Mathematical methods and its applications
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Lecture – 32
Initial and Final value Theorems for Laplace Transforms

So, welcome to a series of lectures on mathematical methods and its applications. We were discussing Laplace transform and their properties. So, in this lecture, we will see that what are the initial and final value theorems on Laplace transforms, and how they are important to find some Laplace transforms of some functions. Now, what is initial value theorem let us see.

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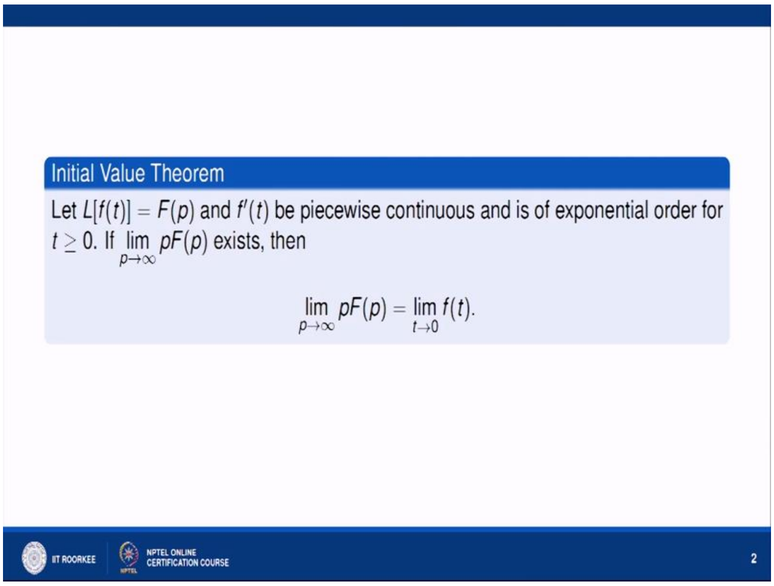

Let Laplace transform of f t is suppose F p and f dash t be piecewise continuous and is of exponential order for t nonnegative, t greater than equal to 0.

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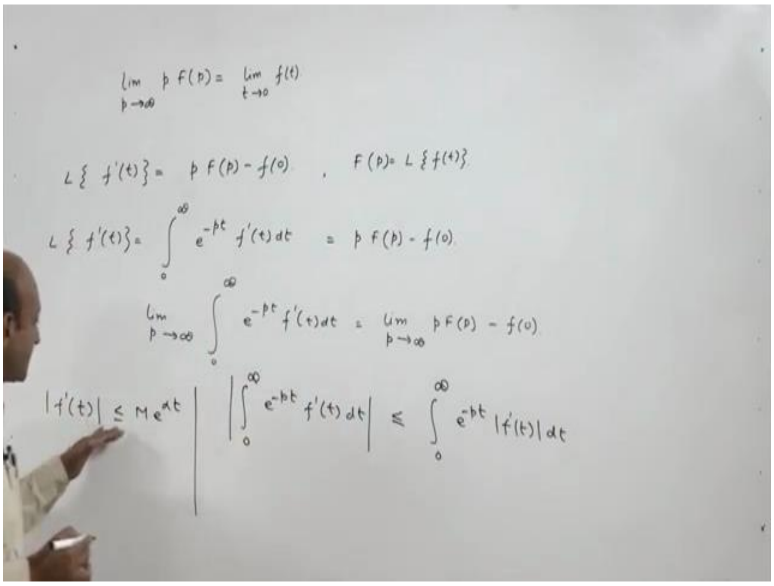

If limit p tends to infinity p F p exists then this theorem is states that limit p tend to infinity p F p is nothing but limit t tend into 0 f t. So, this is initial value theorem. Now, what is the proof of this theorem let us see. So, what is Laplace of f dash t? It is nothing but we already know that it is nothing but p F p minus f 0, Laplace f dash t is nothing but p F p minus f 0, where this F p is nothing but Laplace of f t. So, this F p is nothing but Laplace of f t. Now, let us compute Laplace of f dash t by the definition, and take limit p tend into infinity because in initial value theorem, we have to take limit p tend into infinity.

So, what is Laplace of f dash t by the main definition of Laplace, it is nothing but 0 to infinity e k power minus p t f dash t into dt. And this is which is equal to by this; it is equals to p F p minus f 0. So, take limit p tend to infinity both the sides. So, it is limit p tend to infinity of 0 to infinity e k power minus p t f dash t dt will be equal to limit p tend into infinity p F p minus f 0. Now, let us compute this part, let us compute this part. Now, f dash t is of exponential order and is piecewise continuous. So, mod of f dash t will be nothing but less than equals to some M into e k power M alpha t, there will exists some M and alpha such that mod of f dash t will be less than equal to M into e k power alpha t, since it is exponential order alpha.

