Mathematical methods and its applications
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Lecture – 31
Convolution theorem for Laplace transforms – II

So, welcome to my lecture series on mathematical methods and it is applications. So, we were in the last lecture we were discussing about convolution theorem, I mean convolution function, what do you mean by convolution of two functions.

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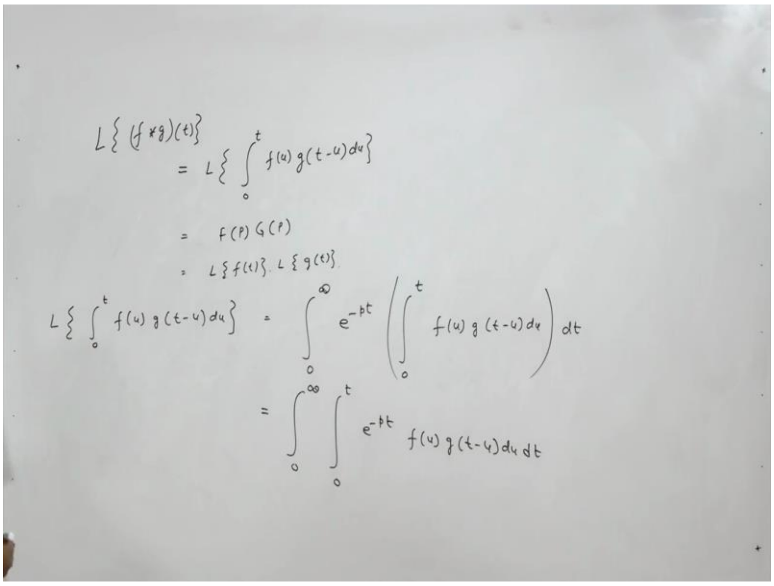

So, I told you that convolution of f and g is nothing but it is 0 to t f u g t minus u du. So, this is how we can define convolution of two functions. So, we have already defined some properties of convolution of two functions; that is it is satisfy commutative property, it is satisfy associative property and distributive property. Also we have seen that convolution of 0 and f is nothing but 0, but convolution of 1 with f may not be 1, may not be f, sorry convolution of 1 with any function f may not be f that we have already studied. Now we will come to the main topic that is the convolution theorem for Laplace transforms.

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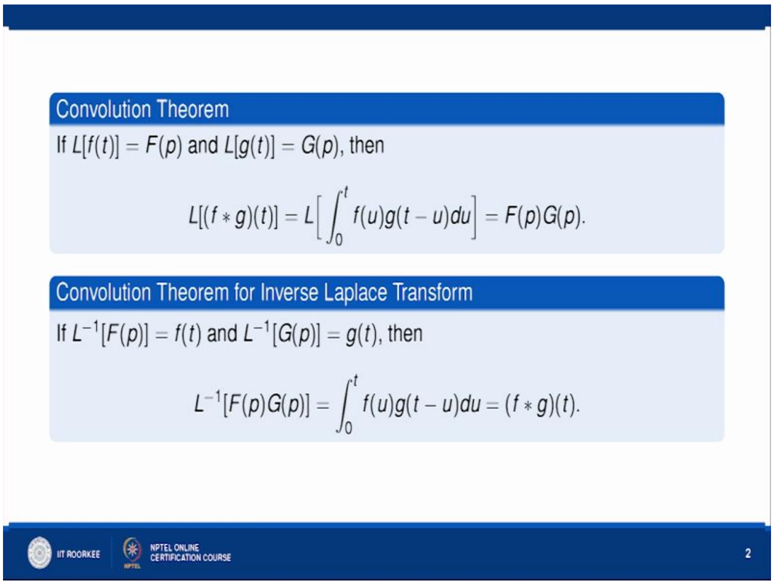

Now, what this theorem states, it is states that if Laplace transform of f t is, say F p and Laplace transform of g t is say G p, then Laplace transform of convolution f and g which is nothing but Laplace transform of integral 0 to t f t g t minus u du, is nothing but F p into g p; that is what convolution theorem is states. Now what is the proof of this theorem? Let us see what is a proof of convolution theorem.

