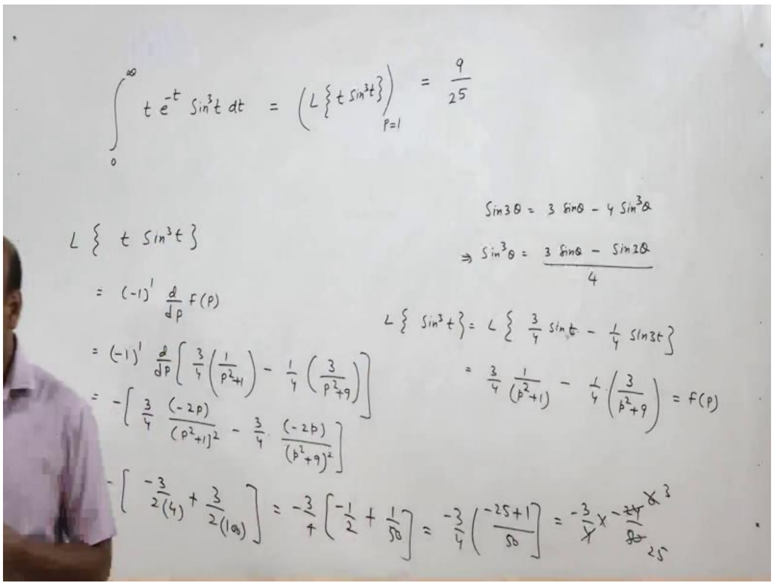
Mathematical methods and its applications
Dr. S. K. Guptha
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 30
Convolution Theorem for Laplace transforms-I

So, we have already studied the properties of Laplace transforms. Now, next topic is convolution theorem for Laplace transforms. So, before stating what convolution theorem is, and how it is important to find out Laplace transform or Laplace inverse transform of some functions, so first we will find some integrals using Laplace transforms.

(Refer Slide Time: 00:49)



So, suppose we have first problem. First problem is 0 to infinity t e k power minus t and it is sin cube t dt. So, how can we find out such integrals using Laplace transforms? Now, this is equal to; now what is Laplace transform of t sin cube t? Laplace transform of t sin cube t is given by 0 to infinity e k power minus p t t sin cube t dt. So, if we want to find out this integral, then if we substitute p equal to 1 over here, so we will get back to this integral. So, this integral is nothing but we can say it is Laplace transform of t sin cube t when p equal to 1. So, substituting p equal to 1 here, we will get back to this integral. So, this integral is nothing but Laplace transform of t sin cube t when p equal to

1. So, basically we have to find out Laplace transform of this function f t, and substitute p equal to 1. So, we will get back to the value of this integral.

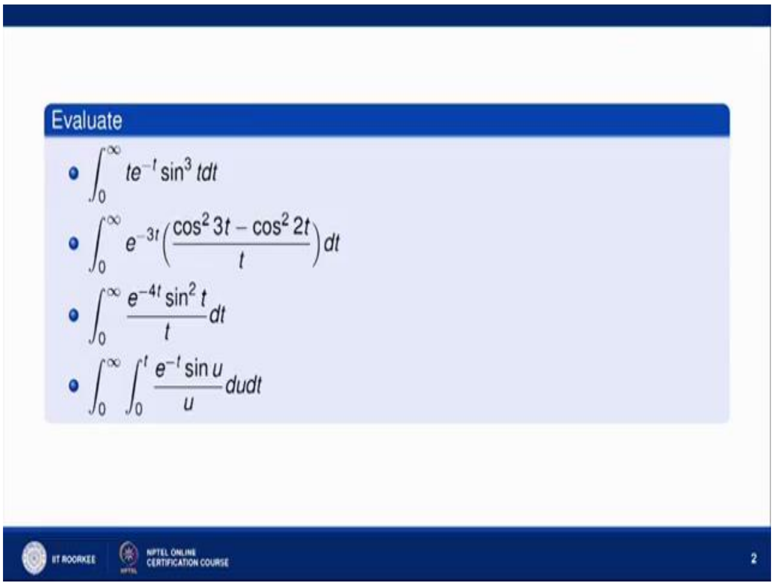
So, how to find out Laplace transform of this, so this we have to already seen how to find Laplace transform of this. So, Laplace transform t f t is given by minus d by dp of F p, where F p is Laplace transform of f t, here f t is sin cube t. So, first will find out Laplace transform of sin cube t, so what is sin 3 theta? It is nothing but 3 sin theta minus 4 sin cube theta. So, what is sin 3 theta from here? It is nothing but 3 sin theta minus sin 3 theta by 4. So, this will be sin cube theta. So, what is Laplace transform of sin cube t, it is nothing but Laplace transform of 3 by 4 sin t sin x or sin t, because it is a function of t minus 1 by 4 sin 3 t. So, it is equals to 3 by 4 Laplace transform of sin t is 1 upon p square plus 1 minus 1 by 4 into Laplace transform of sin 3 t is p upon p square plus 3 square. So, it is p upon p square plus 9. So, this is Laplace transform of sin cube t. So, this is F p, say this is F p.

