Mathematical methods and its applications
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Lecture – 34
Laplace Transforms of unit step function

Welcome to the series of lectures on mathematical methods and it is applications. So, in the last lecture we have studied how to find Laplace transform of periodic functions. In this lecture we will see what are unit step functions, and how to find Laplace transforms of unit step functions. So, what are unit step functions?

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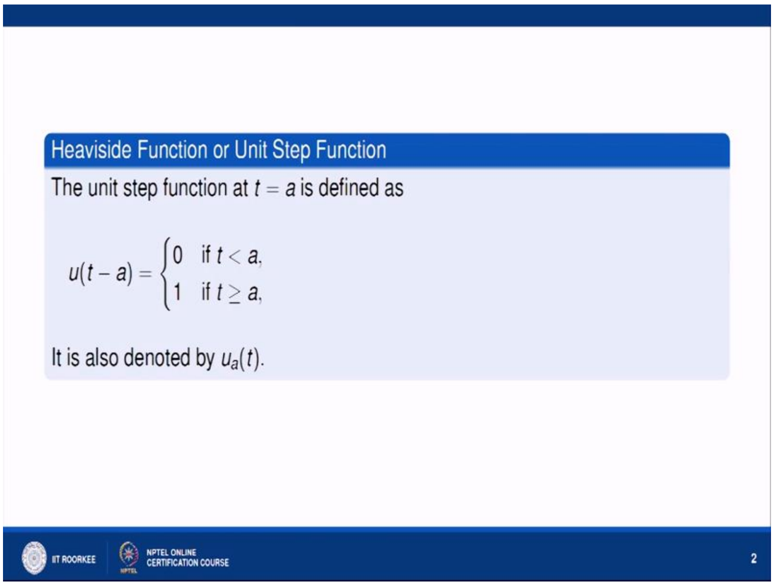

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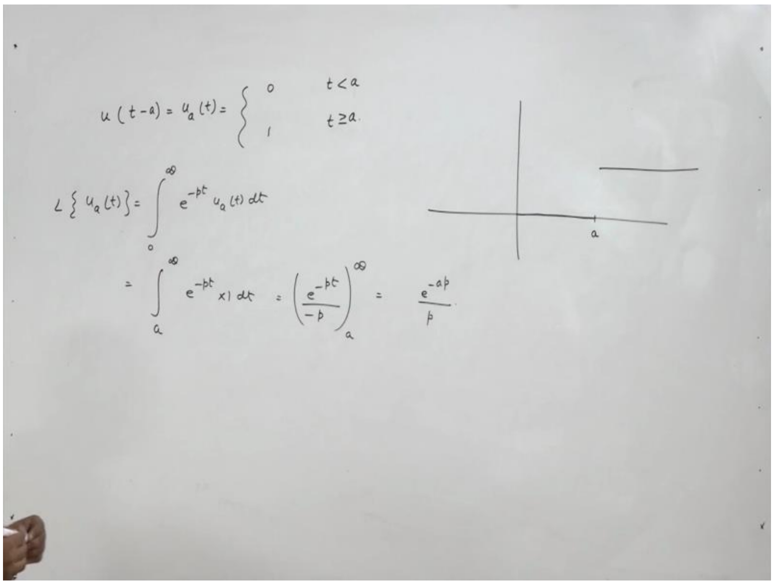

Let us see. So, it is defined like this. Unit step function basically denoted by u t minus a, or it is also denoted by u a t, which is defined like this it is 0 when t less than a. And it is 1 when it is t greater than equal to a. So, it attains this function have only 2 values either 0 or 1. When we are discussing unit step function at a point a, at a point t equal to a; that means, whenever t is less than a, it attains value 0 and when t is greater than equal to a it attains value 1. So, how, what is the graph of this function?

Suppose a is this point. So, when t is less than a, this function has value 0. And when t is greater than a, this value this function is value 1. So, this is unit step function. So, basically this function is used in electrical circuit is and systems. So, whenever we solve some problems on electrical circuits where we have some differential equations having unit step functions. So, therefore, whenever we solve such problems, we have to find Laplace transform of unit step function. So, how to find Laplace of this let us see.

Now Laplace transform unit step function is given as this. Now, what is Laplace transform of this function u a t? Now, one can easily find it is 0 to infinity e k power minus p t into u a t. Now by the definition when t is less than a, u a t is 0. And when t is greater than or equal to u a t is 1. It is basically from a to infinity e k power minus p t into 1. Because otherwise it is 0, and it is nothing but e k power minus p t upon minus p from a to infinity. When t tends to infinity assuming p greater than 0 it is less than 0. And when t equal to a it is nothing but e k power minus a p upon p. So, this will be the Laplace transform of unit step function.