So, convolution theorem basically states that Laplace of f star g, which is a convolution of two functions f and g is nothing but is equal to Laplace of. So, this is the convolution, definition of convolution, it is 0 to t this is f u g t minus u du is equal to F p into G p. So, this F p is nothing but Laplace transform of f t, and this G p is nothing but Laplace transform of g t. So, now, let us see the proof of this theorem, convolution theorem. So, what is Laplace of 0 to t f u g t minus u, d u?

So, this is nothing, this expression is nothing but convolution of f and g; that is why we call it convolution theorem. So, how we define Laplace theorem; Laplace of some function f t. So, Laplace is nothing but it is 0 to infinity e k power minus p t, and instead of f t we have this entire expression. So, this is nothing but integral 0 to t f u g t minus u du and whole multiplied by dt. This integral is over t. So, this is what we define Laplace transform of this function, say capital f t. Now this can be further written as 0 to infinity 0 to t e k power minus p t f u g t minus u du dt.

So, this is how we can define this expression. Now basically we have to show that this expression is nothing but Laplace transform of f t into Laplace transform of g t, this we have to prove. So, in order to evaluate this expression, we have to simplify this expression we will use change of order of integration. Now this is the limit for u. First we have a du this, these are the limit for u, and this is dt, these are the limit for t.

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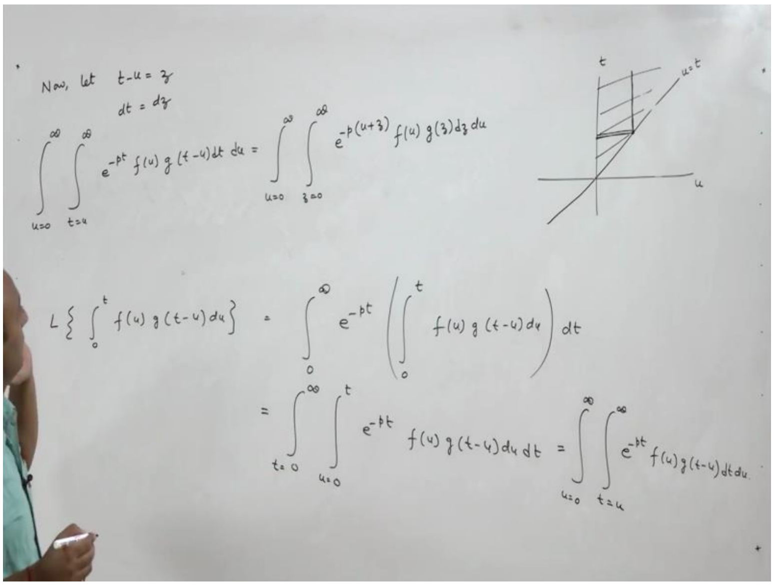

Now, let us draw the like area first what is this area represent this is u, this is t. Now u is varying from 0 to t, u equals to 0 is along this line along t axis, and u equal to t is this line, this is something equal to t. Now, t is varying from 0 to infinity; that is t is varying from 0 to infinity, means this entire expression t is varying from 0 to infinity. Now first we have du here; that means, we have to take strip along u axis. So, if you take a strip along u axis say over here, either it is here or it is here, because both expressions are in positive side. it is 0 to t; that is in the positive side of u, it is 0 to infinity; that is on the positive side of t; that means, it lies in the first quadrant area, whatever area this satisfy line in the first quadrant.

Now, in the first quadrant, either this is this part or this part. Now check whether this is this part or this part, first let us strip parallel to u axis. So, take a strip parallel to u axis. So, I am assuming that this is this part, if this is not satisfied I directly go to this part. Now if you take a strip parallel to u axis over here. So, what is u here 0, what is u here t, u is varying from 0 to t and t is varying from 0 to infinity so; that means, this is the entire region which is covered in this region, covered in this double integral.

Now, in order to change the order of integration I have to make a strip parallel to t axis. So, take a strip parallel to t axis now. So, take a strip parallel to t axis. Now suppose this is a strip parallel to t axis. So, this is the further equal to, now double integral, this double integral. now this is these are the limit for t this are limit for u. Now what is t here, t here is u, what is upper limit of t, it goes up to infinity, and u varying from 0 to infinity, u varying from 0 to infinity entire reach.