So, now to find Laplace transform of this function, it is nothing but it will be equal to minus 1 k power 1 d by dP of F p and what is F p? Here this F p is nothing but Laplace transform of this f t sin cube t. And Laplace of sin cube t is this to this is F p. So, it is nothing but minus 1 k power 1 d by dP of F p is 3 by 4 into 1 by p square plus 1 minus 1 by 4 p upon p square plus 9 sorry a upon it is sin a t is nothing but a upon p square plus a square. So, it is must be 3, so it must be 3. Now, this is nothing but negative of derivative of this respect to p. So, it is 3 by 4 into minus 2 p upon p square plus 1 whole square minus 3 by 4 again minus 2 p upon p square plus 9 whole square. So, this is a derivative of this.

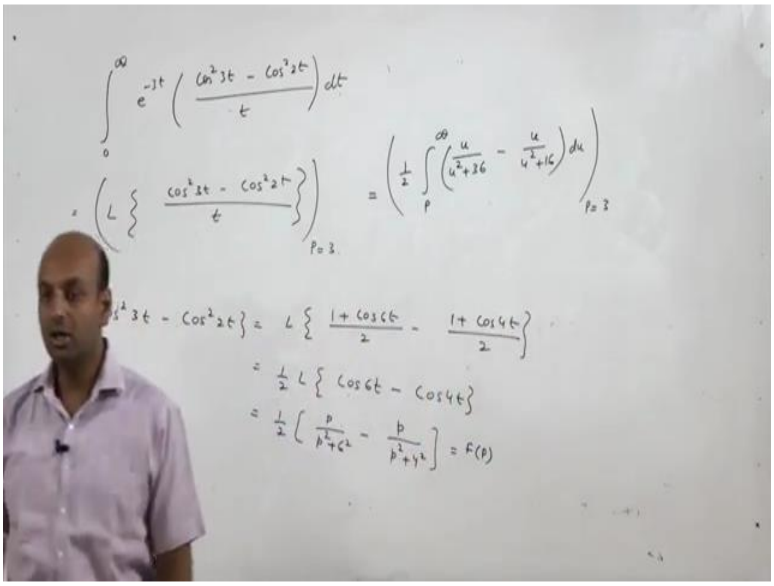
So, now to this is a Laplace transform of t sin cube t, to substitute p equal to 1 here, we will get back to this the value of this integral. So, substitute p equal to 1, when you put p equal to 1 here, so this is nothing but minus of it is minus 3 by 2, p is 1, so it is 4 minus minus - plus, again it is two, two cancels out, so 3 by 2. And p is 1, so 1 plus 9 is 10, 10 square is 100. So, we can simplify this expression. So, this is nothing but 3 by 2, 3 by 4 can be common, and it is nothing but 1 by 2, and it is nothing but plus which is minus it is plus 3 by 4, I have taken common. So, it is 50, 1by 50, and it is nothing but minus 3 by 4 into when you take 50 as LCM. So, it is minus 25 plus 1 upon 50.

So, what is a final answer of this? This is nothing but negative, negative – positive, and they it is 24, 4 6 - 24 and it is, so it is nothing but minus 3 by 4 into minus 24 by 50, and it is 4 6 – 24. And it is 2 3 and 2 25, so it is nothing but 9 by 25, so answer is 9 by 25. So, the final answer is 9 by 25. So, this will be the value of this integral.

