Mathematical methods and its applications

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Lecture – 33
Laplace Transforms of Periodic Functions

So, welcome to the lectures series on mathematical methods and it is applications. In the last lecture we have seen initial and final value theorems and we have solved some problems based on it. So, we have also seen that when they are important; like initial and final value theorem when they are applied basically. In this type of problems we can apply initial and final value theorems.

Now, in this lecture we will see Laplace transforms of periodic functions. So, we already know periodic functions. Periodic functions are those functions which repeat after some t; that means, if function f t is said to be a periodic function, if f of t plus capital T is equals to f t and t is not equal to 0, and that t is a smallest one.

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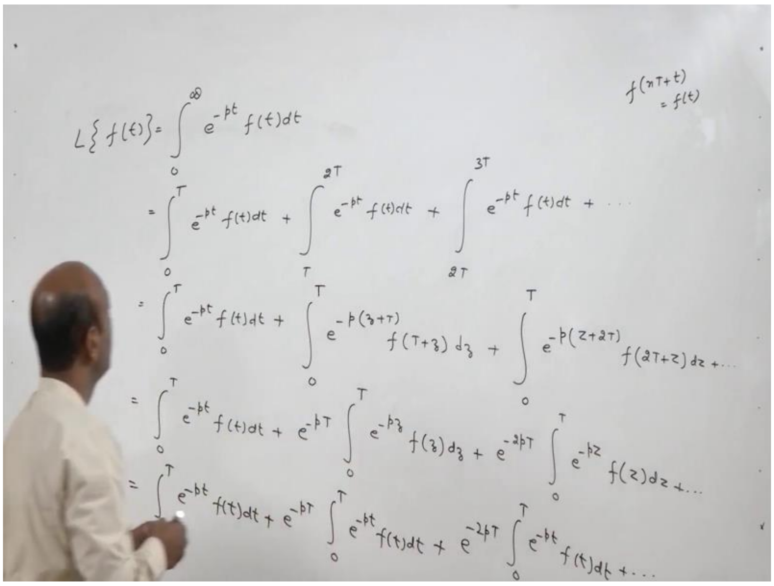

Smallest positive number, is smallest positive real number; such that f t plus t equal to f t, then we say that T is period of the function f. Now to find Laplace transform from a periodic functions. So, we have a direct expression to find out Laplace transform of periodic functions, what is that. Suppose f t has a period t; of course, t greater than 0, then the Laplace transform f t is given this expression. Now how we obtain this expression.

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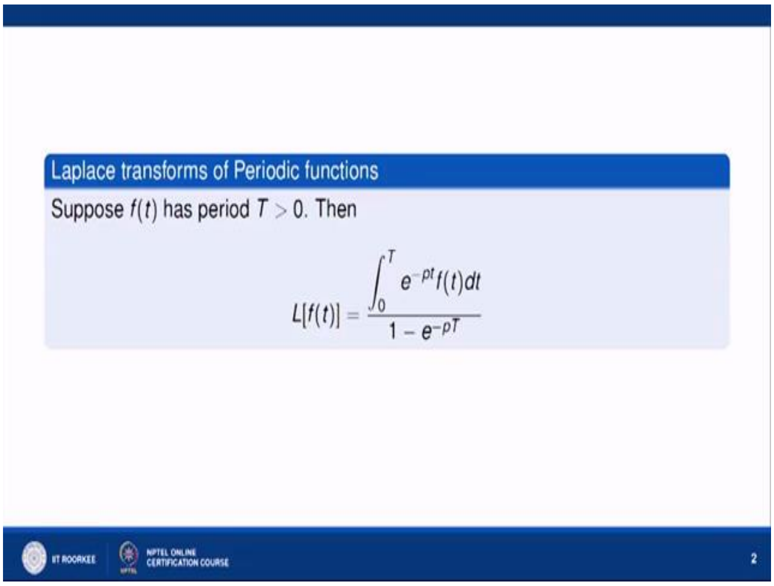

Let us see. So, what is Laplace transform f t, it is nothing, but 0 to infinity e k minus p t f t d t.