So, we can easily write that this value the left hand side, the left hand side 0 to infinity e k power minus p t f dash t dt and its modulus will be less than or equals to 0 to infinity e k power minus p t mod f dash t into dt. And this mod f dash t less than equal to this, because it is of exponential order of alpha.

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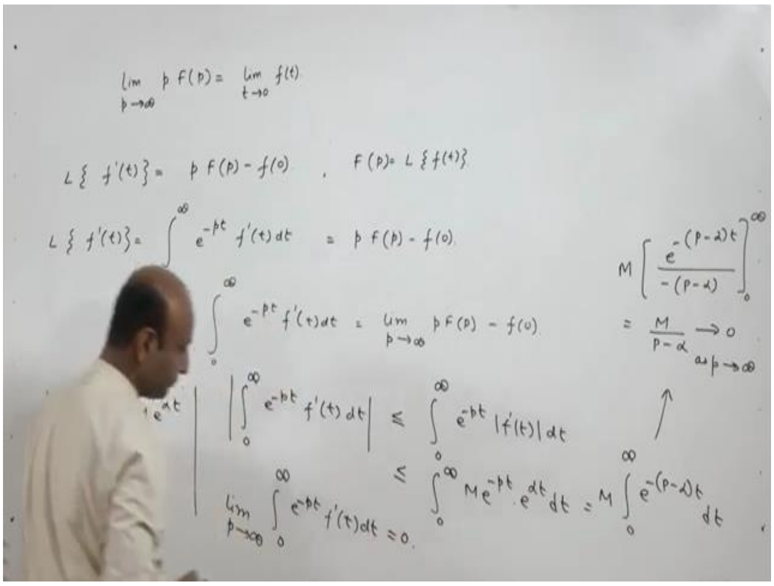

So, it is again less than equals to 0 to infinity M into e k power minus p t into e k power alpha t dt. And this quantity is nothing but M into integral 0 to infinity e k power minus p minus alpha t into dt. So, this integral is nothing but it is M into e k power minus p minus alpha times t upon minus p minus alpha times t 0 to infinity.

So, when p is greater than alpha, so this value is nothing but one M upon p minus alpha, because when p greater than alpha and t tend to infinity, this will tends to 0; and at 0 this will tends to one. So, this will nothing but M upon p minus alpha. So, basically this expression is less than equal to this expression and when p tending to infinity, so this will definitely tend to 0. So, hence this tends to 0, because this expression is less than this expression less than equal to this expression M upon p minus alpha. And as p tending to infinity, this expression will be less than equals to limit p tend to infinity this expression; and as p tend to infinity this is tend in to 0 which is tend in to 0 as p tend in to infinity. So, hence we can say that limit p tending to infinity 0 to infinity e k power minus p t f dash t dt is equal to 0.

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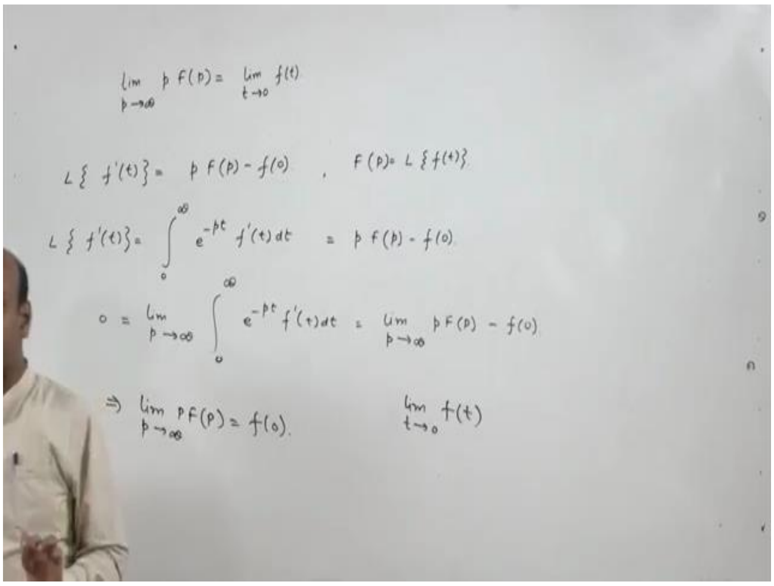

Now, using this fact, in this expression, what you will obtain. So, if you use this fact over here. So, this is nothing but 0. We have just proved that this expression is nothing but 0. So, from here we will obtain that limit t tend into infinity p F p is nothing but f 0. Now, when we proved this property of Laplace transform, Laplace transform of derivatives that Laplace transform of f dash t is nothing but this expression. So, there we are assume that function is continuous for all t greater than equal to 0. So, if function is not continuous that is why we are obtaining f 0 over here because we are assuming f is continuous throughout nonnegative side of real axis, so that is why we obtain f 0 over here.