(Refer Slide Time: 08:11)



(Refer Slide Time: 08:14)

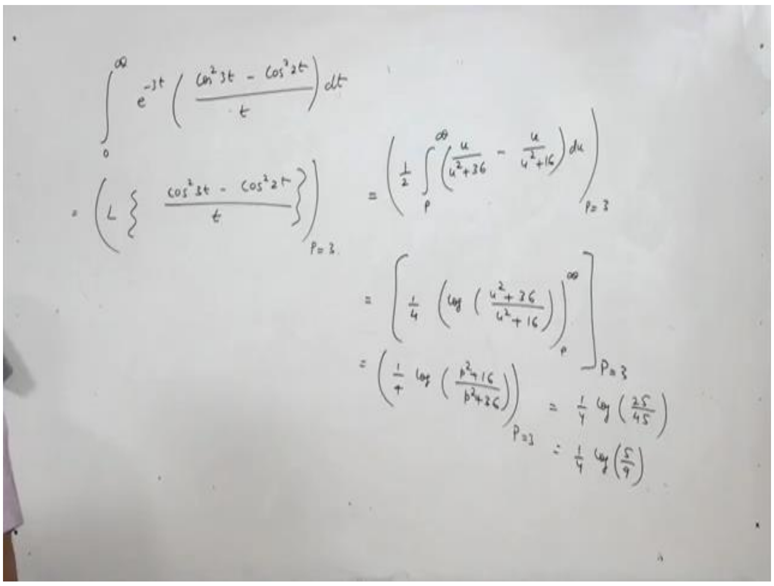


Now, similarly suppose you want to find out solve the second problem, second problem also can be solved on the same lines. So, for the second problem here, for second problem, it is nothing but what is second problem 0 to infinity e k power minus 3 t cos square 3 t minus cos square 2 t by t. So, it is nothing but Laplace transform of cos square 3 t minus cos square 2 t by 2, when p is 3, because Laplace transform of some function of this function is nothing but e k power minus 0 to infinity, e k power minus p t into this function into dt. And here instead of p, we have 3, so substitute p equal to 3 n.

Now, how to find Laplace of this, so for finding Laplace of this expression, it is nothing but. So, first find Laplace of numerator quantity. And we already know that if we have Laplace of f t then Laplace f t by t is nothing but integral p to infinity F p dp, where F p is nothing but Laplace of f t. So, first find Laplace of the numerator quantity. So, Laplace of cos square 3 t minus Laplace of minus cos square 2 t. So, it is nothing but 1 plus cos 6 t by 2 minus 1 plus cos 4 t by 2. Now, half, half cancels out, so it is 1 by 2 Laplace of cos 6 t minus cos 4 t. So, it is nothing but 1 by 2 it is p upon p square plus 6 square minus p upon p square plus 4 square.

Now, this is here this is F p. Now, to find Laplace of F p by t, we again use that property of Laplace transform. So, this is nothing but is equal to integral p to infinity F u du or F p dp both are same, it is nothing but 1 by 2 can be taken out, it is u upon u square plus 36 minus u upon u square plus 16 into du, when p equal to 1. So, we will substitute p equal to 3 later on after finding after solving the entire expression, we will substitute p equal to 3.

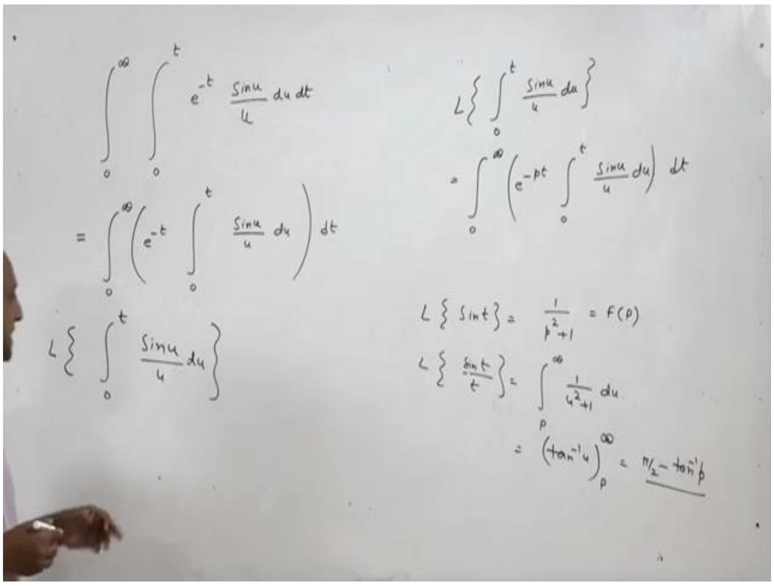
(Refer Slide Time: 11:09)