Now, what is the period of f, a period is T, period T means it is the smallest real number t real, smallest non 0 real number t; such that f t plus T equal to f t that is by the definition of the periodic functions. Now since it is period T, we can write it like this 0 to t e k power minus p t f t d t plus t to 2 t e k power minus p t f t d t plus 2 t to 3 t e k power minus p t f t d t and so on, up to infinity. Now you this first integral remain put it as it is. For a second integral let T minus capital minus as z for second integral. So, what will be expression when small t is T, the z will be 0, and when it is 2 t. So, 2 t minus t is T z will be t, and e p what is t caps small t is z plus T it is z plus T f of, and t is nothing, but z plus t into and d t is, from here d t is nothing, but d z. So, it is d z plus.

Now, for the next expression, for the next integral again suppose say t minus 2 t is capital Z suppose. So, d t will be nothing, but d capital Z, and when a small t is 2 t 2 T, capital Z will be 0. So, it is 0 and where it is 3 t, 3 t minus 2 t is T, capital Z will be t e k power minus p, and small t from here is nothing, but z plus 2 t. So, it is capital Z plus 2 T f of t is nothing, but 2 t plus z into d z plus and. So, on in a same way we will get another terms of this infinite series. So, this is nothing, but this integral remains as it is; 0 to t e k power minus p t f t d t plus. Now this term e k power to minus p T is free from z free from z can be taken out from the integral. So, it is e k power minus p capital Z integral 0 to t e k power minus p z, and since function is periodic has a period T. So, f z plus t is nothing, but f z and d z, since function is periodic. So, f t plus T is f t here t is z. And similarly here e k power minus p into 2 t, we can take outside the integral minus 2 p t from 0 to t e k power minus p capital Z.

Again functions of period t, so this will be nothing, but f z itself into d z, because functions of period t means after every t interval the function repeat itself. So, if function is has a period t so; that means, f; that means, f of n t plus t will be nothing, but f t, because after every t function repeat itself. So, after n interval also function will be itself only, because period is T; that is by definition of periodic functions.

Now, rather you integrate taking variable z or you integrate taking variable t, it is same by the definition of, by the properties of definite integrals. And here also rather you integral with respect to capital Z or small t all are same. So, this is nothing, but this can be written as 0 to t e k power minus p t f t d t, plus e k power minus p t 0 to t. So, we can write it like this e k power minus p t f t d t plus. In this expression you can also write e k power minus p to t 0 to t e k power minus p t f t d t and so on. So, what is the final expression now? So, in the final expression you can easily see that this term is common from all the terms. This expression is common in all the terms. In the first term we have 1, second we have e k power minus p t, the third term here e k power minus 2 p t and so on.

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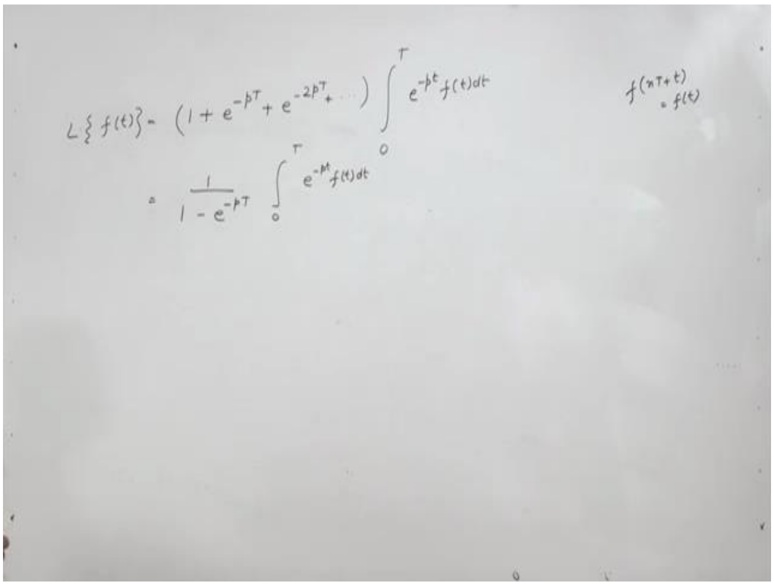

So, what will be the Laplace transform of f t? This we were calculating, this is nothing, but this equal to this. So, when we simplify this. So, we will get 1 plus e k power minus p t plus e k power minus 2 p t plus and so on, and this is common from all the terms 0 to t e k power minus p t f t d t.