If function is not continuous t equal to 0, then instead of obtaining f 0, we will obtain limit t tending to 0 f 0 over here f t over here; instead of this expression over here, we will obtain this expression if f is not continuous at t equal to 0. So, basically things are same, if function is continuous at t equal to 0. So, this is nothing but f 0.

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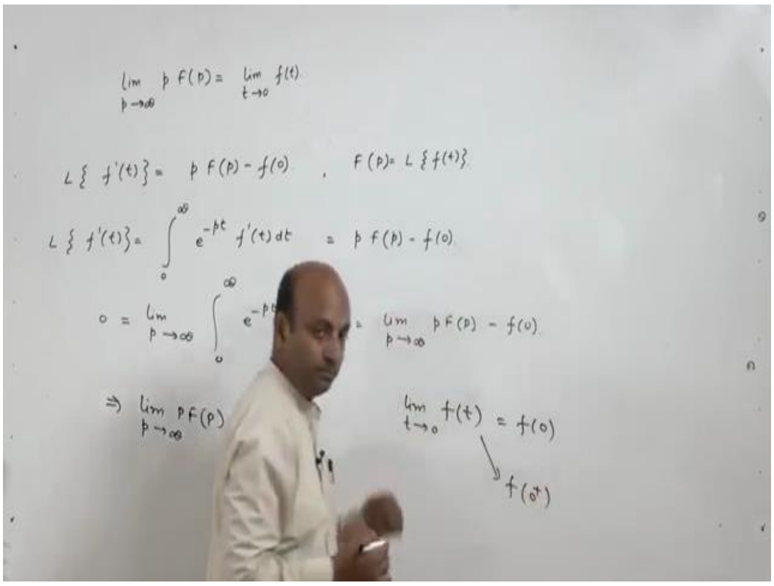

However, a function is not continuous at t equal to 0, so this is nothing but this expression nothing but f 0 plus on the right hand side of t equal to 0, it is a right hand limit of t equal to 0 that is f 0 plus. So, therefore, to generalize this theorem, we are writing on the right hand side, we are writing limit t tending to 0 f t because if function is continuous, so right hand side will definitely be f 0. If function is not continuous at t equal to 0, so right hand side is nothing but f 0 plus. So, in both the case it will work, so that is why we are writing limit t tend into 0 f t in the right hand side instead of writing f 0. So, this is initial value theorem.

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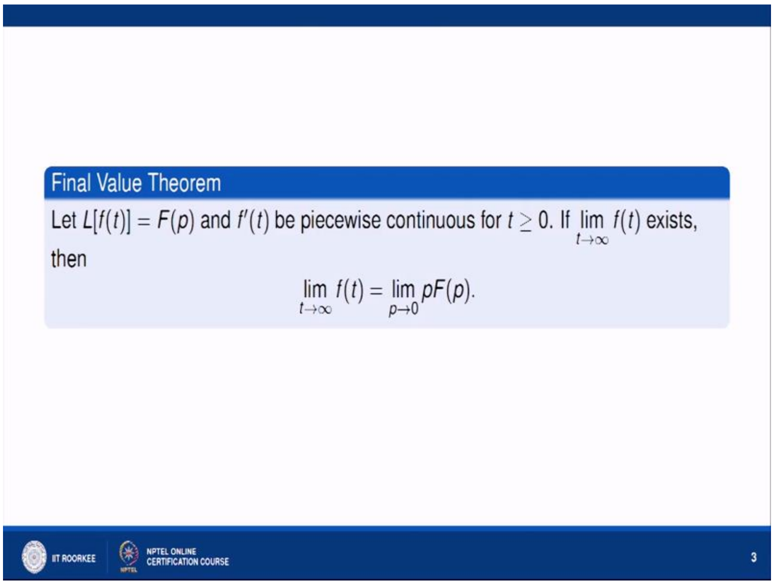

Now, we will come to final value theorem. Now, what final value theorem state let us see if Laplace transform of f t is F p suppose and f dash t is piecewise continuous for t greater than equal to 0, and if limit tend in to infinity f t exists then this result hold.

