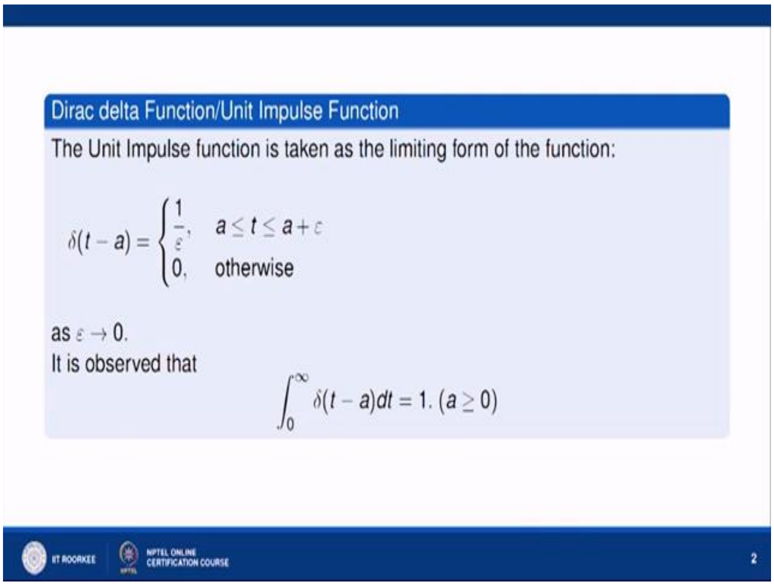
Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 35
Laplace transforms of Dirac delta functions

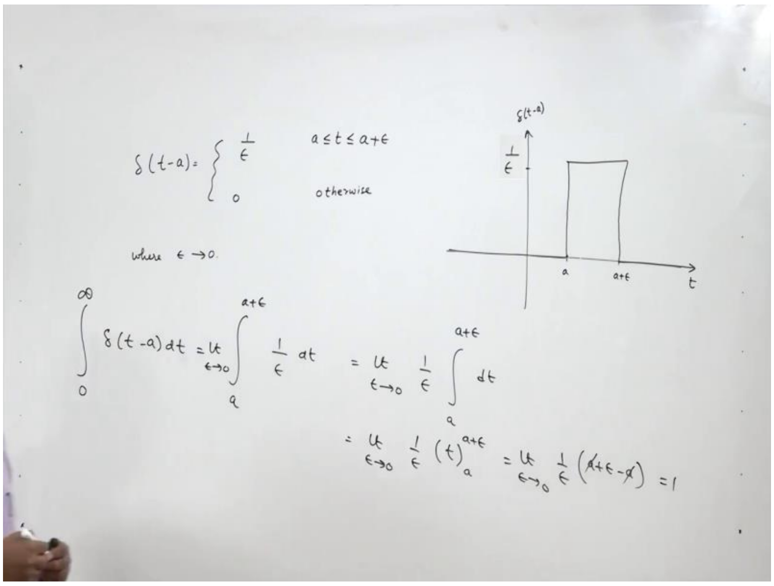
Welcome to the series of lectures on mathematical methods and applications. So, we have seen so many things about Laplace transforms. We have seen what Laplace transforms are, what are the important properties of Laplace transform. How can you find out Laplace transform of some important functions that inverse etcetera? In the last lecture we have seen what unit step function is how to find out Laplace transform unit step of function and some problem based on it.

Now, in this lecture we will see what is Dirac delta functions. And how to find out Laplace transform Dirac delta function and some problems based on it. So, first thing is, what is Dirac delta function or we call it unit impulse function.

(Refer Slide Time: 01:03)



(Refer Slide Time: 01:11)