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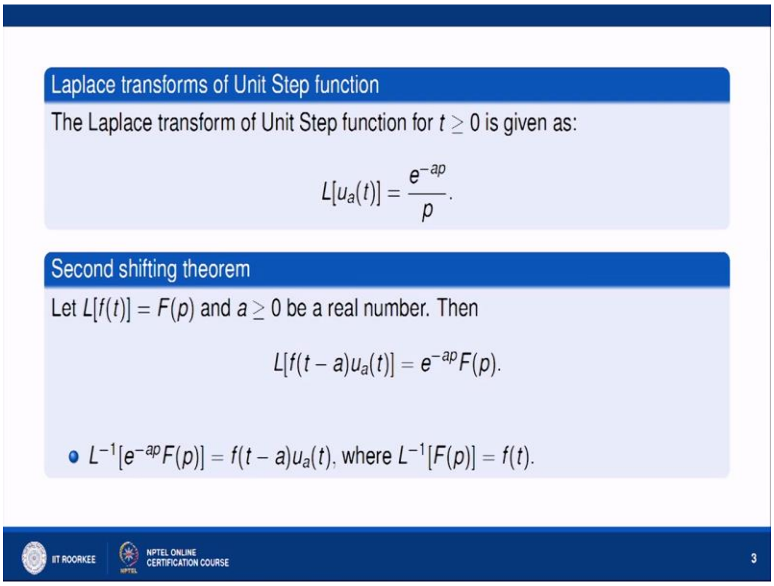

Now, the second shifting property, it is states that if Laplace transform of f t is F p, then Laplace transform f t minus a into u a t is nothing, but e k power minus a p into F p. So, what this property is basically Laplace transform of f t minus a, u a t is nothing, but e k power minus a p, into F p where F p, is Laplace transform of f t. So, this property is called second shifting property.

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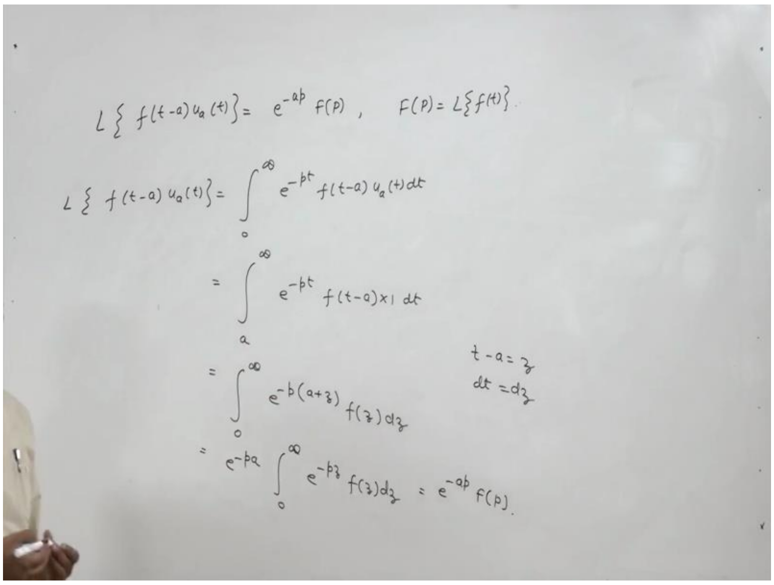

So, how to prove this? Proof is quite simple. This can be written as 0 to infinity e k power minus p t, f t minus a, u a t into d t. Now when t is greater than a, u a t is 1, and when t less than a, u a t is 0, by a definition of unit step function. So, this is a to infinity e k power minus p, t f t minus a, into 1. Because from a, because from 0 to a, u a t is 0. So, that value is 0 now you take t minus a, as some variable z. So, d t will be d z. So, this will be equals to when t is a, it is 0 when t is infinity it is infinity e k power minus p t is a plus z, it is f z into d z. And it is nothing, but e k power minus p a can come out, because it is free from z. So, it is 0 to infinity e k power minus p, z f z d z, and this is nothing, but Laplace transform of f t. So, we can write e k power a p into F p. So, hence we proved second shifting property that, whenever we have to find out Laplace transform of f t minus a, into u a t is nothing, but e k power minus a p into F p where F p is Laplace transform of f t.

Now, the same property can be stated for inverse Laplace transforms. Laplace inverse of e k power minus a p into F p is nothing, but f t minus a u a t. We are Laplace inverse of F p s f t. Now let us solve some problem based on this.