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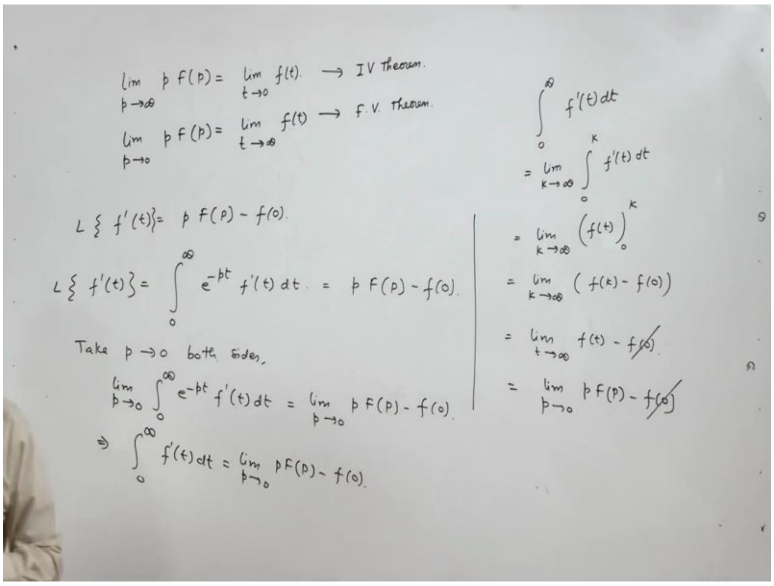

Now, what this result is. So, this was initial value theorem. And what is final value theorem, it is limit p tend into to 0 p F p is nothing but limit t tend in to infinity f t. So, this is final value theorem. So, let us now prove the final value theorem, how we will obtain final value theorem let us see. Now, again we will go to the same expression, what is that limit Laplace transform of f dash t is nothing but p F p minus f 0. So, again this f 0 will be limit t tending to 0 f t if function is not continuous at t equal to 0. And this Laplace of f dash t is nothing but from the definition of Laplace transforms 0 to infinity e k power minus p t f dash t into dt.

Now, we want limit p tend into 0, this side. So, same here, same on this side also. So, this is equal to p F p minus f 0. Here, we are assuming f is continuous at t equal to 0, take limit p tend in to 0 both the sides. So, what will obtain? Limit p tend in to 0 0 to infinity e k power minus p t f dash t dt; and the right hand side will remain the same limit p tend in to 0 p F p minus f 0. So, now, if p tend into 0, this will tend to 1. So, this side is nothing but this implies 0 to infinity f dash t dt which is equals to limit p tend in to 0 p F p minus f 0.

Now, let us see the left hand side what is left hand side left hand side is 0 to infinity f dash t dt this is right left hand side. So, what this expression is basically this is something limit k tend into infinity 0 to k something f dash t dt. Now, since derivative integral is anti-derivative we already know this. So, this is nothing but limit k tending to infinity f t from 0 to k. So, this is nothing but limit k tend into infinity f k minus f 0, and which is nothing but we can simply write limit t tend into infinity f t minus f 0. This we can simply write like this. So, this is left hand side. So, left hand side is left hand side is equal to this expression.

So, when we equate both the sides. So, this will be equal to from this side is nothing but limit p tend into 0 p F p minus f 0. So, f 0, f 0 cancels out both the sides. So, this we obtain this result, which is final value theorem. If function is continuous at t equal to 0, we are having f 0 over here; if it is not continuous at t equal to 0, we will be having f 0 plus, here also, here also. So, in both the cases whether function is continuous or not equal to 0, f 0 or f 0 plus will cancel from both the sides. So, finally, we will be having this expression which we are calling as final value theorem.

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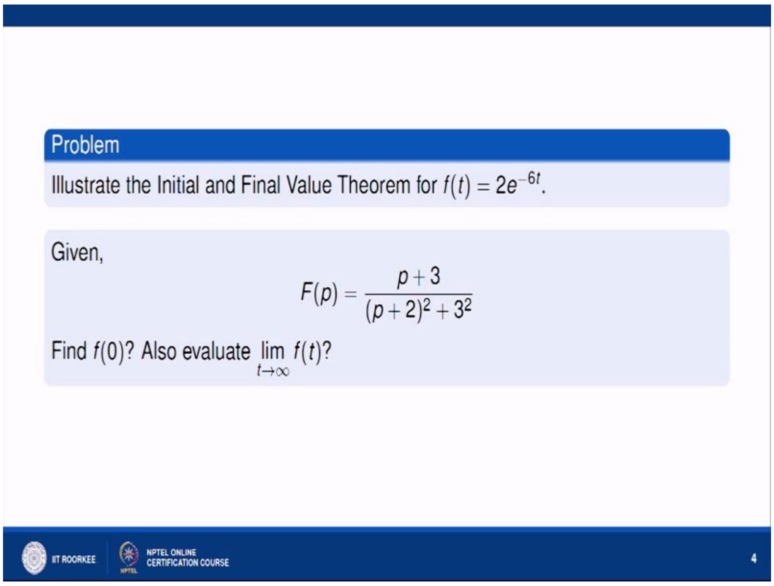

Now, let us see some of the problem based on initial final value theorem. First of all, let us see the simple example of f t equal to 2 e k power minus 6 t. Let us illustrate this using these two theorems. So, one can easily illustrate this very easily. You see that first is suppose initial value theorem, suppose first is initial value theorem.