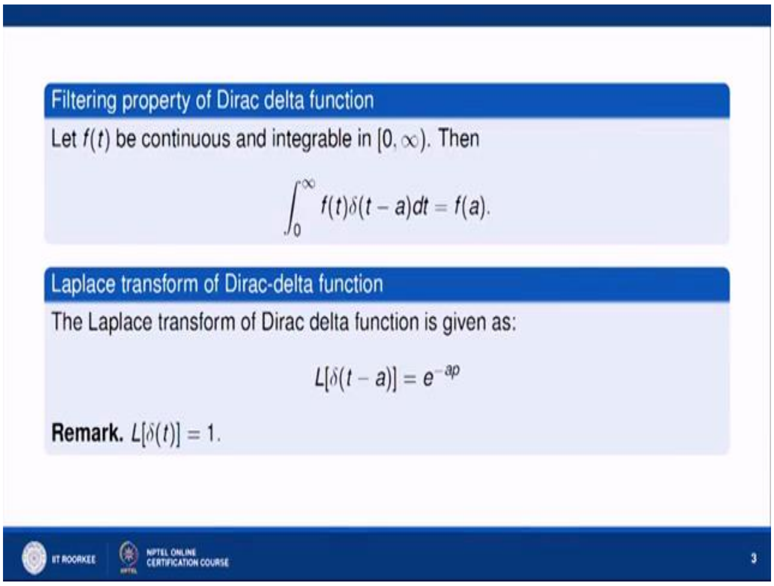
So, it is defined like this. Derived delta at t equal to a is nothing, but 1 upon epsilon where a is tending to t tending from a to a plus epsilon and 0 otherwise, and where epsilon tend to 0. So, this is how we define Dirac delta function, if we roughly speaking if epsilon tend to 0. So, t equal to a and this will tend to infinity.

So, basically if we speak about a definition of Dirac delta function. So, we can say that at t equal to a that is at one point, it is having infinite value and it and all other values at for all other points it is 0. So, whenever we would solve such type of problems or based on Dirac delta, we always use this definition. That is it is 1 by epsilon when t is varying from a to a plus epsilon 0 otherwise. And where epsilon tend to 0. So, what does this definition means, this means when a is varying from a to a plus epsilon, where epsilon tend to 0. Then this value is 1 upon epsilon. So, this value is 1. So, a to a plus epsilon this value is 1 and otherwise this value is 0. So, this is what the graph of Dirac delta function is, this is t this is Dirac delta function s t equal to t minus a. So, this is how we define we can define Dirac delta function.

So, now the important property of delta function is that 0 to infinity the integration of Dirac delta function is always 1. So, that we can easily show 0 to infinity, this integral is equal to now upon the definition of Dirac delta function. It is 1 by epsilon when t varying from a to a plus epsilon, and 0 otherwise. So, this integral has a value only when t varying from a to a plus epsilon. And it is 1 by of 7 d epsilon d t and limit epsilon tend to

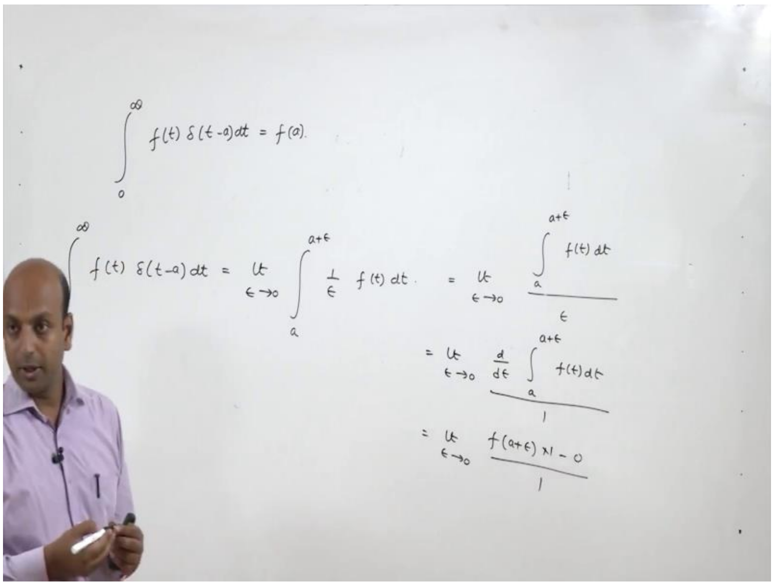
0. Because we epsilon tend to 0. So, this is nothing, but limit epsilon tend to 0, 1 by epsilon integration a to a plus epsilon d t. And this is further equal to limit delta epsilon tend to 0 1 by epsilon t a to a plus epsilon. And it is equals to limit epsilon tend to 0, 1 by epsilon a plus epsilon minus a which is nothing, but 1. So that means, the total area covered by this function, the Dirac delta function from 0 to infinity the total area is always 1. So, we can always define Dirac delta function like this. That this value is 1 by epsilon when t varying a to a plus epsilon 0, otherwise and that we are epsilon tend to 0 and the total area covered by that Dirac delta function is always 1.

(Refer Slide Time: 04:59)



The next property is, we call it filtering property of Dirac delta function it states that, if f t be is a continuous and integrable in 0 to infinity. Then this integral is always equal to f a. So, this also, we can prove easily let us try to prove this result. So, what this property is it is 0 to infinity f t into Dirac delta of t minus a d t is nothing, but f t.

