x p is nothing but 8 p upon minus 17, upon it is nothing but p square minus 3 p minus 4 which can be is the factorize it is nothing but p minus 4 into p plus 1 which is p square minus 3 p minus 4 of course, and y p is nothing but y p is nothing but 3 p minus 22 upon again the same quantity in the denominator.

Now, this is nothing but we can make use of partial fractions again. So, it is p minus 4 it is upon minus 4 it is plus p plus 1. So, we will substitute p as 4 first. So, it is 32. It is 32 minus 17; 32 minus 17 is 5; 15. So, 15 upon when you substitute 4, 5, 3, it is 3, this partial fraction you can also compute compare the coefficients both the sides as you already know now you substitute p as minus 1. So, it is minus 8 minus 17 upon minus 5. So, it is nothing but minus 25 upon minus 5 which is 5. So, it is 5.

Now, we will use a partial fraction again here it is minus p minus 4 plus p plus 1. So, when you substitute p as 4 it is 12 minus 22 upon 5. So, it is minus 10 upon 5 it is minus 2 and when you substitute p as minus 1 it is minus 3 minus 22 upon minus 5 that is minus 25 upon minus 5 it is 5 .

Now, to find x t, x t which is nothing but Laplace inverse of x p, we will make Laplace inverse of this expression 3 upon p minus 4 plus 5 upon p plus 1. So, it is nothing 3 e [FL] power 4 t plus 5 e [FL] power minus t, again to calculate y t we will take Laplace inverse of y p which is nothing but Laplace inverse of this expression and it is nothing but minus 2 e [FL] power 4 t plus 5 e [FL] power minus t. So, these are solutions of this differential equation you can check also F 0 is 8 yeah F 0 is 8 and y 0 is 3 yeah y 0 is 3 and it will also satisfy these equations you can easily verify.

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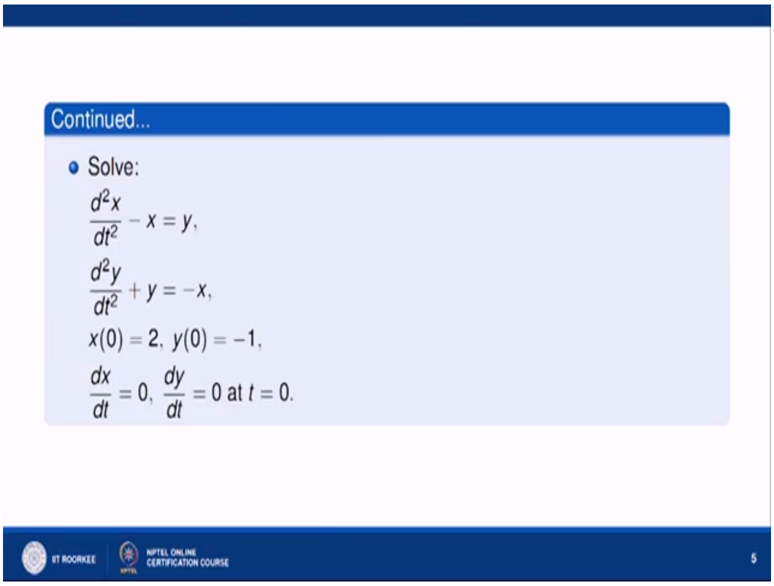

Now, let us discuss one more example based on this which is a second order simultaneous differential equation. So, this also we can solve using Laplace transforms.

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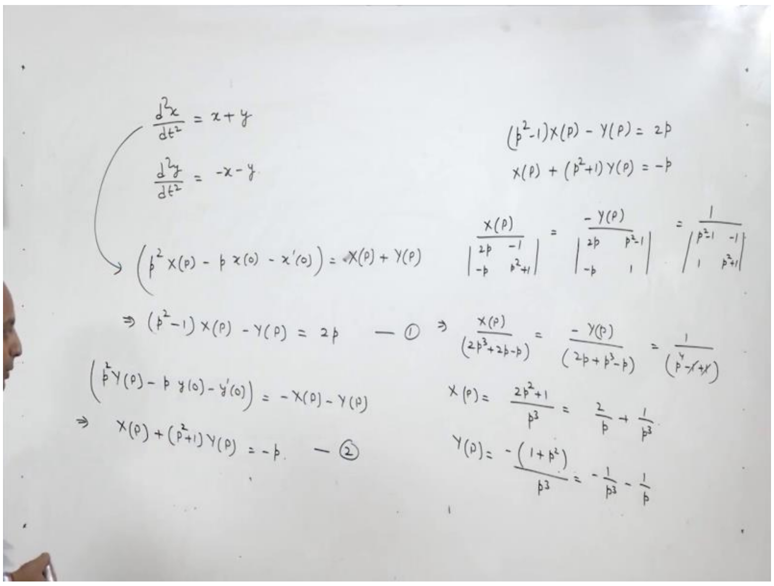

So, how can we solve let us see. So, it is nothing but second derivative of x respect to t which is equals to given as x plus y and d 2 y by d x 2 is nothing but it is minus x minus y and the initial conditions are given to you.