So, this after simplification we get this. Now what is the first, what is the first term? It is a geometric progression. So, how to summit up, using the formula for geometric progression for infinite series, and when we have a. So, it is nothing, but a upon 1 minus r. here a is a first term first term is 1, and 1 minus r r is e k power minus p t integral 0 to t e k power minus p t f t d t. So, this is the required expression, which is same as this expression. So, this is how we can find out Laplace transforms of periodic functions. So, whenever any function is given to you and you know the period of that function. So, you can simply find out Laplace transform of such function by simply using this expression, I mean this formula. So, now let us solve some problems on periodic functions. First is square wave function, how it is defined. It is defined like this when f t is k when t is from 0 to a.

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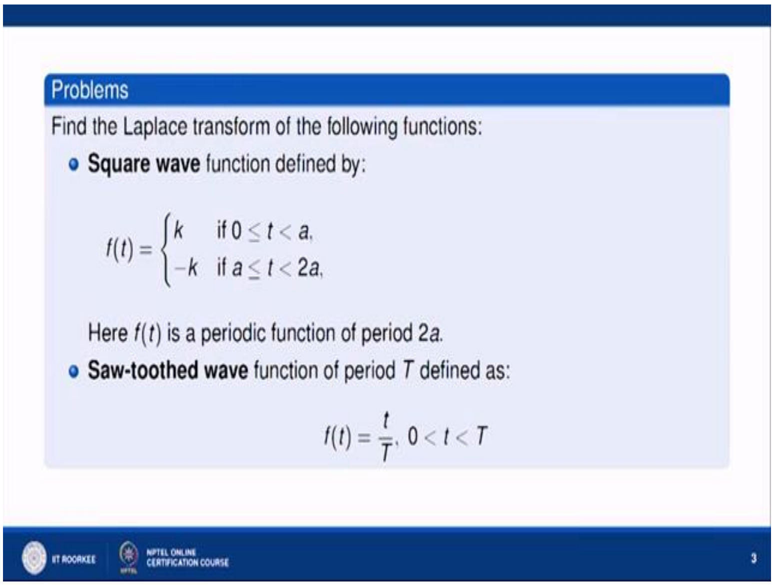

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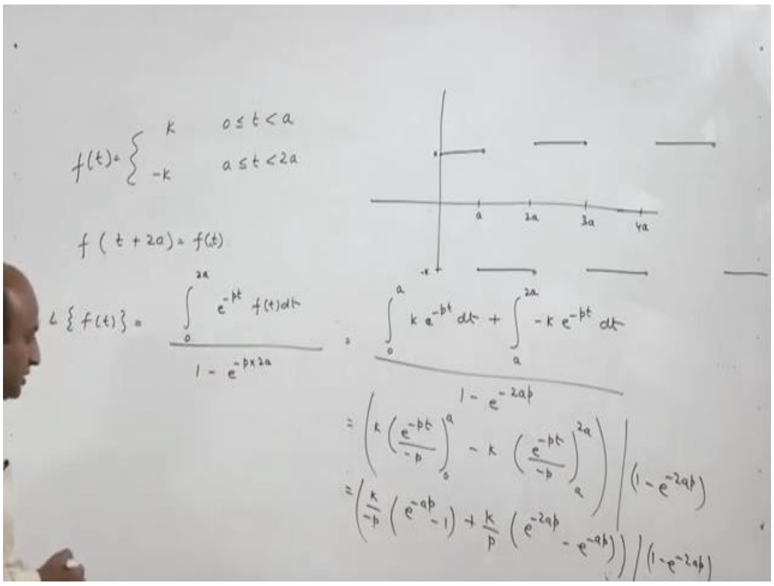

So, it is f t is k, when t varying from 0 to a, and it is minus k when t varying from a to 2 a, and period is 2 a; that means, f of t plus 2 a is f t f t. So, what is the shape of this function. From 0 to it is a, it is 2 a, 3 a, 4 a and so on. Now from 0 to a, it is having a value k, assume k to be positive. So, it is a value k; that means, suppose this is k. So, this is the k, and from a to 2 a it has value minus k. So, suppose minus k is somewhere here, from a to 2 a it has minus k again. So, this repeat itself. So, now, from 2 to a to 3 a, it has a value k. and from 3 a to 4 a it has a value minus k, and the process continues. Since it has a period 2 a.