(Refer Slide Time: 05:15)

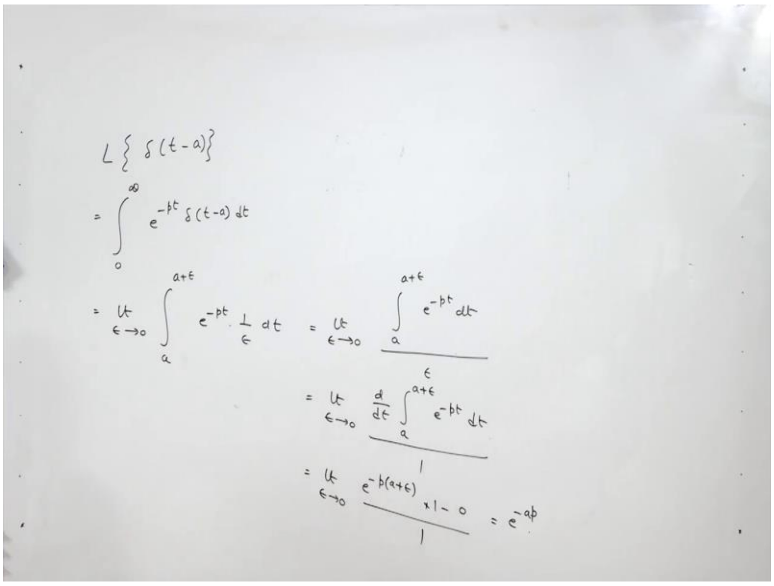


So, this property is call it filtering property of Dirac delta function. So, how we will prove it? Now we will apply the definition of delta function. So, the left hand side is 0 to infinity f t into Dirac delta of t at a into d t. Which is equals to now limit epsilon tend to 0 a to a plus epsilon. It is 1 by epsilon f t into d t. This is by a definition of delta function. Because Dirac delta function attains a value 1 by epsilon, when t varying from a to a plus epsilon and epsilon tend to 0. Now this is equal to limit this tend to 0 a to a plus epsilon into f t d t upon epsilon.

Now, when epsilon tend to 0, this is a to a that is 0 and this is 0, this is 0 by 0 form. And function is continuous and integrable. So, we can apply l hospital rule to find out the value, because it is 0 by 0 form. So, we can apply l hospital rule. So, it is equal to limit epsilon tend to 0, d by s epsilon of a to a plus epsilon, f t, d t, upon 1. We differentiate respect of epsilon the numerator and the denominator. Now this is equal to limit epsilon tend to 0. Now, when you take d by d epsilon of this function with respect of epsilon, so this is nothing, but f t a plus epsilon into 1 minus 0 upon 1 and when epsilon tend to 0, it is nothing, but f a. So, this is called filtering property of Dirac delta function that integration 0 to infinity f t into Dirac delta t minus a, d t is nothing, but f a.

So, this how we can prove this. The next property of Dirac delta is the Laplace transform of Dirac delta is nothing, but e to the power minus a t. So, Laplace transform of Dirac delta function. So, it is nothing, but 0 to infinity, e to the power minus p t, Dirac delta of t minus a into d t; by definition of Laplace transforms.

(Refer Slide Time: 08:00)

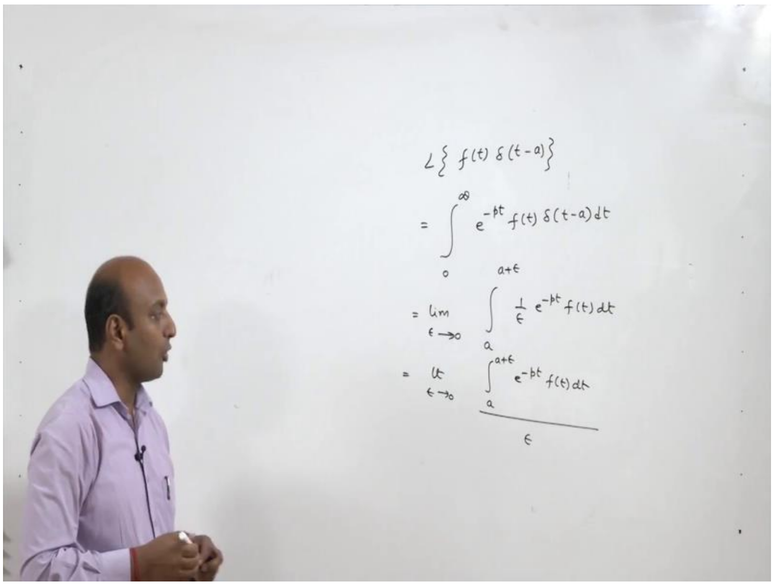


Now, by the definition Dirac delta function, it is equal to limit epsilon tend to 0, a to a plus epsilon, into e to the power minus p t, into 1 by epsilon into d t. And which is further equal to limit epsilon tend to 0 a to a plus epsilon e to the power minus p t d t upon epsilon.