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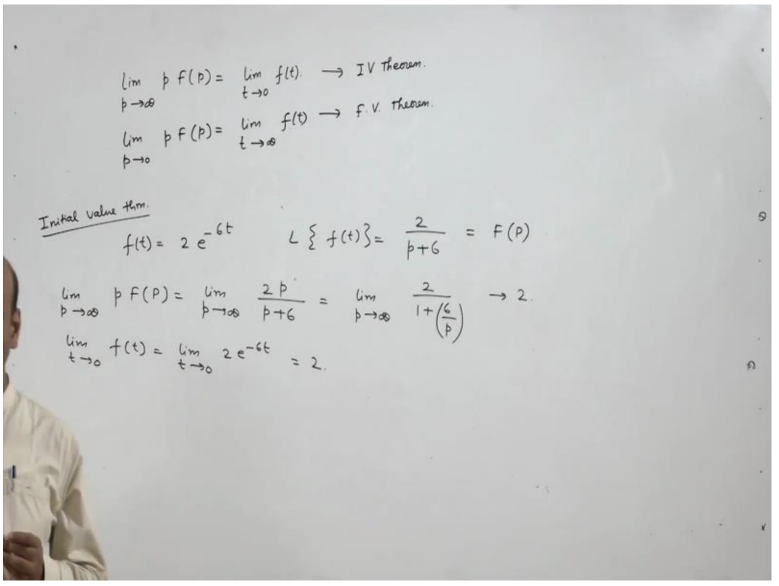

Suppose you want to illustrate this theorem using this example for this example sorry. So, what is f t? f t is 2 e k power minus 6 t. So, what is Laplace of f t, very simple, it is 2 upon p plus 6. So, this we are calling as F p. Now, see initial value theorem, the left hand side of initial value theorem is limit p tending to infinity p F p. So, what is limit tend into infinity p F p, p into F p that is 2 p upon p plus 6. And as p tends to infinity, this will tends to you divide numerator denominator by p, this is this quantity, and this will tends to 2. Now, the left hand side, limit t tending to 0 f t, so limit t tend into 0. What is f t, to e k power minus 6 t? So, it is nothing but 2. So, hence initial value theorem has been verified, because left hand side is equal to right hand side.

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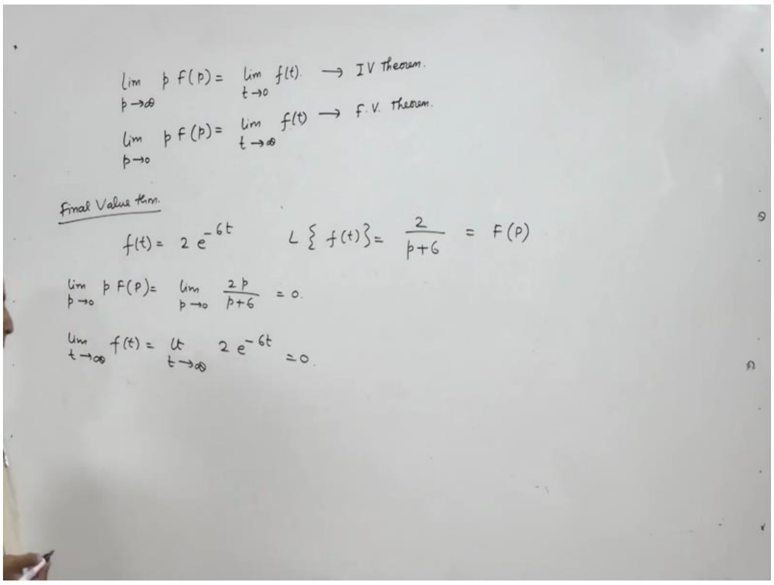

Now, come to final value theorem for the same problem. So, now the final value theorem. So, what is left hand side for final value theorem limit p tend into 0 p F p. So, it is limit p tend into 0 p F p is 2 p upon p plus 6. And it will equal to 0, clearly equal to 0. Now, what is limit t tend into infinity f t, it is limit t tend into infinity f t is 2 e k power minus 6 t which is obviously 0, because at t tends to infinity e k power minus t tends to 0, so that will tends to 0. So, hence since left hand side equal to right hand side, hence the final value theorem has been verified. So, the first problem has been solved. Now, come to second problem given at F p is this find f 0. So, we already know F p which is Laplace transform of some f t. Now, without finding Laplace inverse we can easily find what is f 0, how using initial value theorem. So, here we are assuming that function is continuous at t equal to 0, then only we can obtain f 0.