Now, how to find Laplace of this function? So, finding Laplace of this function is easy, you can simply find Laplace of this f t, by using that expression which is nothing, but 0 to t, t is the T is the period of function, that we have seen. If you see this for expression here T is the period of the function, period of this function is 2 a. So, it is 0 to 2 a e power minus p t f t d t upon 1 minus e k power minus p into, and T and T is period which is 2 a. So, now, we can compute the numerator part, denominator is constant, I mean function of p it is we can we have to simplify. Now again from 0 to a it is f t is k. So, it is k into e k power minus p t d t plus a to 2 a it is minus k minus k into e k power minus p t d t divided by 1 minus e k power minus 2 a p. So, this can be further written as k e k power minus p t upon minus p 0 to a minus k e k power minus p t upon minus p from a to 2 a, and whole divided by a 1 minus e k power minus 2 a p. now when you take the, when you plan the limit is. So, it is k upon minus p minus minus plus it is k by p, numerator is e k power minus 2 a p, denominator is a which is minus e k power minus a p, and this whole expression divided by 1 minus e k power minus 2 a p.

So, you can simplify this expression to get the final answer of Laplace transform of this function. So, what we will obtain. So, when you simplify this. So, Laplace transform of f t will be nothing, but of the first problem, of the first problem is nothing, but you can take k upon p common, it is e k power minus 2 a p.

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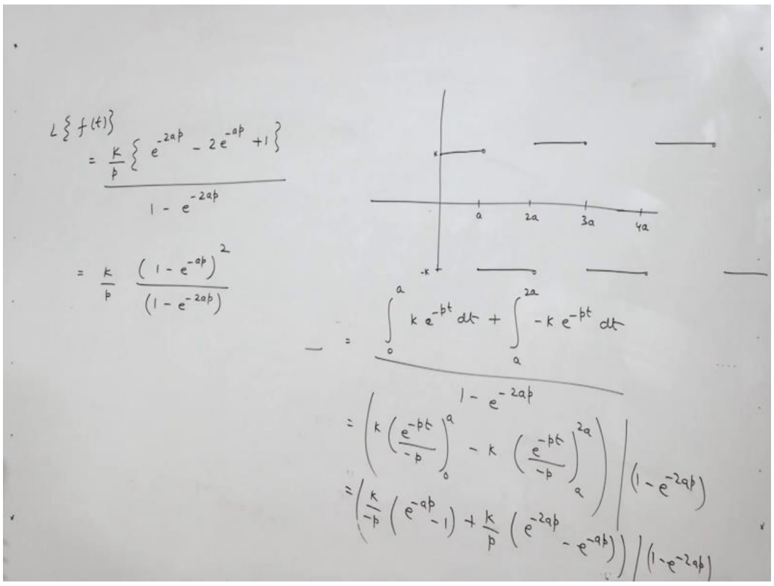

Now, this is negative and this is also negative. So, it is minus 2 e k power minus a p k upon p is common, and this is minus minus plus, it is plus 1, and whole divided by 1 minus e k power minus 2 a p. So, this is nothing, but k upon p. So, it is nothing, but 1 minus e k power minus a p whole square, upon 1 minus e k minus 2 a p. So, this will be the final expression. Now, the second problem, we call it saw toothed wave function which is defined like this. Now what is the shape of this function let us see it has a period T. So, what is the shape of this function functions defined like this it is f t, which is equals to t upon T, when t varying from 0 to t, and period is T.

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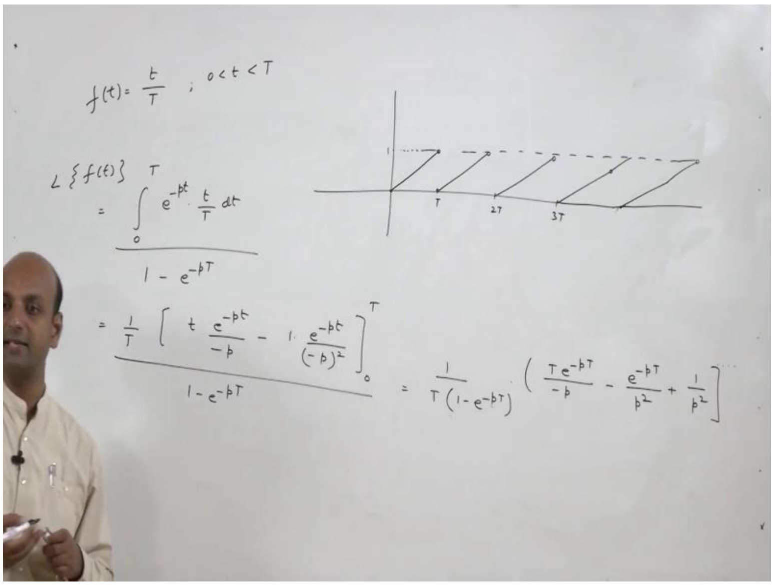