Now, it is again 0 by 0 form. Because there are epsilon tend to 0 the numerator is 0 and the denominator is 0. So, 0 by 0 form. So, again you will apply l hospital rule to simplify this. So, this is nothing, but equal to limit epsilon tend to 0, again d by d epsilon of the numerator quantity upon denominator is 1, the derivative denominator respect to epsilon is 1 and it is nothing, but limit epsilon tend to 0 e to the power minus p. In place of t we have the upper limit which is a plus epsilon. And when into 1 minus 0 of course, upon 1 and when 7 tend to 0 it is nothing, but e to the power minus a p. So, that is how we can find out Laplace transform of Dirac delta function. And it is nothing, but e to the power minus a p. And of course, when a is 0 the Laplace transform of delta t is nothing, but 1. Because when you substitute a equal to 1 in this expression. So, we will get Laplace transform of Dirac delta function at t equal to 0 is nothing, but 1.

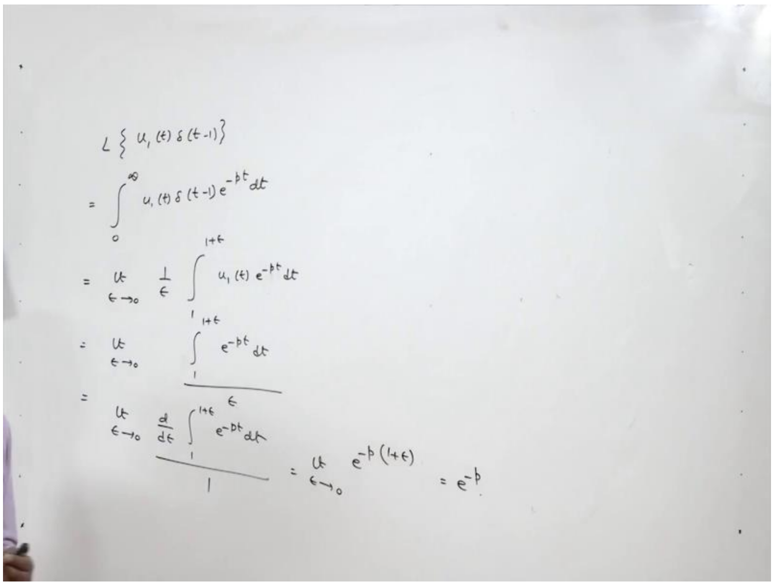
Now, let us solve some problems based on Dirac delta function. Now whenever we want to find out Laplace transform of some f t, into Dirac delta at t equal to a.

(Refer Slide Time: 10:20)



So, this will be equal to 0 to infinity e to the power minus p t, f t into Dirac delta t minus at t equal to a t. And when you apply a definition of definition of Dirac delta function it is nothing, but limit delta tend to 0 a to a plus epsilon into 1 by epsilon e to the power minus p t into f t d t. And which is again equal to epsilon tend to 0, limit epsilon tend to 0 integration a to plus epsilon e to the power minus p t, f t d t upon epsilon. So, it will become it will it will always become 0 by 0 form. So, you will apply l hospital rule to simplify, this expression and then you will you can get the Laplace transform of f t into Dirac delta or t equal to a. So, this is how we can find out Laplace transform of f t into Dirac delta function.

(Refer Slide Time: 11:40)