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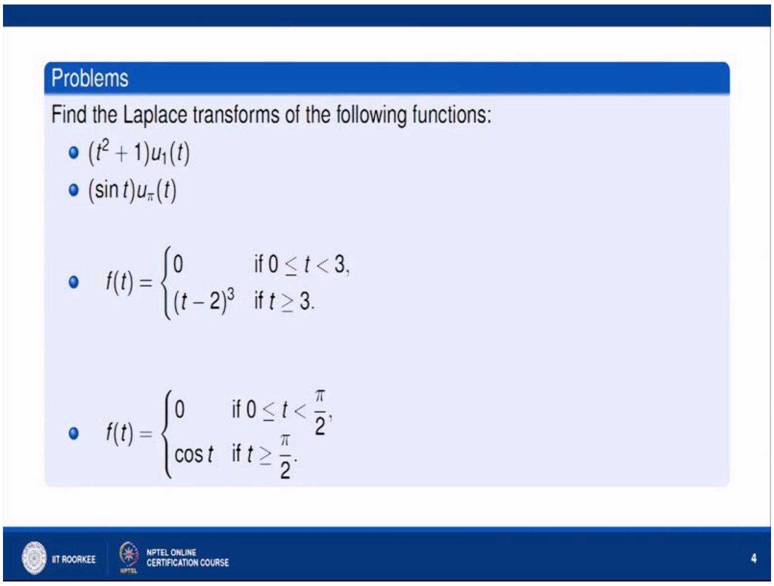

Now, suppose you have to find out Laplace transform of t square plus 1 u 1 t. Laplace transform of t square plus 1 into u t.

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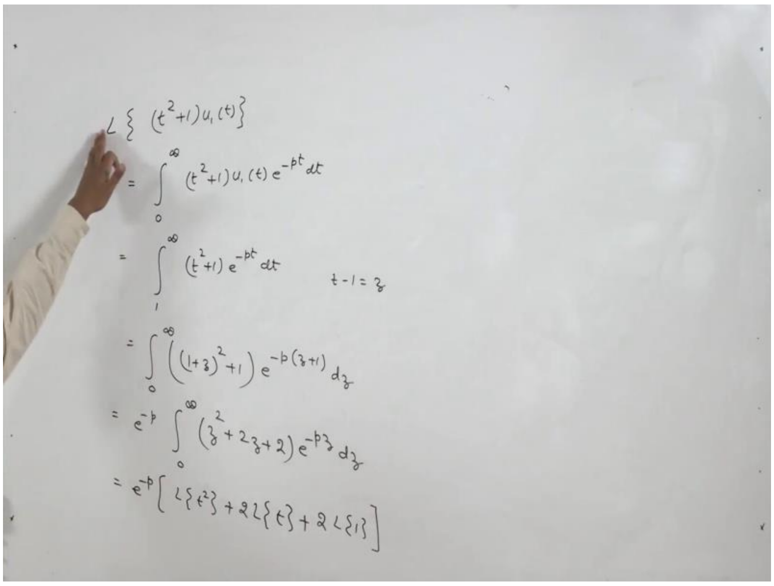

Now, there are 2 ways to solve this problem. Either you applied a definition of Laplace transform, by definition of Laplace transform it is nothing, but 0 to infinity, t square plus 1, u 1 t into e k power minus p t d t. And you know that when t is greater than 1, we know that, when t is greater than 1, it is 1 otherwise 0. So, it is 1 to infinity, t square plus 1 e k power minus p t d t.

Now, take t minus 1 as z. So, it is 0 to infinity and t is 1 plus z whole square plus 1 e k power minus p t is z plus 1 into d z. So, it is nothing, but e k power minus p will come out. So, it is 0 to infinity, it is z square plus 2 z plus 2, into e k power minus p z d z. And it is nothing, but e k power minus p. Now z square into this from 0 to infinity is nothing, but Laplace transform of t square, plus 2 into, z into, this is nothing, but Laplace transform of z that is Laplace transform of t, plus 2 into Laplace transform of 1.

Now, Laplace transform of t square we already know. It is nothing, but 2 upon, t square is nothing, but factorial 2 upon p q. Laplace transform of t is 1 upon p square it is 1 upon t, on we can simplify and we can find out the Laplace transform of this expression. This is the one way you directly apply the definition of Laplace transform and the unit step function. And then find the Laplace transform of this function f t. Now the second way out is he we can use second shifting property.

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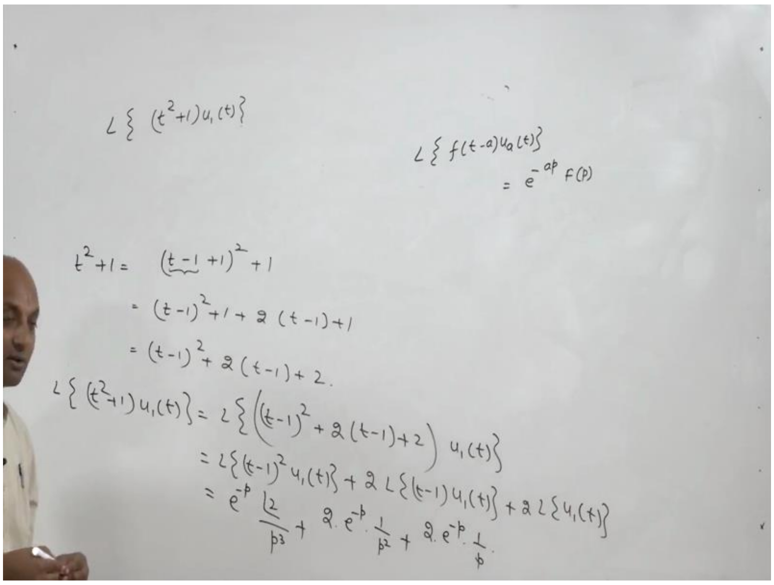

Now, how to use second shifting property let us see. What second shifting property states it is states that Laplace transform of f t minus a u a t must be equals to e to power is equal to e k power minus a p into F p where F p is Laplace transform of this f t.