So, it will be e k power minus p t f u g t minus u. now it is dt into du. So, this is how we can change the order of integration. First we will see that a it is du, to mark that which region this double integral have, you take a strip parallel to u axis first. Now you take a strip parallel to u axis you fix, whether this limit are covered or not. If not then take the other region. If yes then mark this region and in order to change the order of integration, take a strip parallel to t axis now. So, this is equal to this. Now this expression is further equal to. Now in this we make a substitution take t minus u equal to some other variable z. So, now, let t minus u as some variable z. So, what will be dt, it is d z.

So, this expression, this expression, I am talking about, this expression, this expression which is u equal to 0 to infinity, t from u to infinity, and it is e k power minus p t, then it is f u then g t minus u then it is d dt du, will be equal to u remain as it is. Now t when t is u z is 0, and when t is infinity z also tends to infinity. So, it is infinity. It is e k power minus p, t is nothing but, t from here is u plus z u plus z into f of u remain as it is because now, we are not changing u, we are changing t. So, g of t minus u is d z I mean g z and dt is d z and it is du. and this expression is nothing but Laplace of convolution of f and g. So, this is equal to this expression, and this is equal to this expression.

Now, it is further equal to. Now if this expression is further equal to.

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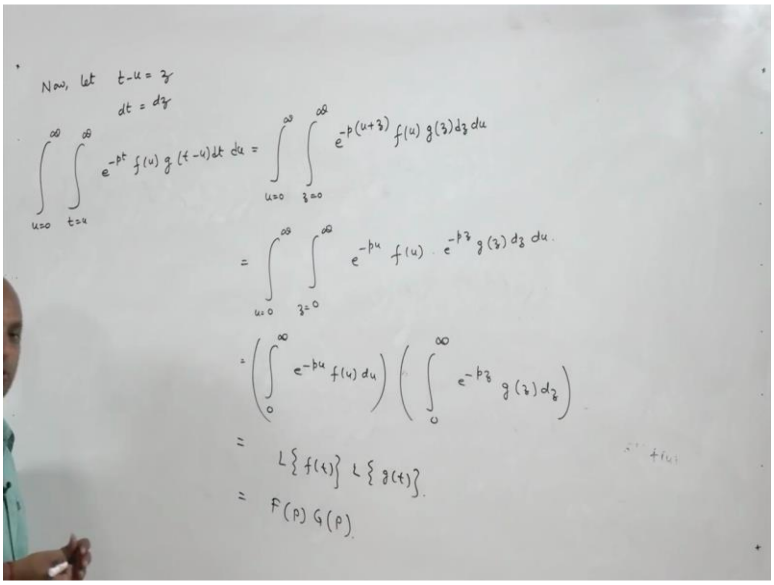

So, this is further equal to 0 to infinity, 0 to infinity again, it is these are the limit for z these are the limit for u e k power minus p u f u, and it is into e k power minus p z d z g z d z du. So, since now the variables are separated u variable and z variables are separated, and limits are constant. So, we can always write it as 0 to infinity e k power minus p u f u du into 0 to infinity, e k power minus p z g z d z, and this is nothing but Laplace transform of f t.

you say f t or you say f u both are same, because it is a change of only variable, and this is nothing but Laplace transform of g t. you call g t or g t or g z all are same. So, this is nothing but F p into G p. So, we have shown that convolution of f and g, which is equal to this expression, is equal to Laplace of f t into Laplace of g t. Hence we have proved this convolution theorem. Now similarly we can also state, we can also state inverse Laplace transforms for this theorem. Now if Laplace inverse of F p is f t, and Laplace inverse of G p is g t, then Laplace inverse of F p into G p, which is a product of 2 p, function functions of 2 p's I mean is nothing but convolution of f and g that is 0 to t f t t g t minus u du, that that is directly from the convolution theorem itself.

So, first let us again rewrite the convolution theorem here; that is Laplace transform of 0 to t f u g t minus u du is nothing but f t into G p, and what is F p here? F p is Laplace transform of f t and G p is Laplace transform of g t, this is by convolution theorem.