Now, again take Laplace transform both the sides first for the first equation when you take Laplace both the sides. So, it will be nothing but p square x p minus p into x 0 minus x dash 0 and it is equal to x p plus y p course x p here is Laplace transform x t and y p is the Laplace transform of y t now x 0 is given to you as 2. So, this value is 2 and x dash 0 is 0. So, this implies p square plus minus 1 times x p minus y p will be equal to x 0 is 2. So, it is 2 and x dash 0 is 0. So, this is the first equation which we obtained.

Now, apply Laplace transform in this equation both the sides when you take Laplace transform both the sides. So, what we will obtain from here. So, when you take Laplace transform both the sides here. So, it is p into y p minus p into y 0 minus y dash 0 is equals to minus x p minus y p. So, this implies now y 0 is minus 1 given to us and y dash 0 is 0. So, it is x p plus p plus 1 times y p y 0 is minus 1. So, it is minus it is minus 1 it is like p. So, it is minus p. So, this is the second equation which we obtained. So, it is it is x p x p will come here it is plus 1 times y p and this y 0 is y 0 is minus 1 minus minus plus p we may come here it is minus p.

Now, in this again verify this equation this is F p square minus 1 time x p and y p x 0 is 2 it is 2 p and minus y p. So, this equation is obtained now here it is p square y p, p square y p minus p y 0 minus y dash 0 equal to this. So, it is nothing but p square y. So, what are 2 equations which we obtained it is p square minus 1 times x p minus y p is equals to 2 p and here it is x p plus p square plus 1 times y p is equals to minus p. So, again we will these are simultaneous linear equations. So, we solve for x p and y p. So, it is x p upon a determinant of 2 p minus p minus 1 p square plus 1 is equals to minus y p upon p determinant of p square minus determinant of 2 p minus p and p square minus 1 upon 1 and 1 upon it is determinant of p square minus 1 minus 1 1 p square plus 1.

So, this implies x p upon this expression is 2 p q plus 2 p minus p this is minus y p upon this expression is 2 p minus minus plus p cube minus p and this is 1 upon p [FL] power 4 upon 1 plus 1. So, clearly x p will be nothing but it is p cancels it is nothing but 2 p square plus 1 upon p cube. So, it is 2 upon plus 1 by p cube and x p y p is nothing but it is negative of it is negative of 1 plus 1 plus p square it is 2 p minus p is p p p cancels and it is 1 plus p square upon p cube. So, it is nothing but minus 1 by p cube minus 1 by p.

So, now x t now x t is nothing but x t we will Laplace inverse of x p and y t is Laplace inverse of y t.

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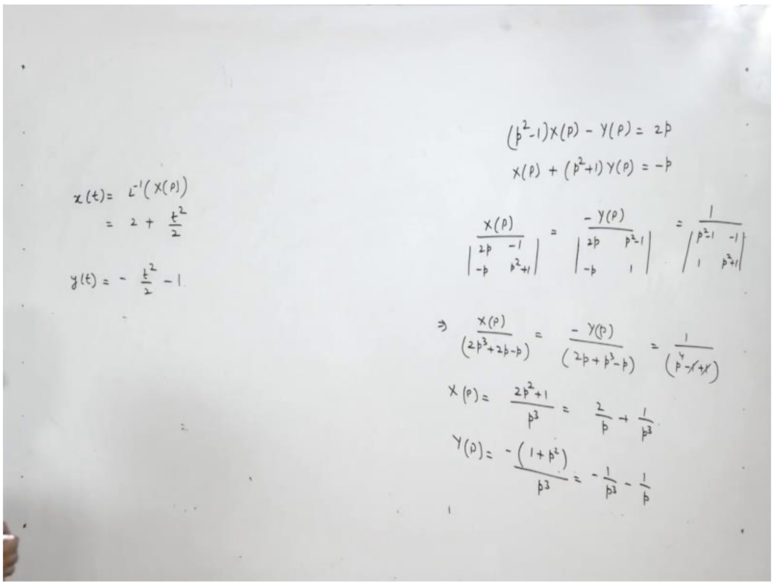

So, what will be x t? X t will be nothing but Laplace inverse of x p which is equals to 2 plus now 1 by p cube 1 by p cube is t square by 2 and y t is will be nothing but minus of 2 by minus of t square by 2 minus 1. So, these are the solutions which we can obtain from here. So, hence using Laplace transforms we can also solve integral equations and simultaneous ordinary differential equation. So, in the next lecture we will also see that how can we use Laplace transform to solve partial differential equations.

Thank you.