Mathematical methods and its applications
Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 03
Solution of second order homogenous linear differential equation with constant coefficients

Hello friends, welcome. In this lecture I will discuss Solution of Second Order Homogenous Linear Differential Equation with Constant Coefficients. This topic will be covered in 2 lectures. We will cover in this lecture and the next lecture. Now let us define a homogenous linear differential equation of second order with constant coefficients. This, such an equation is given by y double dash x, plus a y dash x plus b y x, equal to 0. Where a and b are some real constants and x is any real number belonging to r.

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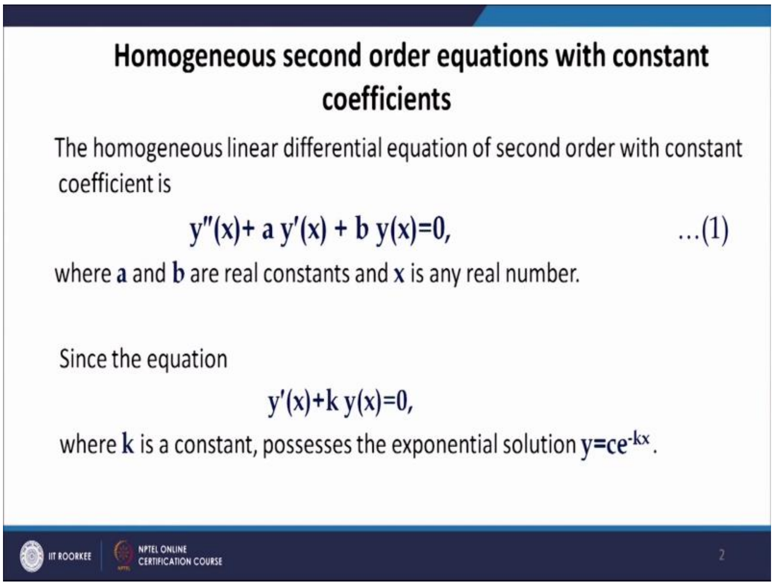

Now, let us look at this. We shall be finding solution of this second order homogenous linear differential equation, by with the help of this solution of first order linear differential equation with constant coefficients.

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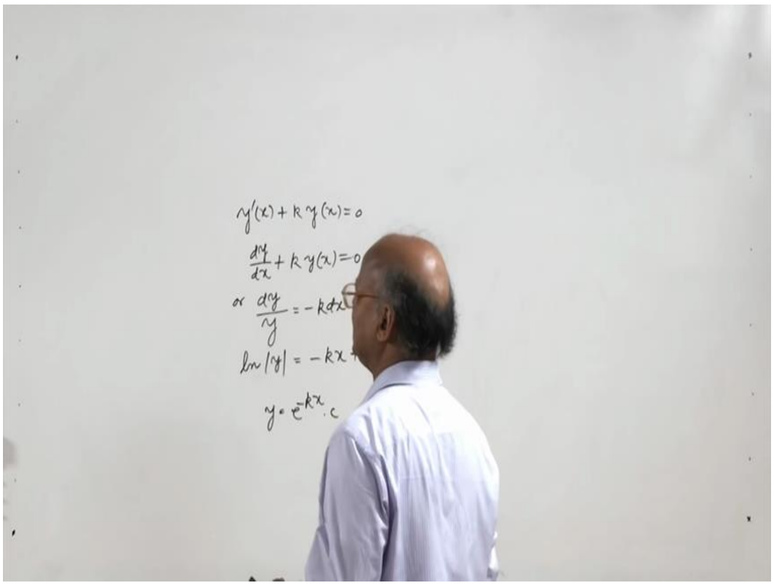

So, let us look at this equation y dash plus k y x equal to 0. We know that it possess a exponential solution, y dash x plus k y x equal to 0. You can write as, or we can write it as, now integrating both sides. So, we can write l n mod y equal to minus k x plus some constant c 1 or we can write y equal to e to the power minus k x into a constant c. So, y equal to c times e to the power minus k x. Here c is an arbitrary constant.

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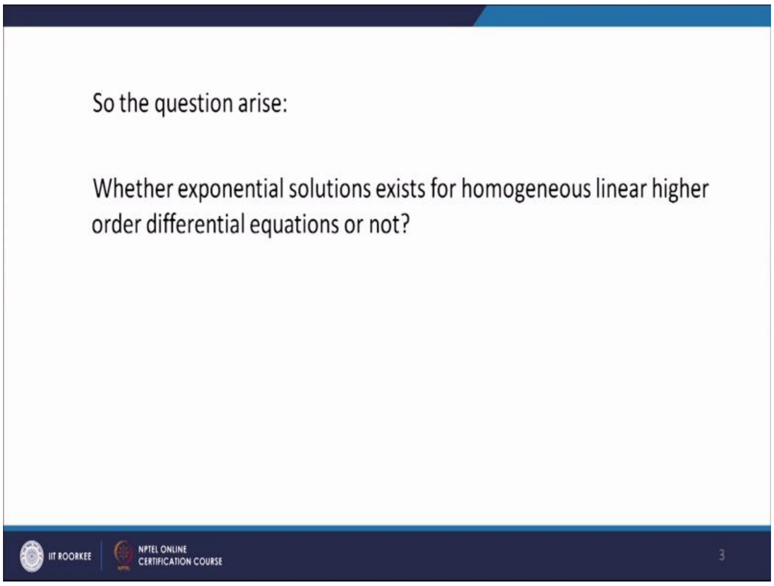