So, it is T it is 2 t, it is 3 and 3 t and. So, on from 0 to t it is t upon T, it is something like a straight line passing through origin. So, it is t to t, when it is T it is 1. Now this value is 1, this basically, and when it is 0 it is 0. So, we have a hollow here we have a hollow here. Now from t to t t, the same will repeat from t to t 2 t itself, because it is a period T as given the problem. So, the same will repeat here also, the same will repeat here also, the same will repeat here also. So, likewise we can draw this function.

So, this is this function we call at, all heights are same, all heights are same. So, this is saw toothed wave. Now how to find Laplace of this function? again we can use the same expression the same formula, Laplace of f t will be nothing, but it is 0 to t T T is T here, e k power minus p t f t, f t is p upon T into d t and 1 upon 1 minus e k power minus p t t is t is here. So, it is nothing, but this T is constant it will come out. And to integrate this we will use by parts, integration by parts. So, what is the expression for this t into e k power minus p t upon minus p minus 1 into e k power minus p t upon minus p square? It is from 0 to T and whole divided by 1 minus e k power minus p t.

So, you can substitute a small t as T, it is 1 upon t into 1 minus e k power minus p t. when you take small t as T it is nothing, but t into e k power minus p t upon minus p and when you take t as 0, so it is 0. Now here where you take t as t, so it is negative is already outside, it is e k power minus p t upon p square and minus minus plus 1 upon p square.

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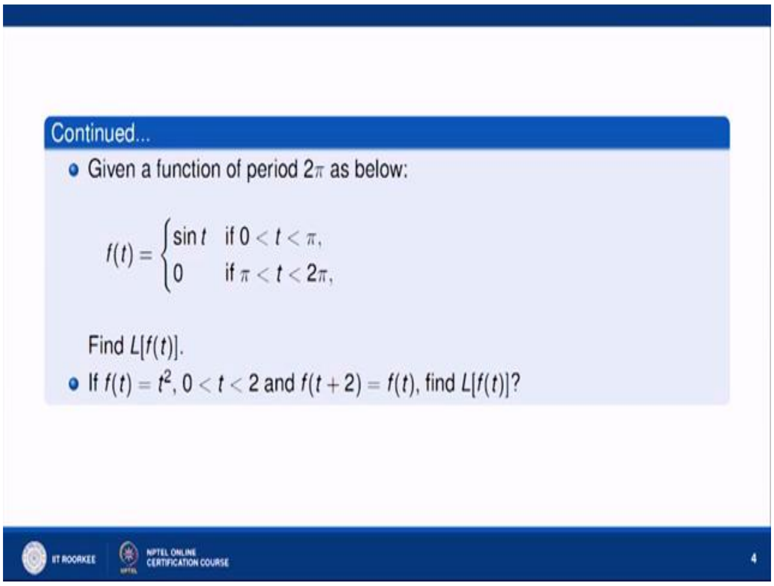

So, you can take l c m and simplify this. So, which will be the Laplace transform of this f t? Similarly we have this problem, period is 2 pi f t it is defined like this, what is f t here. So, here f t from 0 to pi is sin t, and pi to 2 pi is 0.