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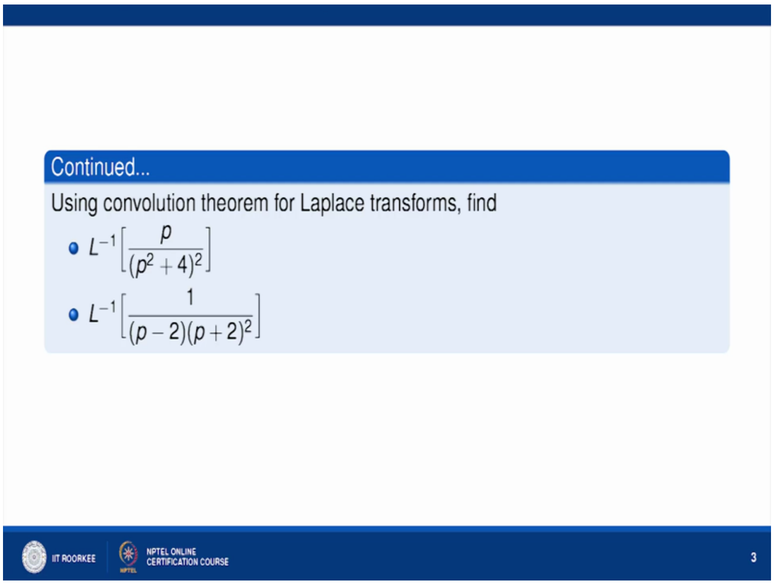

Now, let us use convolution theorem to find out Laplace inverse of some functions. So, the first problem, so this is nothing but p upon p square plus 4 whole square.

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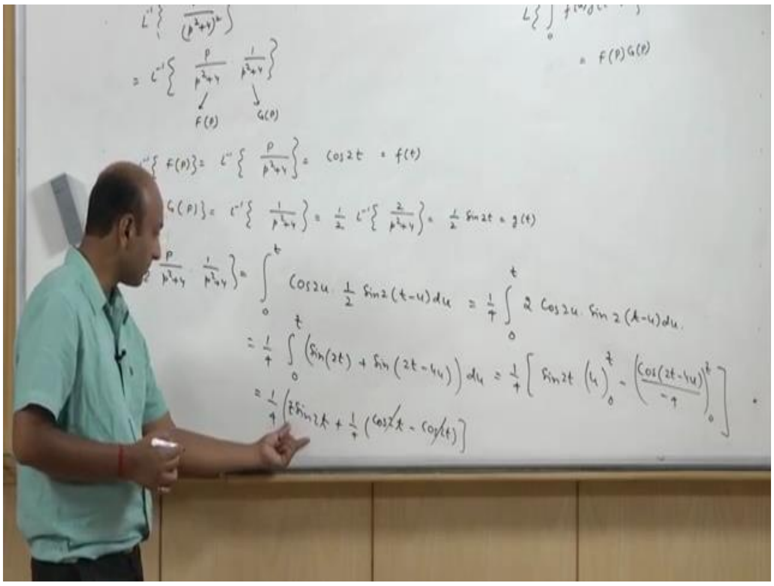

So, this can be solved, this is equals to Laplace inverse of. You can easily find p square by 4 into 1 upon p square by 4. You can easily split this function into two sub functions F p and G p. You can call anyone as F p, you can call the other function as G p. suppose this is F p, suppose I am calling this as F p, and suppose I am calling this as G p.

So, what is Laplace inverse of F p? This is nothing but Laplace inverse of p upon p square plus 4, which is directly we can say it is cos 2 t. So, suppose you are calling it f t Laplace inverse of F p, we call it f t. Now second is Laplace inverse of G p, and G p is nothing but 1 upon p square plus 4. So, you multiply and divide by 2, and this is nothing but sin 2 t. So, this is 1 by 2 sin 2 t. So, say it is g t. Now again to find out Laplace inverse part of these two, we use, if you use convolution theorem. So, this is nothing but the Laplace inverse of this part is nothing but 0 to t f t g t minus u du. So, Laplace inverse of this expression p upon p square plus 4 into 1 upon p square plus 4 is nothing but 0 to t. Now what is f t. f t is Laplace inverse of this function.