The important thing to note here is, a is same. Whatever a we have we are having here, the same a we are having here. Now here we have to we are a here a is 1. Now to applying second shifting property, this f t square plus 1 must be in the powers of t minus 1, because a is 1. So, t square plus 1 we have to express any way in the powers of t minus 1. So, it has it has a quality expression. So, we can write it like this, t minus 1 plus 1 whole square. This and plus 1 this you can take as 1 function. So, it is a square plus b square, plus 2 a b plus 1. And it is t minus 1 whole square plus 2, into 2 minus 1 plus 2. So, this expression can be written as this in in the powers of t minus 1.

Now, Laplace of t square plus 1 into u 1 t. This can written as Laplace of t minus 1 whole square plus 2 into t minus 1 plus 2, whole multiplied by e u 1 t. Now this can be done as Laplace of t minus 1 whole square, u 1 t plus 2 into Laplace of t minus 1, into u 1 t plus 2 time Laplace of u 1 t. Now by the shifting property, second shifting property, this expression is nothing, but e power minus a p, a is 1 to e k power minus p, F p, F p is Laplace transform of f t. Here f t is t square. So, Laplace transform of t square is factorial 2 upon p q, plus now 2 into again e k power minus a p by this property.

Now here F p is t, here f t is t and the Laplace transform of t is 1 by t square plus 2 into, and Laplace transform of 1 is 1 into. So, 1 is 1 by p e k power minus p into 1 by p. So, that is how we can find out the Laplace transform of same function Laplace transform using second shifting property. So, it is our choice whether we use the direct definition of Laplace transform or we can directly use the second shifting property to find out the Laplace transform of such functions.

Now, suppose you have a second problem. So, again a second problem can be solved, using second shifting property. So, how to find? So, Laplace transform of it is sin t into u pi t.

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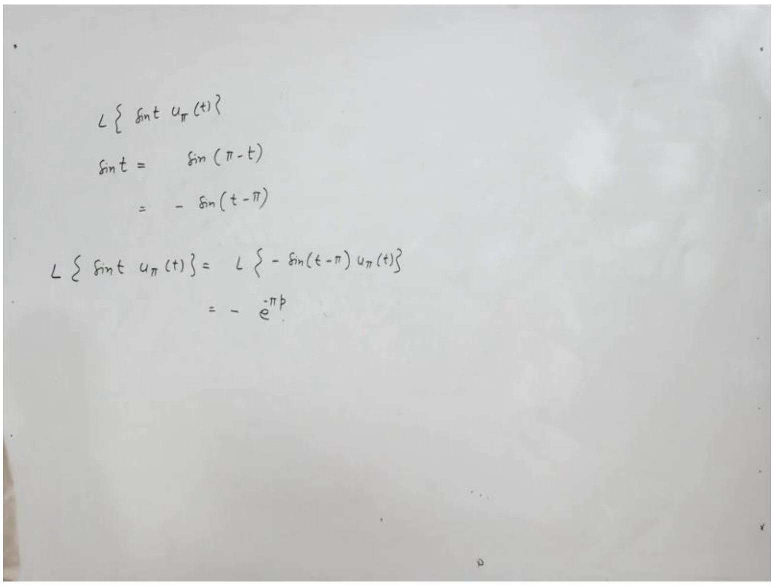

So, here a is pi. So, you have to express this function f t in terms of t minus pi. Then only you can apply second shifting property, because by second shifting property we have Laplace transform of t minus a into u a which is equals to e k power minus a p into f t into F p. So, you have to express the sin t in the in the as a function of t minus pi. So, how to write this? Sin t is same as sin pi minus t, sin theta is same as sin pi minus theta, and it is nothing, but negative of sin minus pi.

So, that is how we can express sin t in that in terms of t minus pi. Now Laplace of sin t into u pi t is nothing, but Laplace of negative of sin t minus pi, into u pi t. Negative can come out and by the second shifting property it is nothing, but e k power minus a p a is pi, pi minus a p into F p, F p is Laplace transform f t. Here f t is sin t. So, Laplace of sin t is 1 upon p square plus. So, that how we can find out Laplace transform of this function.

Now, similarly Laplace transform of third problem can be find out, because what is the what is the what is the third problem basically.