Now, it is nothing but is equals to it is 1 by 4 log of u square plus 36 minus u square plus 16 from p to infinity. So, where you have taken p equal to 3, you can put p equal to 3 afterwards also or here also. So, it is nothing but when it is infinity, it tends to 0; when it is p because of negative sin it is 1 by 4 log of p square plus 16 upon p square plus 36 now you also substitute p equal to 3.

So, this will be equals to 9 plus 16 – 25, 25 upon it is 45, 9 plus 36 is 45. So, it is nothing but 1 by 4 log of 5 by 9. So, this will be the answer of this problem. So, similarly the third problem can be solved using Laplace transforms, because you first find Laplace transform of sin square t by t, and then substitute p equal to 4. Now, to find Laplace transform of sin square t by t, you first find Laplace transform of sin square t by converting it to double, 1 minus cos 2 t by t. And then for division by t, you again use this property integral p to infinity F u du, where F u is nothing but Laplace transform of sin square t.

(Refer Slide Time: 13:10)



Now, let us try to solve the last problem - the fourth problem. So, what is a fourth one? It is nothing but 0 to infinity integral 0 to t e k power minus t sin u upon u dt du dt. Now, how to find value of this expression using Laplace transforms. So, we can easily see that this nothing but 0 to infinity e k power minus t and 0 to t, it is sin u upon u du and whole multiplied by dt. So, this can be easily we can easily write this double integral in this form, because here this is the function of t only, so it can be taken out from this integral because this integral involves only u and this is the function of only t. So, it can be taken out from this integral.

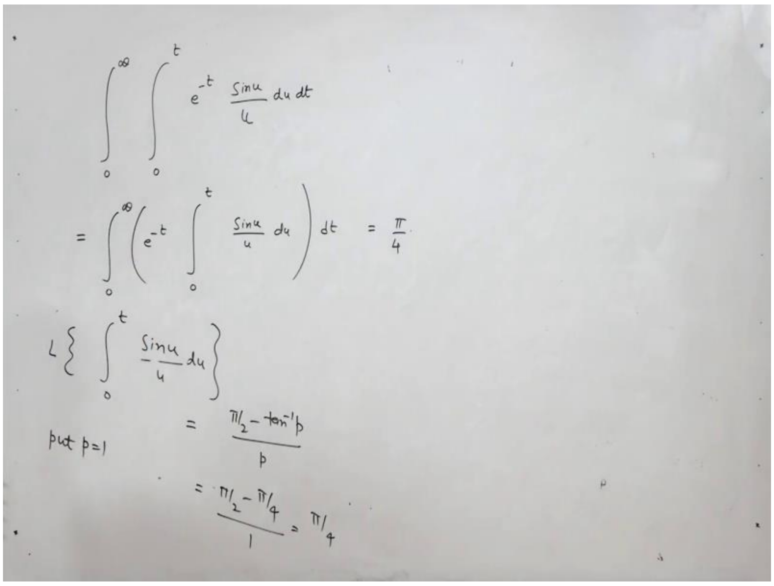
Now, again if we see Laplace transform of 0 to t sin u by u. So, what it is by definition of Laplace transform? It is nothing but 0 to infinity e k power minus p t 0 to t sin u by u du. So, if you compare du into dt of course, so if you compare this with this expression, so we can easily see that p is equal to 1 to that means, if you find Laplace transform of this expression and substitute p equal to 1, so we will get back to the value of this integral. Now, the question is we have to find out the Laplace transform of this f t and substitute p equal to 1 in that expression to find out the value of this integral. Now, how to find Laplace transform of 0 to t sin u by u into du. The only thing is we have to find out Laplace transform of this function f t and then substitute p equal to 1.