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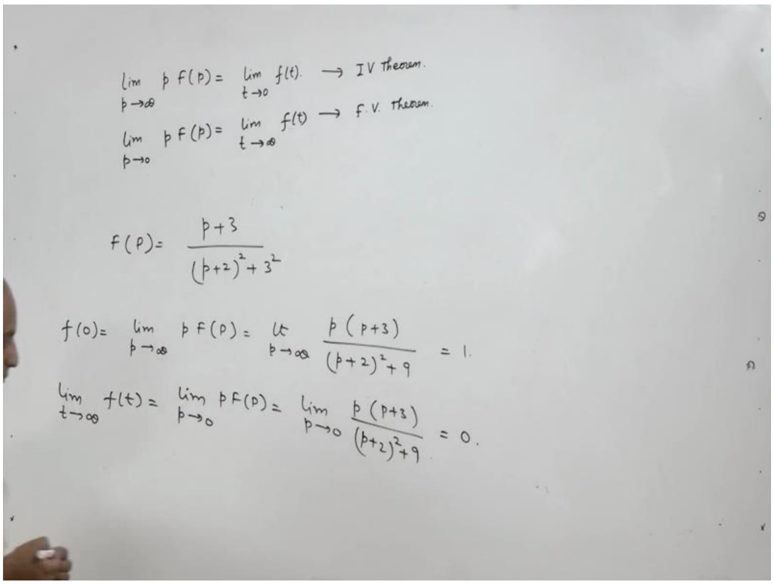

So, what is F p for this problem, what is F p given to us it is p plus 3 upon p plus 2 whole square plus 3 square, this is F p. Now, what is f 0, f 0 is nothing but by the initial value theorem is f is continuous at t equal to 0, f 0 is nothing but limit p tending to infinity p F p. So, it is nothing but limit p tending to infinity p into F p. So, this you can easily see that this limit will tends to 1. So, it is equal to 1. So, what is a value of f 0, this one. So, without finding Laplace inverse of this function, we can easily find out, what is the value of f 0 using initial value theorem.

Similarly, if you want to find out limit t tend to infinity f t, so that also we can find out using final value theorem. So, limit t tend in to infinity f t will be nothing but using final value theorem, it is equal to this expression limit p tend in to 0 p F p. And it is nothing but limit p tend into 0 p into F p is p plus 3 upon p plus 2 whole square plus 9, and it is nothing but is equal to 0. So, limit t tend to infinity f t is nothing but 0, so that also we can find out using initial and final value theorems.

Now, let us solve some problems, which we are already discussed in the previous lectures that how to find Laplace transform of these functions, but let us find these function using initial value theorem, let us solve first problem. So, I am telling these problem. So, here because these theorems will play an important role while we solve some deferential equations, so that is why we must have information about what initial final value theorems are, and how they are important and how we can solve problem based on this.

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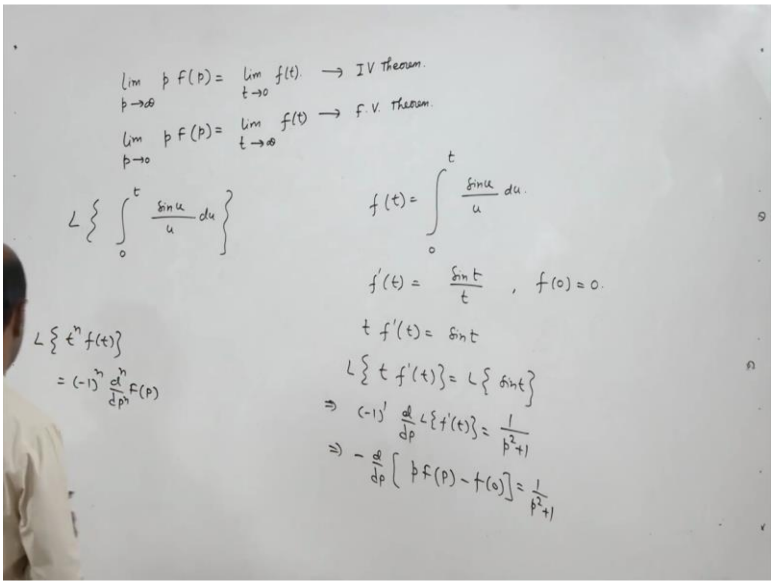

So, what how to find Laplace transform of this function 0 to t it is sin u upon u du. So, we already see that how to find Laplace of this, we first find Laplace of sin u sin t which is 1 upon p square plus 1, if sin t upon t is giver by integral p to infinity F p dp. And then for 0 to t it is nothing but F p by p, which is F p is Laplace of this entire function, so that we have already discussed in the last lectures. Now, how to find Laplace of these functions using initial value theorem. So, let us see. Now, let us call this whole function as f t, so f t is equals to 0 to t sin u upon u du. So, what is f dash u derivative of f respect to t that we can find out using Leibniz theorem. So, what will be the derivative of this, it will be sin t by t into 1 minus 0, so that is quite obvious that this is will be this. And of course, one can easily see that f 0 is nothing but 0. Because when you take t tend into 0 or t equal to 0, so this lower and upper limit both are same value will be 0.