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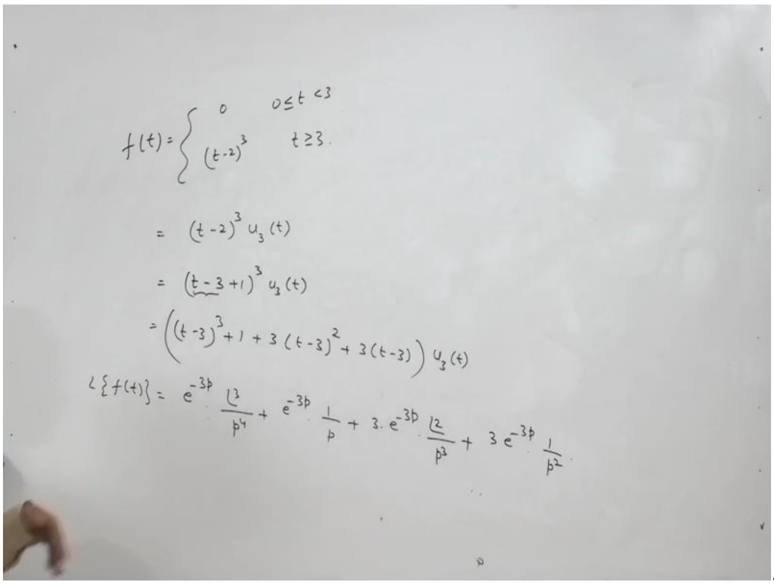

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If you see the third problem carefully, so it is 0 when t is less than 3 greater than or equal to 0, and it is t minus 2 whole cube when t is greater than or equal to t. So, if you see this problem carefully it is nothing, but t minus 2 whole cube into unit step function a t equal to 3. Because when t is less than 3, it is 0. 0 into this is 0, and when t is greater than 3 it is 1. So, 1 into this is this. So, basically this function is nothing, but this function, now to find out Laplace transform of this function and you have to express this function this f t in the powers of t minus 3.

So, how we can express this function power of t minus 3? You can take it this t minus 3 plus 1 whole cube. So, it is now you can take it a and it is b. So, a cube, plus b cube, plus 3 a square b, plus 3 a b square, and u 3 t. Now the Laplace transform of this function f t will be nothing, but Laplace transform of this into this which is nothing, but 8 power minus a p a is 3 p and t cube is factorial 3 upon p to the power 4, plus 1 one is e k power minus 3 p 1 is 1 by p plus 3 into e k power minus a p e k power minus 3 p into, t minus 3 whole square is factorial 2 upon t cube, plus 3 into e k power minus 3 p. Again t minus 3 f t is t. So, Laplace transform of t is 1 by p square. So, that is how now we can simplify this. So, that will be Laplace transform of this function.

Now, similarly in the last problem is nothing but, cos t into unit step function at t equal to pi by 2. And again we have to express cos t in terms of t minus pi by 2, to apply shift second shifting property. Or the other way is you can apply you can use the definition of Laplace transform and directly you can find out the Laplace transform of such functions.

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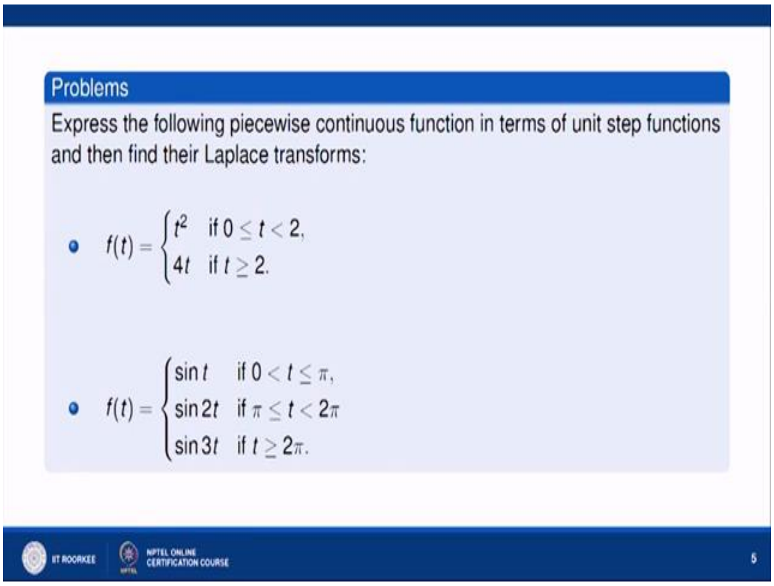

Now, suppose this function is given to you and you want to express this function in terms of unit step function. And then you have to find out the Laplace of this function. How to solve these such problems?