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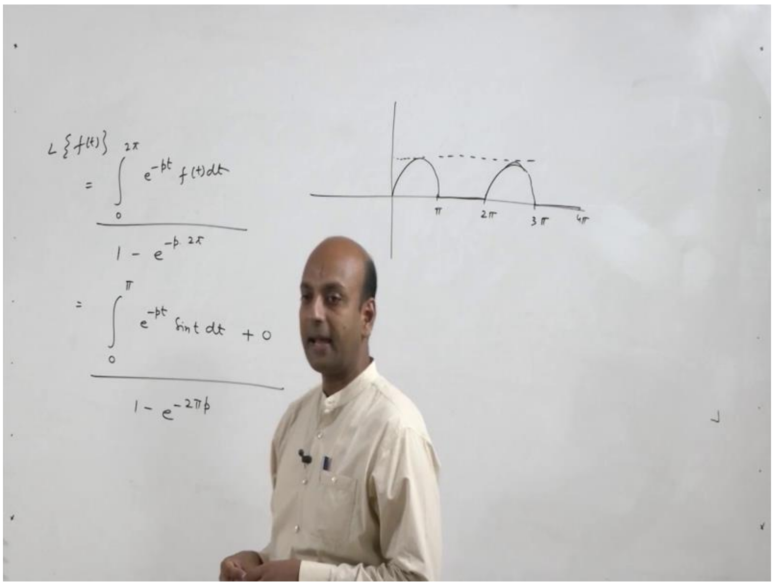

So, it is pi it is 2 pi. So, from 0 to pi it is sin t, it is something like this and from pi to 2 pi it is 0 to 0. Again it has a period 2 pi, same will repeat after 2 pi also. So, from 2 pi to 3 pi it is sin t, having the same heights, and from 3 pi to 4 pi it is 0. Again from 4 pi to 3 5 pi it is sin t, and from 5 pi to 6 pi it is 0.

So, how to find Laplace of this function? Again we will use a same expression Laplace of f t will be nothing, but it is 0 to T, T is 2 pi here e k power minus p t into f t d t upon 1 minus e k power minus p into 2 pi. you can break it from 0 to pi e k power minus p t 0 to pi it is sin t d t, and pi to 2 pi it is 0 so plus 0, upon 1 minus e k power minus 2 pi p.

Now, one can easily solve this expression by applying integration by parts, and we can get the final answer. now the next problem if f t equal to t square, when t varying from 0 to t, and f t plus 2 is 2; that means, period is 2. So, what is the shape of this function? First we can ritualize what is the shape, and then we can try to find out the Laplace of this function. So, what is the shape of this function? This is from 0 to 2 0 to 4 then 6 and so on.

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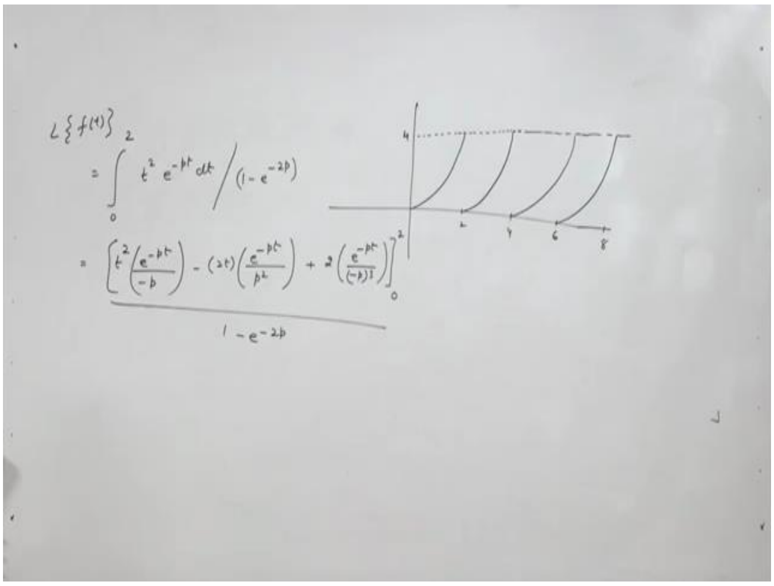

So, from 0 to 2 it is t square; that is parabola, and when it is 2 it is 4. So, it is something like this expression, and this is 4. When t is 2 f t is 4. Now from 2 to 4 again it is parabola. So, it is something like this expression, height remains same and 4 6 it is something like this expression. So, it is the same expression. Again from 6 to 8 it is something like expression.

So, that, this will be the rough shape of this function f t. Now how to find Laplace of this, again using the same expression? So, Laplace of this f t will be nothing, but 0 to t, here t is 2 period is 2 and it is t square e k power minus p t d t and whole divided by 1 minus e k power minus 2 p, because T is 2.