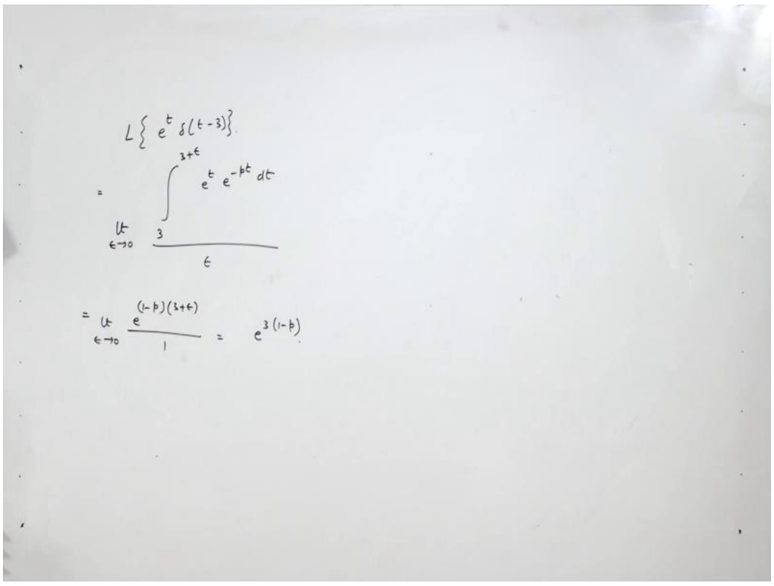


So, now let us try to find out Laplace transform of these problems. First is Laplace transform of unit step function at t equal to 1 into Dirac delta t equal to 1 this. So, this is nothing, but 0 to infinity u 1 t that is f t, f t is a entire expression, into e to the power minus p t into d t which is further equal to first apply definition of Dirac delta function. By definition of Dirac delta function is nothing, but limit epsilon tends to 0 1 by epsilon a to a plus epsilon u here a is 1. So, this will be 1 only and this is 1, and u 1 t into e to the power minus p t into d t. Now apply definition of u 1 t, so u 1. So, u 1 t at t equal to t more than 1 by definition of unit step function is 1.

So, it is nothing, but limit epsilon tends to 0, 1 by epsilon 1 to 1 plus epsilon into e to the power minus p t d t. Because u 1 t is 1, when t is greater than 1 otherwise it is 0. So, now, it is nothing, but limit epsilon tends to 0 1 by epsilon, it is now 0 by 0 form because you can always write it like this. So, it is 0 by 0 form. So, you have to apply l hospital rule to simplify this. So, it is nothing, but limit epsilon tends to 0, d by d epsilon of integral 1 to 1 plus epsilon e to the power minus p t d t upon 1, and which is nothing, but limit epsilon tend to 0 e to the power minus p and t is nothing, but 1 plus epsilon and it is nothing, but e k power minus p. So, that will be the Laplace transform of the first problem.

Now, let us compute Laplace transform of second problem. It is again on Dirac delta function. So, the second problem is basically power t involved in this problem. So, it is Laplace transform of e k power t into Dirac delta at t equal at a equal to 3.

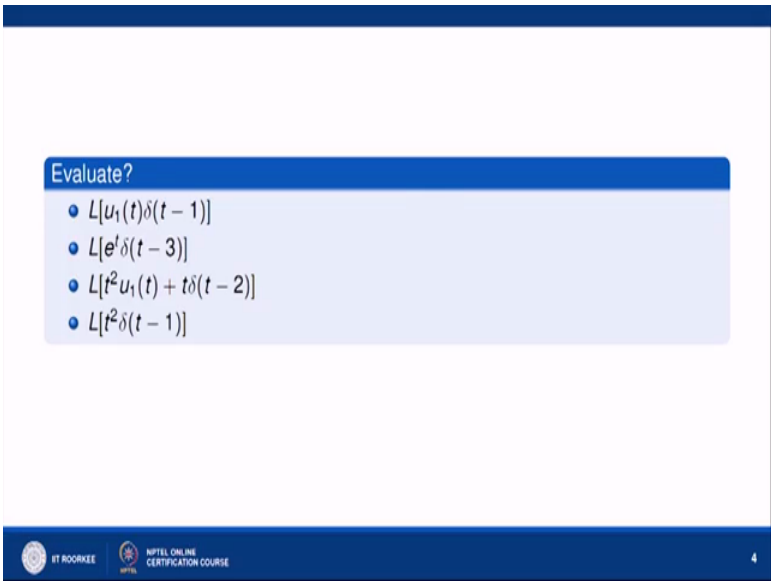
(Refer Slide Time: 14:09)