Now, how to find Laplace transform of this? So, first Laplace transform of 0 to t F u du is nothing but it is F p by p, where F p is nothing but Laplace transform of this f t or this F u. And for this, first you find the Laplace transform of numerator and division by t or u, we find by finding integral p to infinity F u du. So, first you find Laplace transform of sin u Laplace transform of sin u is or sin t, one in a same thing. So, it is nothing but 1 upon p square plus 1. Now, Laplace transform of sin t by t. So, we know that it is nothing but integral p to infinity and Laplace transform of numerator, sin t Laplace transform sin t is F P. So, Laplace this is F p basically.

So, Laplace transforms of sin t is this. So, it is 1 upon u square plus 1 du, because Laplace transform sin t by t is nothing but integral p to infinity F u du, where F u is nothing but Laplace transform of numerator quantity and numerator is sin t. So, Laplace transform of sin t is nothing but 1 upon u square plus 1. Now, integral of 1 upon u square plus 1 is tan inverse u from p to infinity. So, this is nothing but pi by 2 minus tan inverse p. So, this is Laplace transform of this, this function F u. Now, to find out the Laplace transform of 0 to t F u du 0 to t, so it is nothing but F p by p, where F p is Laplace transform of this entire function not this F p, the entire function f t which is inside this integral.

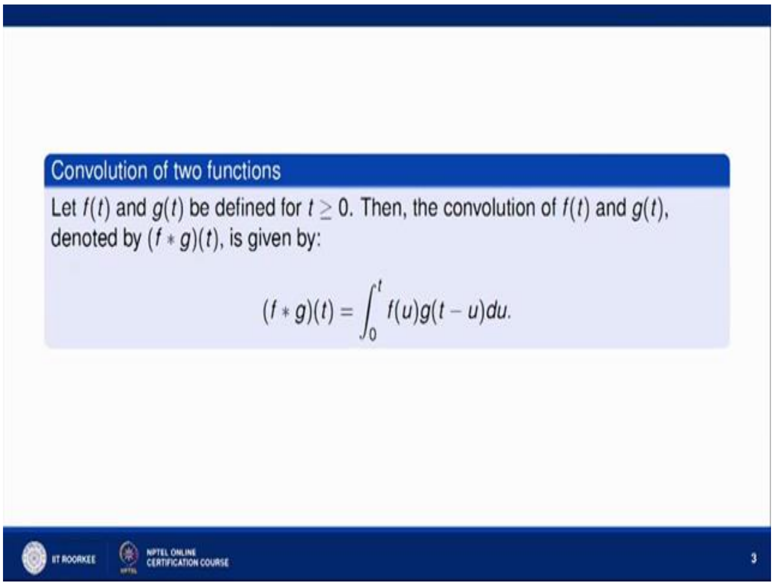
So, Laplace transform of this is nothing but this will be equals to pi by 2 minus tan inverse p upon p. Because Laplace transform of 0 to t f t dt is F p by p, where F p is the Laplace transform of f t, here f t is this entire expression. So, this is Laplace transform of this. Now, to find out Laplace transform of this expression simply substitute p equal to 1. So, put p equal to 1, so we will get the value of this expression which is nothing but.

(Refer Slide Time: 18:53)



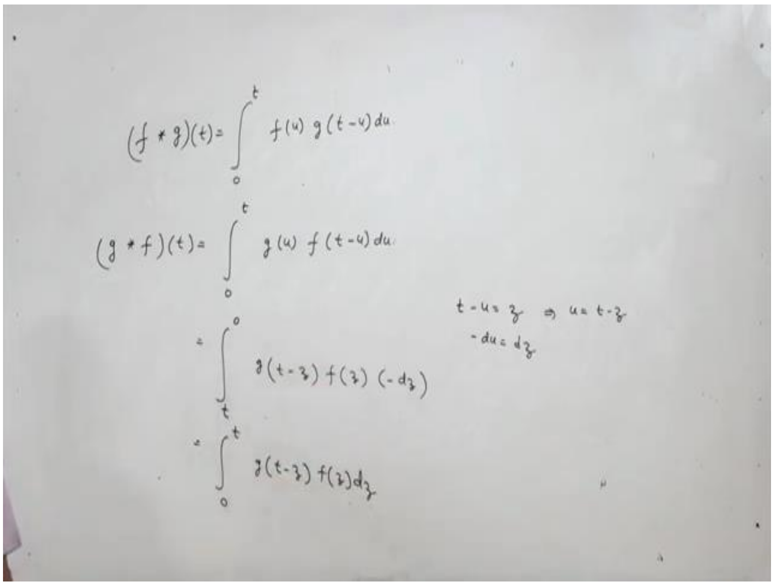
So, when you put p equal to 1, we will get back to pi by 2 minus pi by 4 upon 1 which is nothing but pi by 4. So, the value of this integral is nothing but pi by 4. So, hence we can find out the integrals of such type using Laplace transforms. Now, we come to convolution theorem. So, hence Laplace transform can also be used to find out integral from 0 to infinity some integral from 0 to infinity types. Now, come to convolution of two functions first and once we define this then we come to convolution theorem of Laplace transforms.