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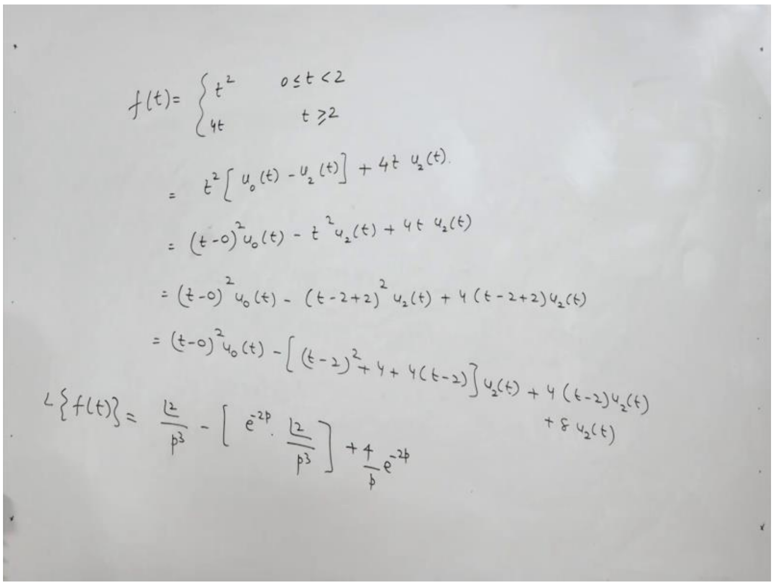

So, what is this function? F t is t square when t varying from 0 to t. And it is 4 t when t is more than 2, more than or equal to, now first we have to express this function in terms of unit step function. So, directly if you want to see to we are unable to express function unit step function directly, but how can we simplify this. Now this function can be written as t square into the lower limit is a t is equal to 0, to write like this u 0 t, minus u 2 t, why I am doing this I will explain it now.

Basically first function u at lower limit minus u at upper limit, plus 4 t into u 2 t. Now if suppose t lying between 0 to t 0 to 2, if t lying between 0 to 2. So, this will be 1, because t is more than 0 and since t is less than 2. So, this is 0 and this is 0. So, we are have we are having only t square. So, t square when t lying between 0 to t 2. Because when t is lying between 0 to 2 t is less than 2 and when t is less than 2 u 2 t is 0 it is 0, it is 0, it is 1 because t is more than 1. So, it is nothing, but t square, now when t is greater than equal to 2. So, it is greater than 0 also, when t is greater than or equals 2 it is more than 0 also. So, it is 1 it is 1 it is 1. 1 1 cancel out. So, it is nothing, but 4 t, so that how we can express this function in terms of unit step function.

Now, how to find Laplace transform of this function? We have to use second shifting property. For second shifting property we have to make the same a f t minus say into 2 a t. Now here t is 0 here it is also 0. So, no problem, it is this is nothing, but t minus 0 whole square and u 0 t here. We know here we have no problem. So, it is t square u 2, t plus 4 t into u 2 t. Now this is t, minus 0 whole square or t square u 0 t. Now we have to express this t square in the powers of t minus 2. So, it is t minus 2 plus 2 whole square we can write and here also t minus 2 plus 2. So, it is nothing, but t minus 0 whole square, u 0 t and it is a square plus b square plus 2 a b. And it is 4 times t minus 2 u 2 t, plus 4 2 z a 8 u 2 t.

Now, the Laplace transform of this f t is nothing, but using second shifting property. It is nothing, but e k power minus a p a 0. So, e k power 0 is 1, and F p f p is Laplace transform f t, f t is t square, t square is 2 factorial upon p cube, minus, now it is 2 minus 2 whole square into u 2 t. So, it is nothing, but it is nothing, but e k power minus 2 p into Laplace transform of t square. Which is nothing factorial 2 upon p cube this expression. Next is minus 4, minus 4 plus 8 is plus 4. Plus 4 times u 2 t is nothing, but 1 by p to the power minus 2 p. Next is minus 4 plus 4 for that is that cancels out. So, this will be the Laplace transform of this function.

So, basically when we have a function in break up form. So, the function can be rewritten in terms of unit step function and then can and then the Laplace transform of such functions can be find using second shifting property. So, the Laplace transform of such functions can be find out first we expressed source function in terms of unit step function. And then we using second shifting property Laplace transform such functions can be find.

Now, similarly we can solve the second problem. Also first we have to convert this function in terms of unit step function. So, I will convert this in terms of unit step function. And then we can find it is Laplace transform using second shifting property. So, what is this function will be basically in terms of unit step function? Sin t into u 0 t minus u pi t, plus next is sin 2 t, upper limit is u pi minus lower limit is u 2 pi t, plus sin 3 t and u 2 pi t.