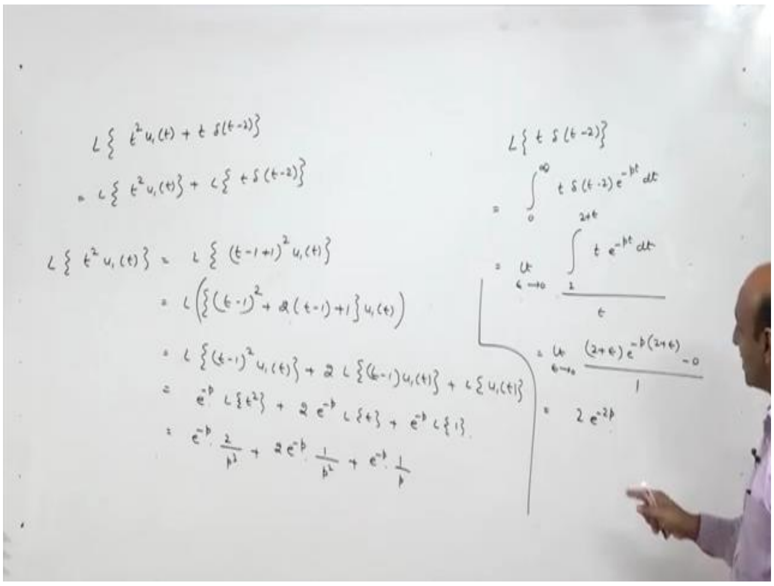
So, it is nothing, but. So, I can write directly it is limit epsilon tend to 0. Instead of a, I have 3 here. So, it is 3 to 3 plus epsilon, 3 to 3 plus epsilon f t, is e to the power minus e to the power t and e to the power minus p t is here also upon epsilon. Because it is a Laplace transform, and I directly applied definition of Dirac delta function. So, it is epsilon tend to 0, 3 to 3 plus epsilon. It is limit epsilon tend to 0, 3 to 3 plus epsilon e to the power 3 into e to the power minus p t d t upon epsilon.

So, it is 0 by 0 form. So, you will again apply l hospital rule to simplify this. So, it is or you can first integrate because it is integrable. It is 1 minus p into 3 plus epsilon upon 1 limit epsilon tends to 0 and it is nothing, but e to the power 3 1 minus p is it. So, this is the final expression which we can obtain using this by applying l hospital rule.

(Refer Slide Time: 15:40)



(Refer Slide Time: 15:42)



Now, the next problem. Next problem is Laplace transform of t square into u 1 t plus t into Dirac delta at t at t equal to 2. So, this is nothing, but Laplace transform of t square into u 1 t, plus Laplace transform of t into Dirac delta at t equal to 2.

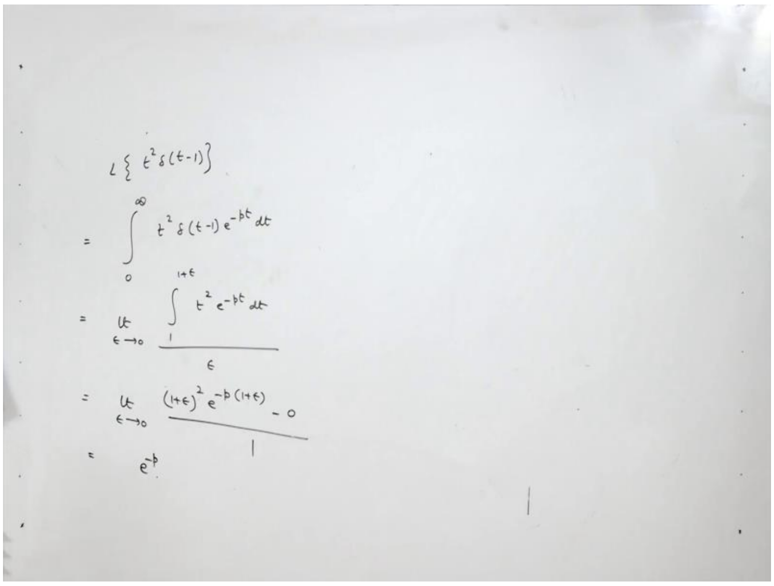
Now, to compute the first expression we will use second shifting property of unit step function which states that, Laplace of f t minus into u t minus a is nothing, but e to the power minus a p f p where this f p is nothing, but Laplace of f t. So, we have to break this t square in the powers of t minus 1, because it is f t minus a and a is 1. So, first we find the Laplace of this then Laplace of this the addition of this 2 will give the Laplace of the entire expression. So, Laplace of t square into u 1 t, this can be find out Laplace of I can write it like this t minus 1 plus 1 whole square u 1 t.