(Refer Slide Time: 19:43)



Now, let f t and g t we define for t non-negative, t greater than equal to 0 then the convolution of f t and g t which is denoted by f is star g is given by this expression. So, convolution of two functions we call it convolution of two functions.

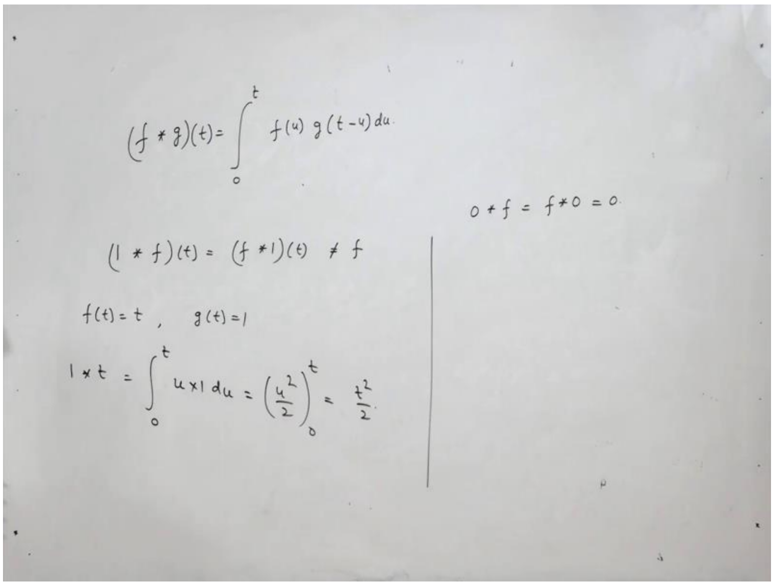
(Refer Slide Time: 20:02)



So, convolution of two function f t and g t is nothing but 0 to t f u g t minus du. So, this we call as convolution of f and g. So, this is star is only a notation. How this notation is defined this is defined like this 0 to t f of u f of first function f u of first function, and g t minus u of second function du. So, this is how we can define convolution of two functions. Now, from here we can easily see that if you find g star f of t that is convolution of g and f, so it is nothing but 0 to t g of u into f of t minus u du by the same definition of convolution of two functions. Here the f and g interchanged.

Now, if you take t minus u as suppose new variable z. So, minus du is nothing but d z. So, it is nothing but when u is 0. So, it is t, z is t and when it is z. So, it is when it is t, u is t, so z is 0. And g of what is u? u from here is nothing but u from here is nothing but t minus z, so it is t minus z, and it is f of z. And du is nothing but minus d z. So, negative will interchange the integrals limits of the integrals. So, by the property of def integral, so 0 to t f of t minus z into f z d z, so this is nothing but the same expression as in this; instead of u we have z, so it hardly matters. So, we can say that f star g is same as g star f that is convolution of f and g or convolution of g and f, both are same both are equal so that means, this star or the convolution of two functions satisfy a commutative property commutative means a star b is equals to b star a.