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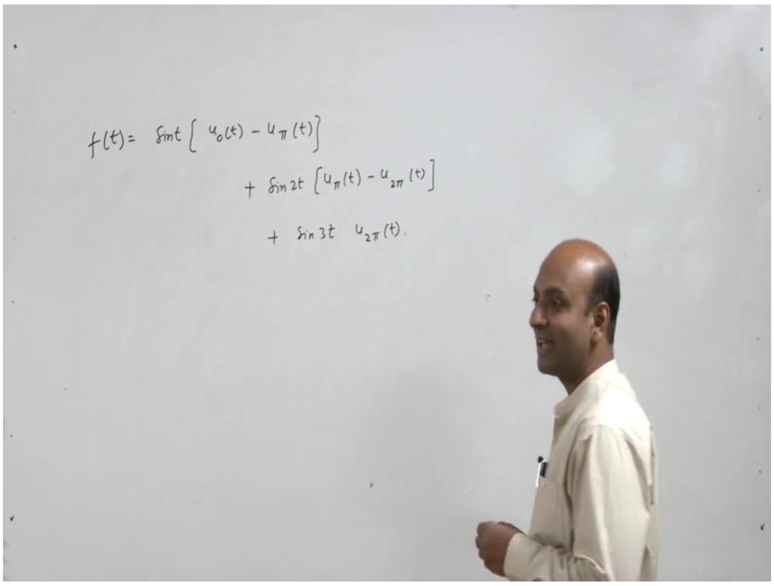

So, in this way we can express this function in terms of a unit step function. You can easily see when t is more than 0 and less than pi. So, it is 1 it is 0. So, you will be having only sin t, that is the first part of this expression of this problem.

When t is more than pi and less than 2 pi. So, it is 1 it is 1, 1 1 cancel out. And it is 1 it is 0 it is 0 because t is less than 2 pi. So, we will be having only sin 2 t which is second part of this expression. And when t is more than 2 pi, it is 1 it is 1 it is 1 it is 1. So, 1 1 cancel out, 1 1 cancel out, it is 1. Only we are having with sin t 3 which is a third part of this expression. So, in this way we can express this function in terms of unit step function. And in order to find the Laplace transform of this f t, express this if you multiply this leave as it is, by multiply this with this express sin t in the terms of t minus pi and when you multiply this with this or this with this express f t with f t minus a whenever you have u have u a t. Here a is pi express this in in the form of t minus pi, when you multiply with this 2 pi express this in the in terms of t minus 2 pi. Then you can easily apply second shifting property to find out the Laplace transform of this function.

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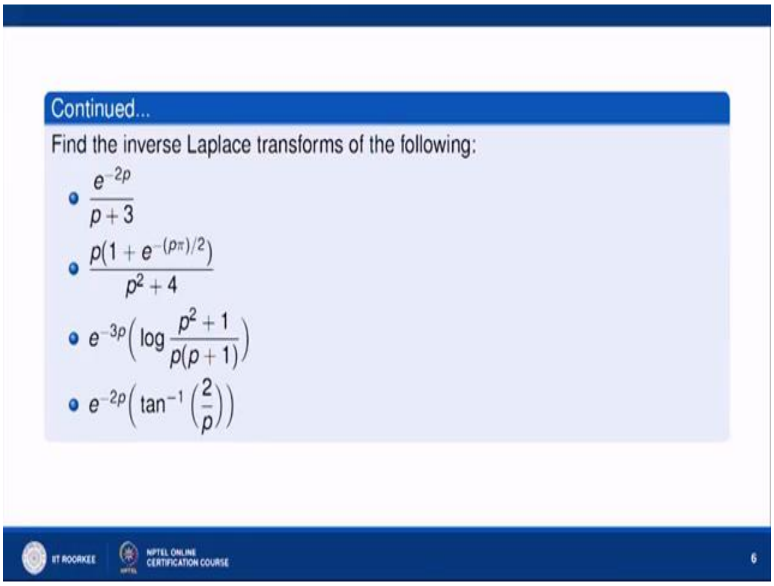

Now, let us find inverse Laplace transform also of such functions. So, what is the inverse Laplace transform of unit step function like?

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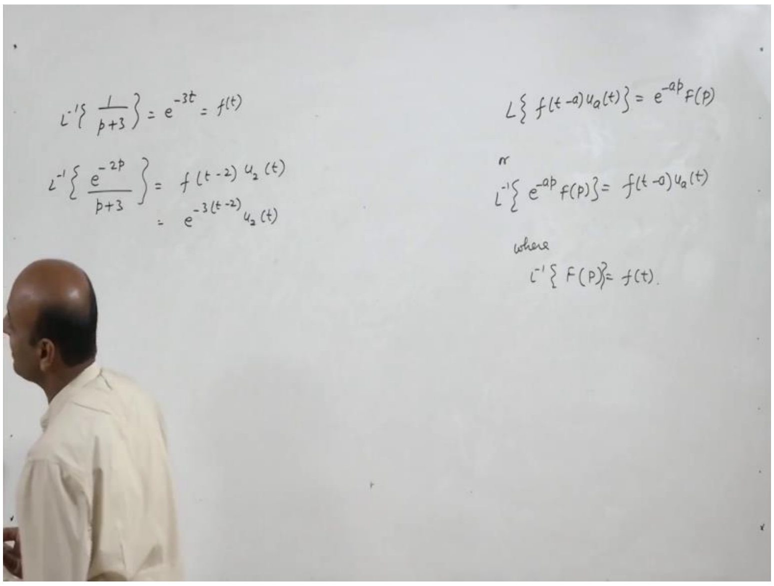