So, this you can integrate by applying integration by parts. So, how to find integration is simple t square e k power 2 p p t upon minus p then negative and 2 t e k minus p t upon p square, then plus it is 2 e k power minus p t upon minus p cube. Here I have applied the concept of partial integration of parts directly, you can find it by applying integration by parts also, and then the whole expression divided by 1 minus e k power minus 2 p. So, this you can simplify, and you can find out the Laplace transform of this function f t by substituting t equal to t and t equal to 0.

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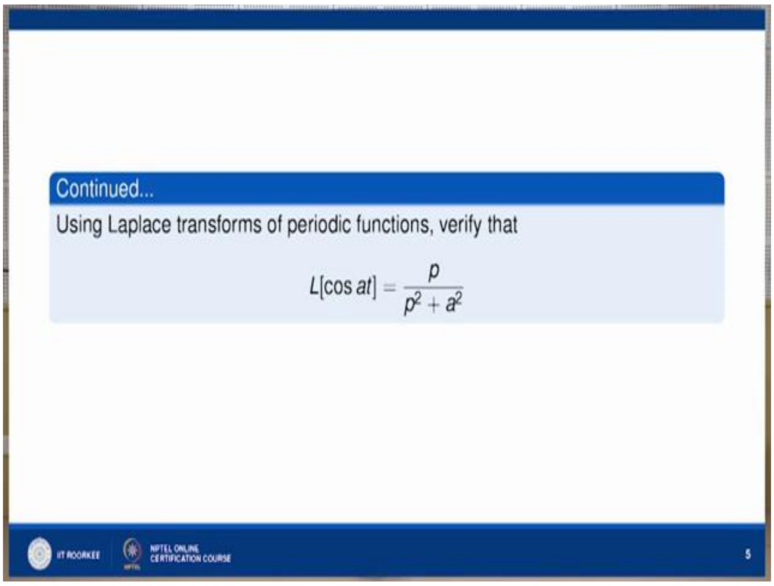

So, if you want to verify, say Laplace transform cos at which we have already find out, which is p upon p square plus a square that also we can verify by using periodicity also. So, period of cos a t is what, is 2 pi upon a. So, using periodicity also, we can find out Laplace transform of cos a t. So, what will be the expression for this? If you use the periodicity of cos a t, f t is f t is cos a t. So, it is period is 2 pi upon a, a should not equal to; of course, a should not equal to 0, 2 pi upon a.

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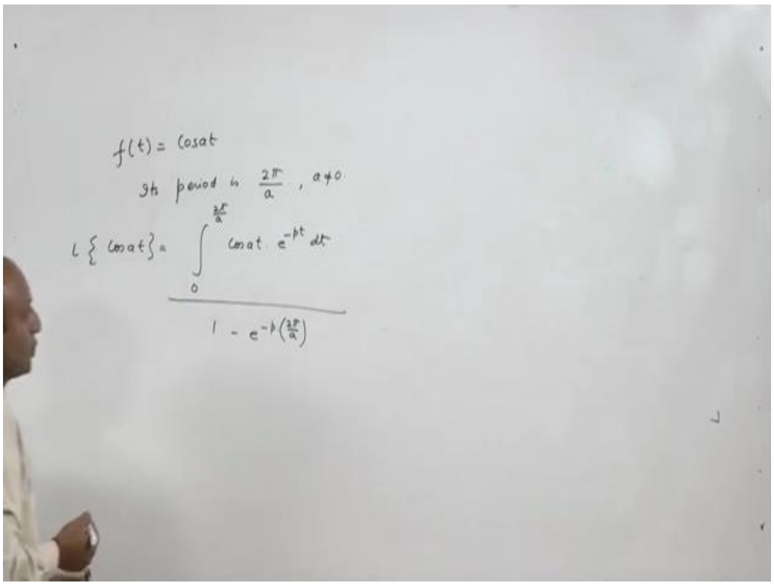

So, how to find Laplace of cos a t now? Using property of Laplace transform of periodic functions, so it is nothing, but 0 to T, T is 2 pi upon a and it is cos a t f t into e k power minus p t d t and whole divided by 1 minus e k power minus p into t t is 2 pi by a. So, you can easily verify, you can apply integration by parts over here, and when you simplify. So, you will derive you will definitely get, you will get p upon p square plus a square which is the Laplace transform of cos a t. Now the last problem, suppose we have to find out Laplace transform of this triangular wave function.