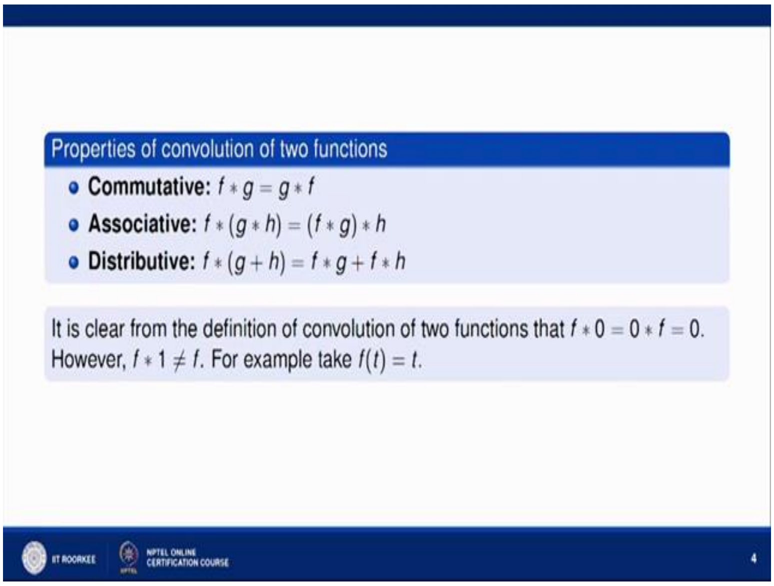
(Refer Slide Time: 22:59)



If you find 1 star suppose f of t, so that you can easily see is equals to f star 1 of t that is obvious because convolution of two functions satisfy commutative property. So, if I take one function as one then 1 star f 1 convolution of one with f same as convolution of f with one both are of course equal. Now, are they equal to f, are they equal to f? So, the answer is may or may not be. So, it is then may not equal to f. Now, we have an examples suppose we take suppose f t equal to t and g t as 1. So, 1 star t that is the convolution of one with t is nothing but 0 to t f u f u is 1, f u is u. And g t minus u is one because g t is 1, so it is 1 into du. So, it is nothing but u square by 2 0 to t is equal to t square by 2, so it is not equal 2 f. Of course, this is not this is not equal to t. So, we can say that the convolution of one with f may not be equals to f, may not be equals to f.

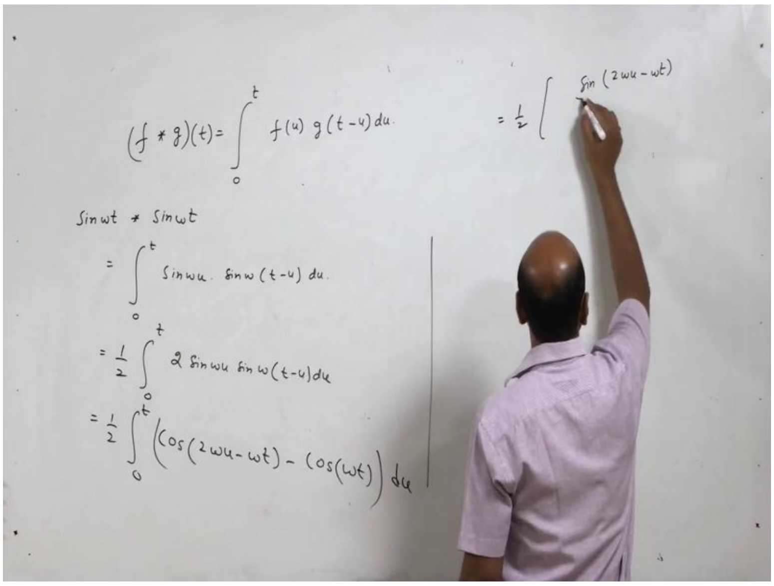
However, if you find the 0 convolution with f or f convolution with 0 is always 0, because when you convolute with 0 then f convolution with 0 one function is 0. So, multiplication will be, of course will be 0. And f convolution of 0 with f is same as convolution of f with 0 because of the commutative property. So, convolution of f is 0 is equal to 0. However, convolution of 1 with f may not equal to f. So, these are the properties of convolution of two functions.

(Refer Slide Time: 25:39)