This is nothing, but Laplace of a square plus 2 a b plus b square. And this is nothing, but Laplace of t minus 1 whole square u 1 t, plus 2 times Laplace of t minus 1, u 1 t plus Laplace of u 1 t. Now using that that property, it is nothing, but e to the power minus a p a is 1. So, it is minus p into Laplace of f t. So, Laplace of t square plus 2 into e to the power minus p again and Laplace of t plus, and this is nothing, but e to the power minus p into Laplace of 1. So, this is nothing, but e to the power minus p into t square is nothing, but 2 upon p cube plus 2 e to the power minus p into t is nothing, but 1 p square plus e to the power minus p into 1 by p. So, that is the Laplace of the first expression Laplace of t square u 1 t.

Now, we will compute Laplace of second expression. How we compute Laplace of second expression? It is Laplace of t into Dirac delta t minus 2. So, this can be written as 0 to infinity t Dirac delta t minus 2 e to the power minus p t into d t. And applying the definition of Dirac delta function it is nothing, but limit epsilon tend to 0 2 to 2 plus epsilon t e to the power minus p t d t upon epsilon. So, it is 0 by 0 form. Again we can apply l hospital rule to simplify this expression. So, we will obtain this will be equal to limit epsilon tend to 0 2 plus epsilon into e to the power minus 2 p into 2 plus epsilon into 1 minus 0 upon 1. So, when you take epsilon tend to 0 here, it is nothing, but 2 into e to the power minus 2 p. So, the sum of these 2 expressions will give us the epsilon the Laplace transform of this expression.

Now, the last problem of this slide, Laplace transform of t square into Dirac delta at t equal to t equal to 1. So, this problem also can be solved easily it is 0 to infinity t square Dirac delta minus 1 into e to the power minus p t d t.

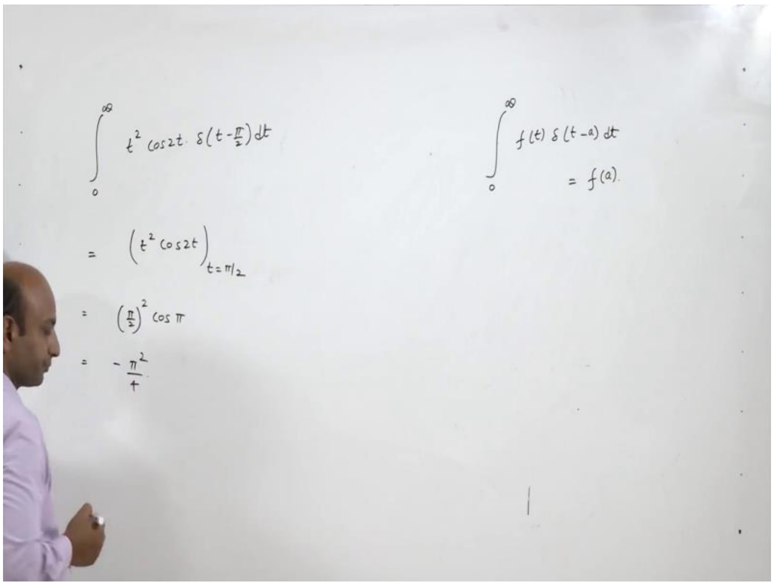
(Refer Slide Time: 20:23)



And when you apply definition of Dirac delta it is nothing, but limit epsilon tends to 0 1 to 1 plus epsilon t square e to the power minus p t d t upon epsilon. Now it is 0 by 0 form. So, you can directly apply l hospital rule to simplify this. So, this is nothing, but limit epsilon tends to 0 1 plus epsilon whole square e to the power minus p into 1 plus epsilon, minus 0 upon 1. And when you take epsilon tend to 0 it is nothing, but e to the power minus p. So, that will be the Laplace transform of t square into Dirac delta at t equal to 1. So, that is how we can solve the problems based on Dirac delta function. I mean Laplace to find out the function with Dirac delta function.

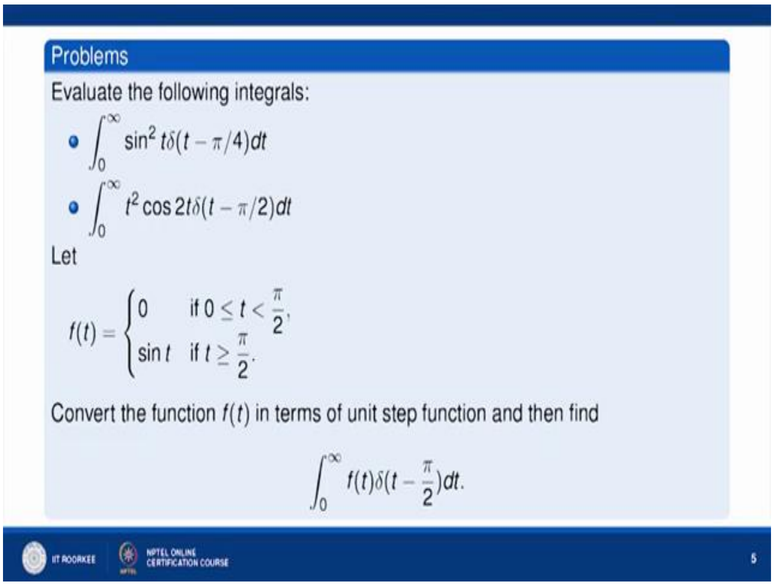
Now, let us evaluate the following integrals, 0 to infinity which are having the involvement of Dirac delta. So, we can directly use filtering property of Dirac delta function. What are filtering property that we have already proved?