This is nothing but cos 2 t cos 2 u into. So, it is cos 2 u into 1 by 2 sin 2 t minus u du. Now in order to simplify this expression, this is nothing but 1 by 4 integral 0 to t 2 times cos 2 u into sin 2 t minus u du. So, this is 2 sin a cos v 2 sin a cos b 1 by 4 integral 0 to t. So, it is sin a plus p. So, sin a plus b is 2 t. So, you can call it as a f b and sin a plus p plus sin a minus p into du. This is nothing but 1 by 4. So, it is sin 2 t will come out. So, integral of du is u from 0 to t, and sin 2 t minus 4 u is nothing but integral is minus cos 2 t minus 4 u with respect of u this is minus 4 come in denominator, and it is 0 to t.

So, if we simplify we will get the final answer. So, final answer will be nothing but 1 by 4 sin 2 t into t and minus minus plus 1 by 4 multiplication with, when you take u as t it is cos 2 t when you take u as 0 it is it is minus cos 2 t. So, this term will cancel out. So, this will be the final answer of this problem .Now similarly the last problem we can solved using convolution theorem.

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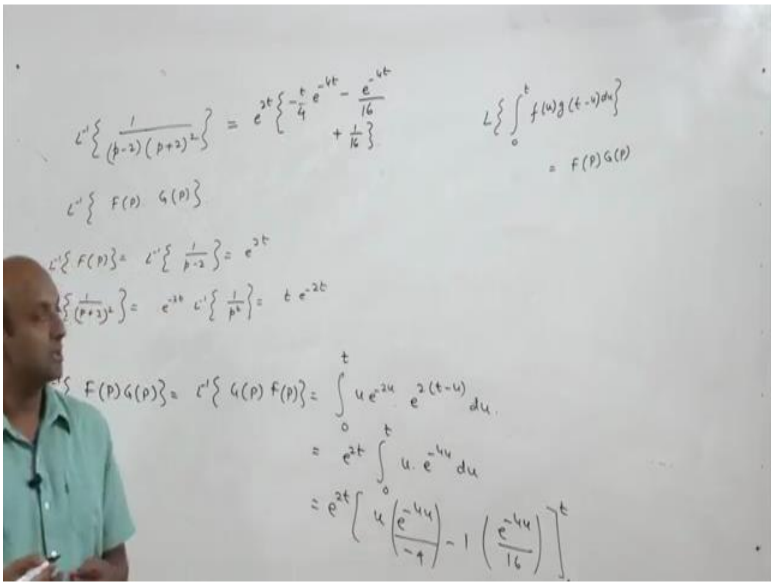

So, what is it is Laplace inverse of 1 upon p minus 2 into p plus 2 whole square. So, this is Laplace inverse of F p into G p. So, Laplace inverse of F p will be nothing but Laplace inverse of 1 upon p minus 2, which is nothing but e k power 2 t, and Laplace inverse of 1 upon p plus 2 whole square, will be nothing but e k power minus 2 t. You will apply shifting property e k minus 2 t in to Laplace inverse of 1 by p square, which is nothing but t. So, it is t e k power minus 2 t.

So, this is, you can say anyone as f t and anyone as g t, because convolution of two functions satisfy commutative property. So, Laplace inverse of F p into G p is same as Laplace inverse of G p into F p. So, this is nothing but 0 to t f u g t minus u, g t minus u du. So, this is by convolution theorem. You can use this also no problem. The only thing is I have simplified it this, because otherwise t minus u comes in both the expressions. So, this will be nothing but e k power 2 t will come out. So, it is 0 to t u into e k power minus 4 u du. And now you will apply integration by parts. So, it is nothing but e k power 2 t.

So, when you integrate by a part it is u e k power minus 4 u upon minus 4 minus, this derivative integral of second is e k power minus 4 u upon 16, and the whole expression from 0 to t. So, when you integrate from 0 to t. So, the final answer of this expression will be nothing but e k power 2 t will come out. When you take, when you substitute p u as t it is minus e k power minus 4 t, and when it is 0. So, it is minus minus plus, when u is 0 it is 0. Now when it is t it is minus of e k power minus 4 t upon 16, and when it is 0. So, minus minus plus it is 1 by 16. So, I simplify this, this will be the final answer of this problem, this is how we can find out Laplace inverse of this expression.