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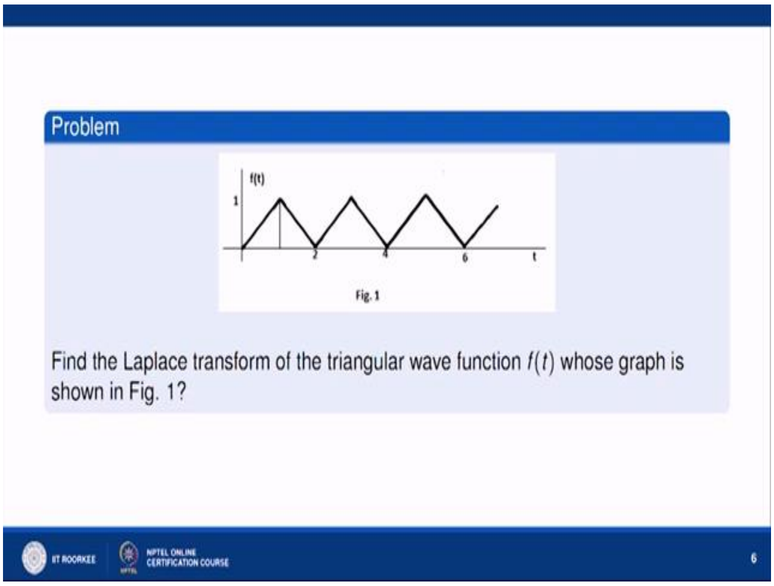

So, first we have to construct the function. So, one can easily see that the period of this function is 2, because after every t equal to 2 function repeats, and what is the nature of the function at t equal to, from 0 to t. So, that we have to construct that one can easily see, that from 0 to t. So, f t is nothing, but. When t is varying from 0 to 1, it is straight line passing through origin and from 0.1 comma 1.

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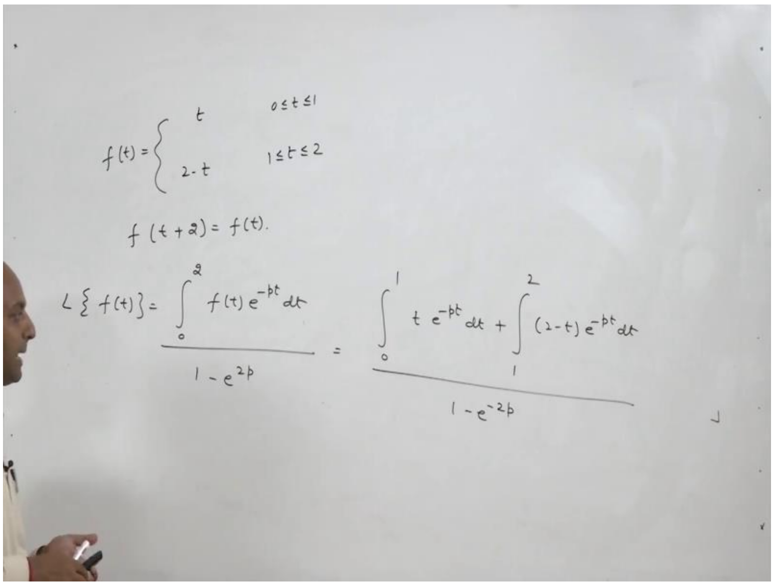

So, it is remain 1, when t is varying from 0 to 1. When t is 0 it is 0 when t is 1 it is 1, and from 1 to 2 it is a decreasing straight line. So, it is something like 2 minus t when t varying from 1 to 2, because when t is 1. So, f t is 1 which is a tip point of this figure, and when t is 2 it is 0. So, that will be the function, of this function, and with f t plus 2 is nothing, but f t. So, this is this function.

Now, we can easily find out Laplace transform of this function using the expression for Laplace transforms of periodic functions. So, what will Laplace transform of this f t. This is nothing, but 0 to T, f t e k power minus p t d t upon 1 minus e k power minus 2 p which is equal to. Now that to break it 0 to 1, 0 to 1 f t is nothing, but t, it is t e k power minus p t d t plus, from 1 to 2 1 to 2 it is nothing, but 2 minus t. So, 2 minus t e k power minus p t d t and whole divided by 1 minus e k power minus 2 p. So, you can apply by parts here and here also, and simplify to get the Laplace transform of this triangular wave function.

Thank you very much.