Now, here t f dash t is nothing but sin t t f dash t is nothing but sin t. Take Laplace both the sides. So, Laplace of t f dash t is nothing but Laplace of sin t, where interested to find out Laplace of f t, we have to find out Laplace of f t using initial value theorem. Now, Laplace of t into some function is nothing but we already know that Laplace of t k power n in to some function say f t it is nothing but minus 1 k power n d by d p of F p where F p is nothing but Laplace transform n th derivative sorry nth derivative of F p where F p is nothing but Laplace of this f t. Now, here in place of f t we have f dash t.

So, what will be this expression will be minus 1 k power 1, n is 1. So, it is d by d p of Laplace of f dash t. And what is Laplace of sin t it is 1 upon p square plus 1. Now, this is further equal to minus of d by dp of Laplace of f dash t is Laplace of f dash t is p F p minus f 0 is equals to 1 upon p square plus 1. So, this expression further can be written as

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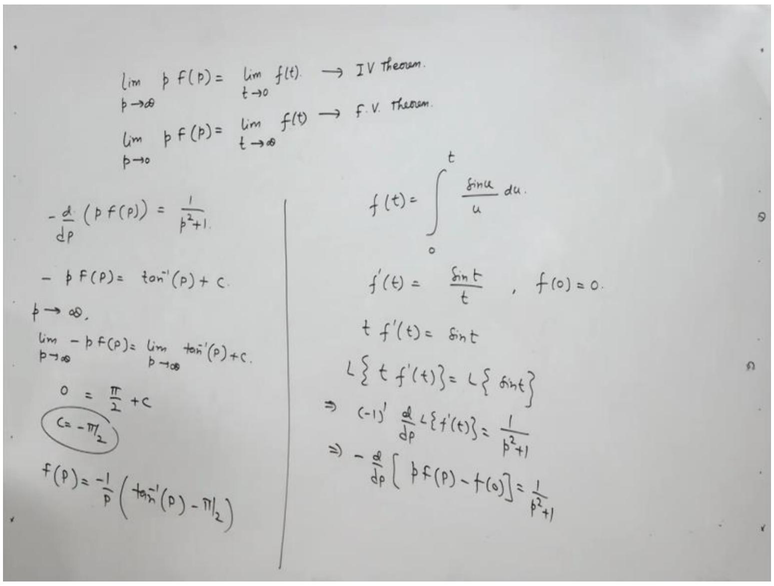

minus d by d p of p F p is equal to 1 upon p square plus 1 derivative of this with respect to the p will be 0, it is a constant quantity to derivative will be 0. Now, it is nothing but minus p F p is equals to take integration both the sides respect to p. So, it is tans inverse p plus some arbitrary constant c because it is an indefinite integral.

Now, how to find c, so to find c take limit p tending to infinity both the sides limit p tend in to infinity n both the sides. So, it is limit p tend to infinity minus p F p it is limit p tend into infinity tan inverse p plus c. Now, this quantity is by the initial value theorem is nothing but limit t tend to 0 f t. And limit t tend to 0 f t which is f 0 here is nothing but 0 which is nothing but 0. So, we can say that this is 0, because this quantity this expression p F p limit t tend into infinity is nothing but t tend to 0 f t, and limit t is tend to 0 f t is f 0 which is 0. So, this is 0, and this is pi by 2 plus c. So, c is nothing but minus pi by 2. So, what will be F p from here? So, F p will be nothing but minus 1 by p times tan inverse p minus pi by 2. So, what is F p is Laplace transform of f t? So, hence Laplace transform of this f t is nothing but this. So, basically initial, final value theorem have important to find out this arbitrary constant c; otherwise it is difficult to find. Hence, this Laplace transform of this f t is nothing but this expression.