So, we have already seen that f t minus a into u a t is nothing, but e k power minus a p into F p where F p is Laplace transform of this f t. So, in terms of inverse the, what we can say Laplace inverse of e k power minus a p into F p is nothing, but f t minus a, into u a t where Laplace inverse of F p is f t. So, this is a implication of second shifting property.

Now, if we want to solve out the first problem which is very simple it is e power minus a p, and F p here is 1 upon p plus 3, the first in the first problem. So, F p is 1 upon p plus 3. And what is the Laplace inverse of 1 upon p plus 3; it is e power minus 3 t. So, basically Laplace inverse of 1 upon p plus 3 for the first problem is e power minus 3 t. So, what will the Laplace inverse of e k power minus 2 p upon p plus 3 it is nothing, but F p minus 8, this is f t. And a is 2. So, this is nothing, but f t minus 2, into u 2 t, by this property. So, it is e k power minus 3 t minus 2 u 2 t. So, this will be Laplace inverse of the first problem.

Similarly, we can find out Laplace inverse of the second problem also. Now let us solve the third and fourth part of similar. So, let us try to solve out any one of the parts say third part we will solve. So, basically whenever there is a involvement of e k power minus a p with any F p, so this over unit step function. So, first find out Laplace inverse of F p, and then using second shifting property you can find out the Laplace inverse of e k power minus a p into F p. Now in the third problem F p is what? F p is log p square plus 1 upon p into p plus 1.

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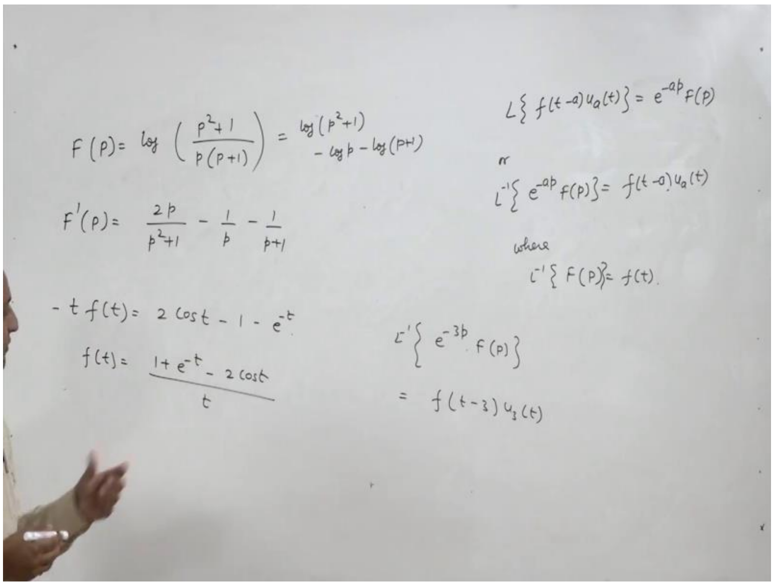

First find out Laplace inverse of this F p, which we call as f t. And e k power minus a p, a is 3 from the problem. If we compare this problem with that problem, so a is 3 after finding f t simply replace t by t minus a, that is t minus 3 into u 3 t. So, the only thing to find is what is f t. So, f t we can find out for such problems we have to use derivative. So, what is f dash p, f dash of this function is nothing, but 2 p upon p square plus 1, minus by p minus 1 upon p plus 1, because this function nothing, but log p square plus 1, minus log p minus log p plus 1. So, derivative will be this. And what is the Laplace inverse take Laplace inverse both the sides.

So, Laplace inverse of f dash p is minus t f t, and this is 2 cos t this is minus 1. Laplace inverse of 1 by is 1 and Laplace inverse of 1 upon 1 plus p is first e k power minus t shifting property. And then 1 upon p is nothing, but 1, it is of course, it for minus t. So, f t will be nothing, but 1 plus e k power minus t minus 2 cos t upon t. So, this is f t. So, what is the Laplace inverse of this? Laplace inverse of e to the minus 3 p into this F p will be nothing, but f of t minus 3 into u 3 t. So, in this f t simply replace t by t minus 3, you will get the Laplace inverse of this function.

Similarly, to solve the last problem, find out the Laplace inverse of 10 inverse 2 by p by making use of derivatives. And then use second shifting property to find out Laplace inverse of e power minus a p into F p, here a is 2. So, that is how we can find out Laplace transform of unit step functions and inverse Laplace transforms involving unit step function.

Thank you very much.