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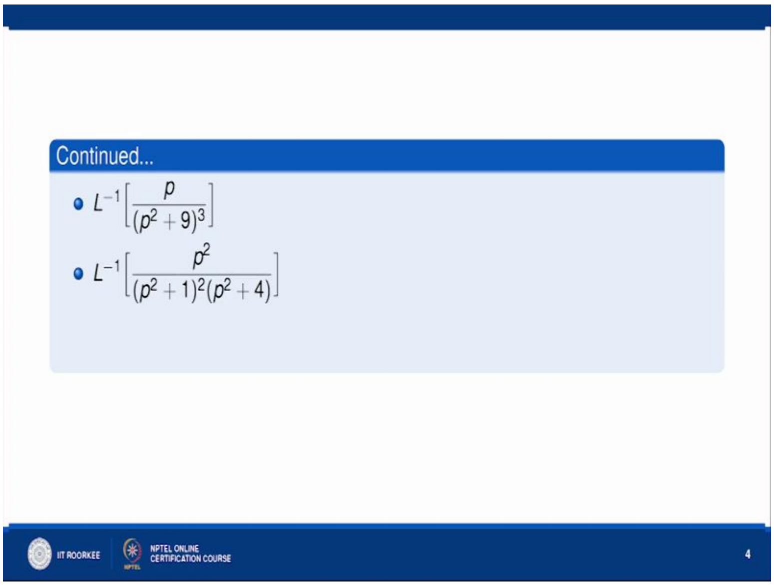

Now, let us see some more problems based on convolution theorem, which otherwise is very difficult to actually find out. So, let us see Laplace inverse of these functions also, using convolution theorem.

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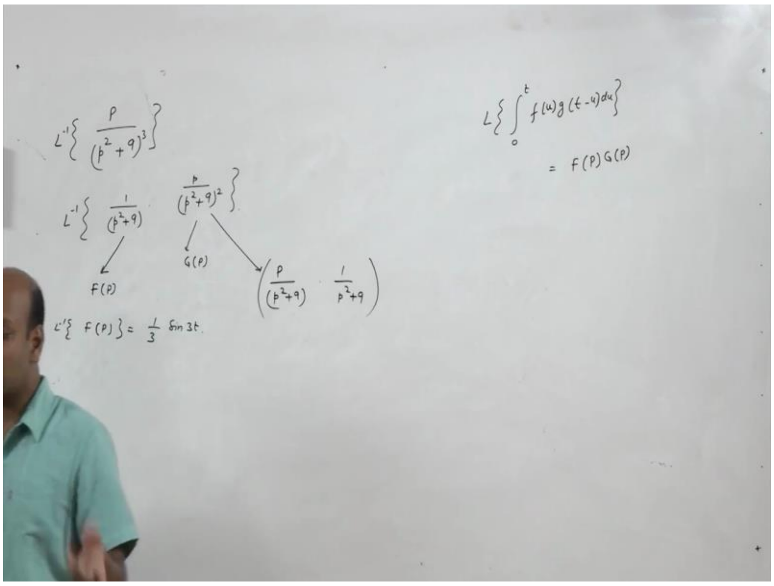

So, what is a first problem let us see, Laplace inverse of p upon p square plus 9 whole cube. Now how to find Laplace inverse of this expression? So, it can be break into two functions, suppose one function is p upon p square upon 9, and suppose other function is 1 upon p square plus 9 whole square. Now if you split the in to this form, if you split into this form. So, finding Laplace inverse of this is difficult.

We take p over here and 1 over here. So, that because otherwise 1 upon p square plus 9 whole square Laplace inverse is difficult. So, now let us suppose, we already know that Laplace inverse of this is, suppose this F p and suppose this is G p. So, Laplace inverse of F p we already know it is nothing but 1 by 3 sin 3 t, and that we can easily find out. Now you to find Laplace inverse of this expression we have two methods, either we again apply convolution theorem in this expression separately, find Laplace inverse of G p, which again can be written as function of a product of two functions. Are you getting my point, I want to say that let us write this as p upon p square by 9 in 2 1 upon p square plus 9. And suppose this is some function, this is some function and again apply convolution theorem to find out Laplace inverse of this expression, or the other way out is, whatever matters we have studied, whatever properties we have studied for Laplace transforms, find out Laplace inverse of this function, using those properties, and directly substitute it over here. So, choice is ours.