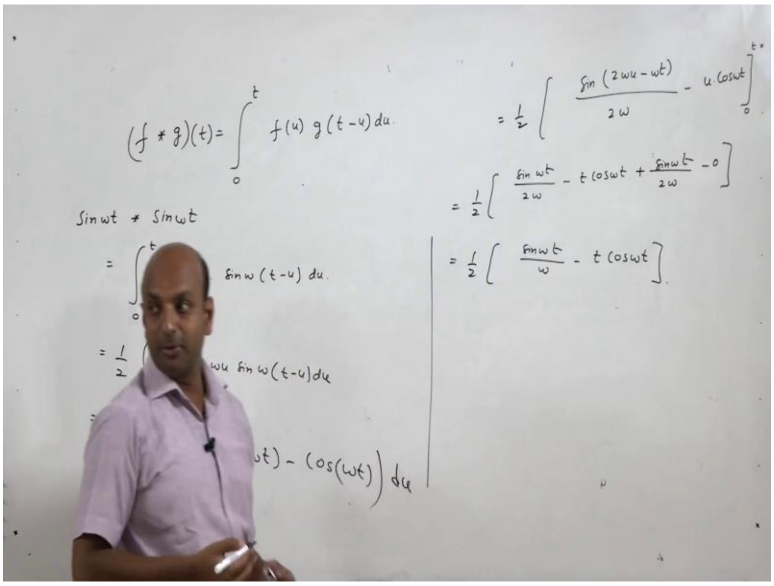
It also satisfies a associative property convolution also satisfy associative property and distributive also. Now, let us find convolution of some functions f and g.

(Refer Slide Time: 25:58)



So, suppose we have first problem convolution of sin omega t with itself. So, by definition, it is 0 to t f u, f u is sin omega u, and g t minus u sin omega t minus u into du. Now, you have to simply find the value of this integral, but that will be the convolution of sin omega d with sin omega t. So, how to find this integral, you divide them and multiply by 2. Now, 2 sin is sin b is nothing but cos a minus v, so it is 2 omega u minus omega t and minus cos a plus b, and cos a plus b is nothing but omega t whole v du.

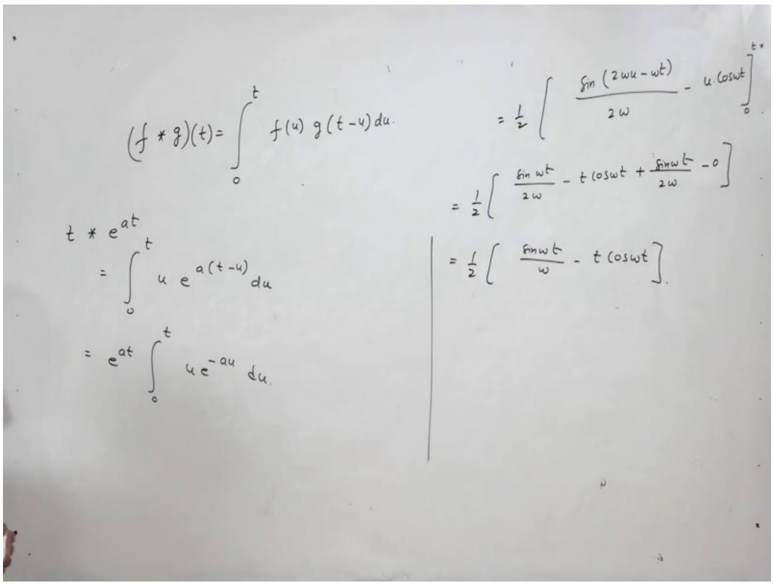
(Refer Slide Time: 27:09)



So, this is nothing but is 1 by 2 is equals to 1 by 2 integral of cos is sin. So, it is sin 2 omega u minus omega t. So, it is respect to u coefficient of u will be in that denominator. So, 2 omega minus, now this will be a constant respect to u. So, it is u times cos omega t from 0 to t. So, it is nothing but 1 by 2 it is when you substitute u as t. So, it is sin omega t upon 2 omega minus t cos omega t and when t equal to 0, when u equal to 0.

So, when u equal to 0, it is minus minus - plus sin omega t upon 2 omega; and when u equal to 0, it is 0. So, this is nothing but 1 by 2 sin omega t by omega minus t cos omega. So, this will be the value of this problem convolution of sin omega t with itself. Let us find convolution of t with e k power a t also. Basically, convolution of functions will be used in convolution theorem for Laplace transforms. So, first we should know that what is convolution of two functions is and then we can state convolution theorem for Laplace transforms.

(Refer Slide Time: 28:54)



So, let us solve this problem also convolution of this. So, this nothing but 0 to t u e k power a t minus u du, so it is nothing but e k power a t 0 to t u into e k power minus a u du. So, one can easily integrate this with respect to u and find out the convolution of t with e k power a t. Similarly, third problem can be solved by this definition of convolution of two functions.

Thank you very much.