(Refer Slide Time: 22:08)



So, this is a filtering property of Dirac delta function that is 0 to infinity f t into Dirac delta of t minus a d t is nothing, but f a.

(Refer Slide Time: 22:22)

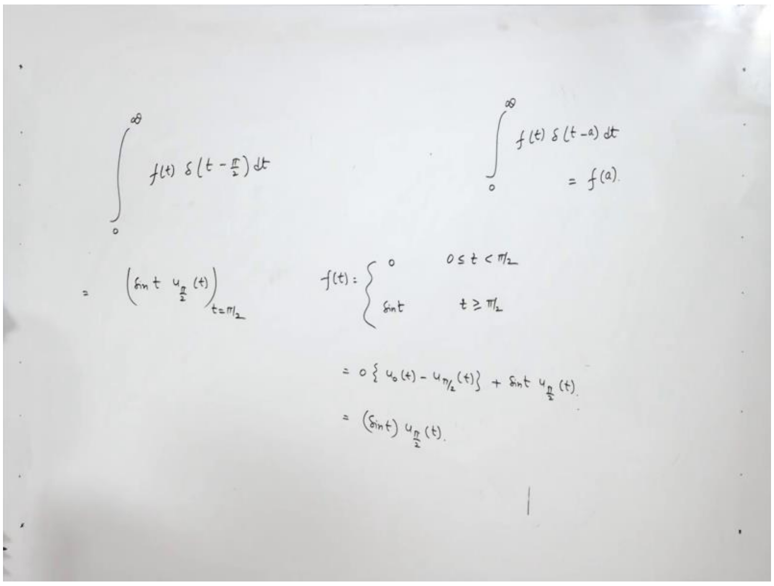


So, suppose want to solve first problem. The first problem is 0 to infinity sin square t Dirac delta at pi by 4. So, if we compare with this expression. So, f t is nothing, but sin square t. And a is nothing, but pi by 4. So, directly using this filtering property, this expression will be nothing, but sin square pi by 4. Now sin pi by 4 is 1 by under root 2, it is whole square which is nothing but 1 by 2. So, directly using filtering property we can find out the value of this problem.

Now, similarly if we take the second problem the second problem is nothing, but it is 0 to infinity t square cos 2 t into Dirac delta at t equal to pi by 2, again if we compare this with this expression. So, f t is nothing but t square cos 2 t, and a is pi by 2. So, by using this property, this expression will have the value equal to t square cos 2 t at t equal to pi by 2. So, this nothing, but pi by 2 whole square cos pi and cos pi is minus 1, it is minus pi square by 4. It is minus pi square by 4.

Now, let us see the third problem of the slide that is 0 to infinity f t into derived delta at t pi by 2 d t. And f t is defined like this f t is given to us it is 0.

(Refer Slide Time: 24:14)



When t lying between 0 and pi by 2 and it is sin t when t is greater than equal to pi by 2. So, basically what this function is, it is 0 into u 0 t minus u pi by 2 t, plus sin t into u pi by 2 t. So, I express this function in terms of unit step function. So, it is nothing, but sin t into u at pi by 2 into t. So, now, this is equal to by filtering property, this will be nothing, but the value of f t equal to a, and here a is pi by 2, so the value of f at pi by 2. So, value of f that is sin t into u pi by 2 into t at t equal to pi by 2.

Now, at t equal to pi by 2, this is 1 it is a unit step function the value will be 1. So, it is nothing, but sin pi by 2 into 1 which is nothing, but 1. So, hence we can find out the value of this expression. So, we have seen that what is Dirac delta function, how we can solve some problems based on this. How we can solve these type of integral using filtering property or delta function. So, in the next lecture we will see that what are the various applications of the Laplace transforms.

Thank you.