So, let us find out Laplace inverse of this function, using the earlier properties which we have already discussed.

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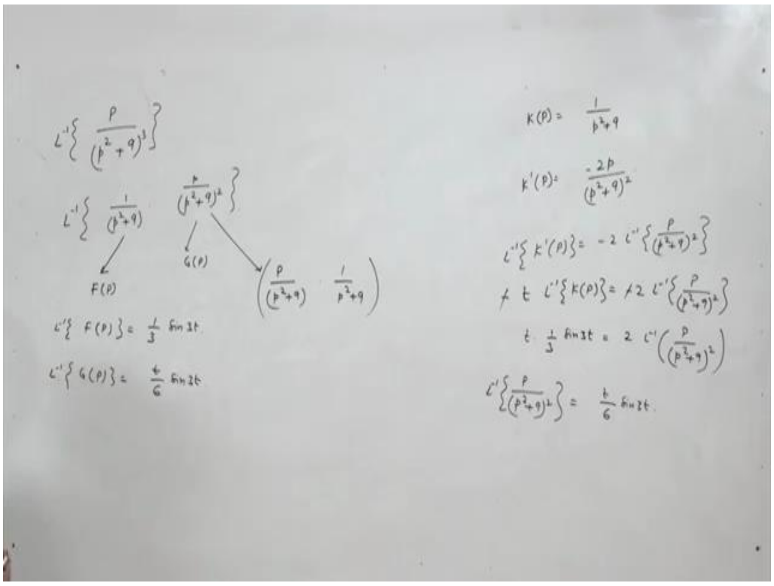

So, though this is, suppose this is, suppose some k p function of k p, let us suppose, because F p we have already used we are using some other function of p is 1 upon p square plus 9. So, k dash p is nothing but minus 2 p upon p square plus 9 whole square. So, now, Laplace inverse of k dash p is nothing but minus 2 times Laplace inverse of p upon p square plus 9 whole square. You want to find out Laplace inverse of this expression.

So, what is f dash. Ok a Laplace of t f t is nothing but minus 1 k power 1 f dash p. So, actually we have to use this property, actually we have to use this property. So, Laplace inverse of f dash p is nothing but minus t into f t, and f t is Laplace inverse of k p. if minus 2 Laplace inverse of p upon p square plus 9 whole square. So, minus minus cancels out, this is nothing but t into. Now what is Laplace inverse of this expression? This is 1 by 3 sin 3 t. So, it is 2 Laplace inverse of this expression. So, Laplace inverse of this expression is nothing but t by 6 sin 3 t. So, this is Laplace inverse of this expression.

So, basically Laplace inverse of G p is nothing but t by 6 sin 3 t. So, this you can also find out using convolution theorem I told you, this we can easily find out in convolution theorem also. Now to find Laplace inverse of the product of these two; that is F p into G p, Laplace inverse of G p into F p here F p into G p both are same.

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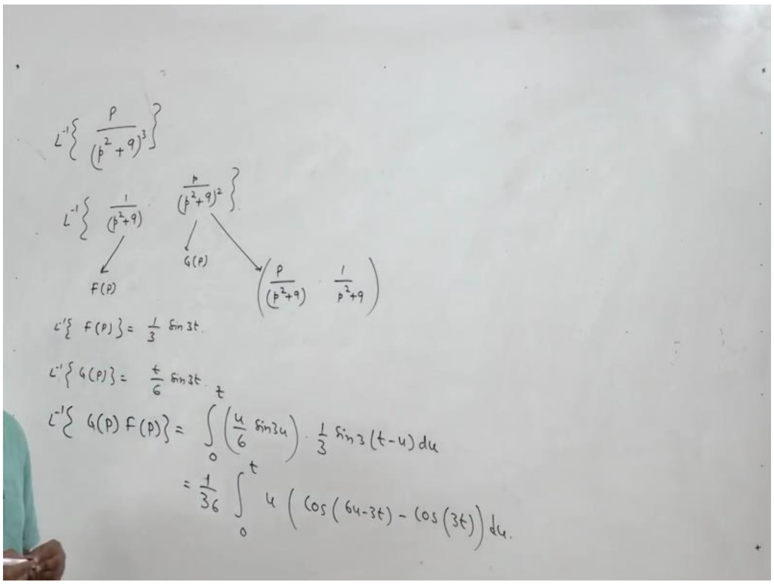