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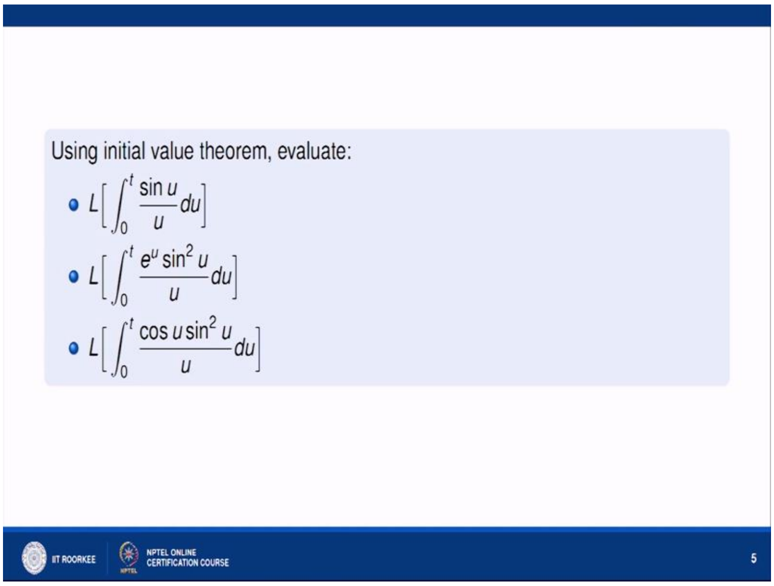

Now, let us compute the second problem.

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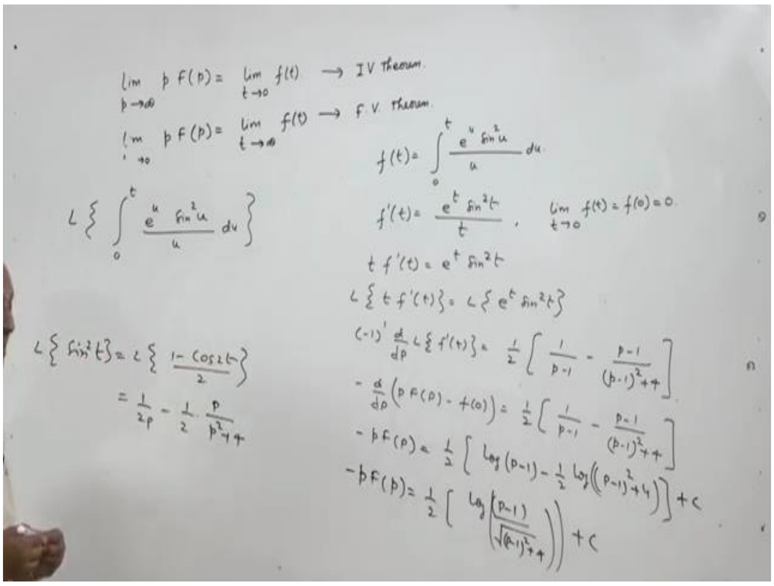

This problem can be solved also using initial value theorem e k power it is u sin square u upon cos upon u sorry upon u into du. So, again you will take this function as say f t and try to find out it Laplace transform using initial value theorem on the same lines we did earlier. So, take this is f t, what is f t, f t is 0 to t e k power u into sin square u upon u into du. So, what is f dash t is e k power t sin square t by t by Leibniz theorem. And of course, limit t tend to 0 f t or we are calling as f 0 is nothing but 0.

So, take t in the left hand side. So, it is this expression take Laplace transform both the sides. Laplace transform of left hand side is minus 1 k power 1 d by d p of Laplace transform of f dash t. Now, to find Laplace transform of this expression, first in Laplace transform of sin square t and then apply first shifting property. So, what is Laplace transform of sin square t? Laplace transform of sin square t, this we can finding out Laplace transform of 1 minus cos 2 t by 2, and it is nothing but 1 by 2 p minus 1 by 2 in to p upon p square plus 4. So, it is nothing but and again apply shifting property. So, you replace p by p minus 1. So, it is 1 upon two times 1 upon p minus 1 minus p minus 1 upon p minus 1 whole square plus 4 you replace p by p minus 1.

Now, minus of d by dp of it is p F p minus f 0 which is 1 by 2 of 1 upon p minus 1 minus p minus 1 upon p minus 1 whole square plus 4, derivative of this respect to p 0. So, it is minus p F p, you integrate both sides respect to p and derivative of this is 0. So, this is 1 by 2 comes out, this is log of p minus 1 and multiply and divide by 2. So, it is minus 1 by 2 log of p minus 1 whole square plus 4. So, from here F p is nothing but and plus c also, because it is an indefinite integral. So, F p is nothing but minus 1 by 2 p, take p over here only and 1 by 2, it is log p minus 1 upon under root p minus 1 whole square plus 4 and plus c.

Now, you can use initial value theorem to find out the value of c take limit p tending to infinity both the sides as p tend to infinity this side will tends to this side will be equal to limit t tending to 0 f t which is nothing but 0. So, this side will be 0 as p tend to infinity this will tends to one and log 1 is 0, so c is nothing but 0. So, c will be 0. And F p can be find out, the F p will be nothing but you divide this by minus p. So, F p is nothing but Laplace transform of this f t. So, hence we can find out Laplace transform of this function. So, similarly the third problem can also be solved using the same concept and initial value theorem.

Thank you very much.