So, F p into G p is nothing but 0 to t. Now the first function is this is f u u by 6 sin 3 u into g t minus u du into g t minus u; that is 1 by 3 sin 3 t minus u into du. So, this is nothing but u by, 1 by 18. You again multiply and divided by 2 integrate this. So, it is 1 by 36 I can write here 1 by 36, it is 0 to t u. Now when you multiply and divided by 2 it is 2 sin a sin b which is nothing but cos a minus b minus cos a plus b. So, it is cos a minus b cos a minus b is nothing but 6 u minus 3 t and minus cos a plus b. So, a plus b is nothing but 3 t into du. So, now, this you can easily integrate, using method of, using by parts, and you can easily find out Laplace inverse of the first problem.

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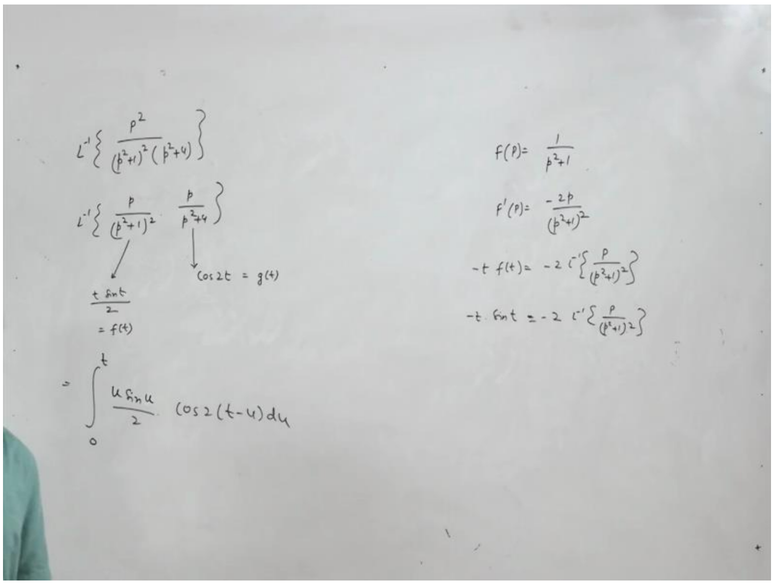

Now, second problem, let us say second problem, Laplace inverse of second problem, Laplace inverse of p square upon p square plus 1 whole square into p square plus 4. Again we have to split into 2 in such a way that Laplace inverse of both the functions you can easily find out, then only we can apply convolution theorem. So, it is p square upon p square plus 1 whole square into p square plus 4. So, it is Laplace inverse of p upon p square plus. So, either you can split in to 3, and apply convolution theorem 2 times, again it is our choice, all you split into two, and Laplace theorem only once, I mean convolution theorem only once. So, this you can do like this, this into this, and this into this.

Now, Laplace inverse of this we already know this is nothing but cos 2 t, which is same which is f t or g t anything. Now Laplace inverse of this, Laplace inverse of this again you can find out, using the previous problems, as in the previous problems we did finding f dash, we can easily find out, because you can take F p as 1 upon p square plus 1, and then find the derivative as f dash p is equals to minus 2 p upon p square plus 1 whole square. Then taking the Laplace inverse both the side it is minus t F p is equals to 2 minus 2 times Laplace inverse of p upon p square plus 1 whole square, and minus t f t is nothing but Laplace inverse of this is sin t it equal to minus 2, Laplace inverse of p upon p square plus 1 whole square.

So, Laplace inverse of this is nothing but t sin t by 2. So, this is where this is f t. So, to find out Laplace inverse of these two is nothing but 0 to t f t and g t minus f u, sorry f u. The first always involved u f u, g t minus u; that is cos of 2 t minus u into du. So, one can easily integrate it again by integration by parts, and find out the Laplace inverse of this expression.

Thank you very much.