Now, the question arises whether exponential solutions exist for homogenous linear higher order differential equation or not. So, we will see that the solutions of higher order homogenous linear differential equations are also either exponential functions or they are constructed out of exponential function. Let us substitute y equal to e to the power x e to the power m x in the second order homogenous linear differential equation with constant coefficients.

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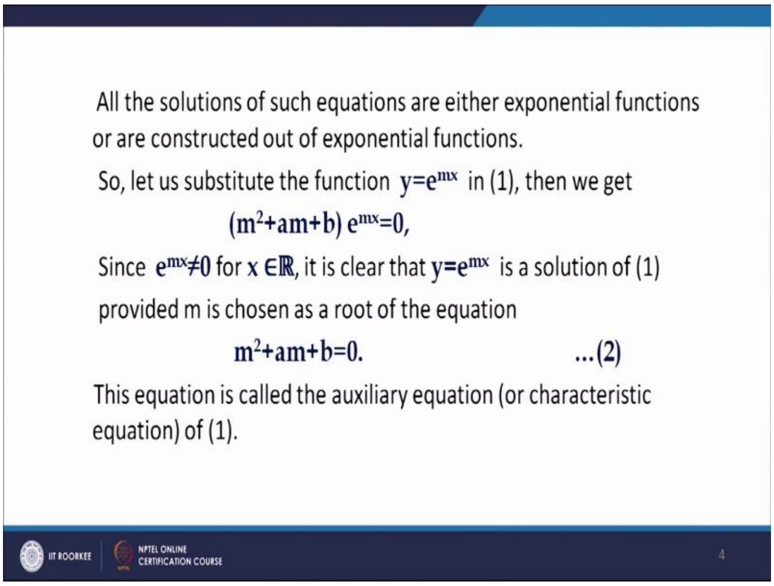

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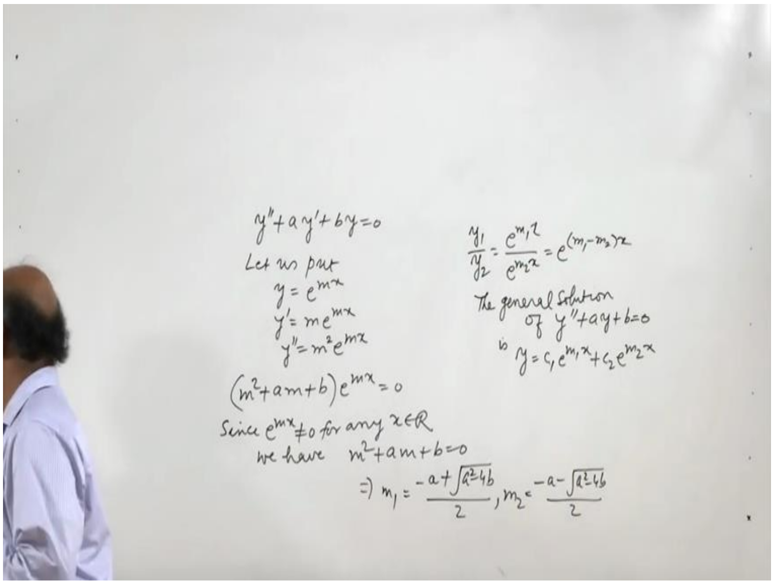

Y double dash plus a y dash plus b y equal to 0. This our second order homogenous linear differential equation with constant coefficients a and b. So, let us put y equal to e to the power m x in this. We get y dash equal to m times e to the power m x, y double dash equal to m square e to the power m x. So, substituting the values of y y dash and y double dash in the equation y double dash plus a y dash plus b y equal to 0, we get m square plus a m plus b into e to the power m x equal to 0.

Now since e to the power m x is not 0, is never 0 for any x belonging to r. We have m square plus a m plus b equal to 0 which is a quadratic equation of second order in m. This equation is called as the characteristic equation or auxiliary equation. So, by obtaining the roots of this equation say the roots are m 1 and m 2. We can say that we get solutions y equal to e to the power m 1 x and e to the power m 2 x of the given differential equation.

Now, there arise the following values of m 1 and m 2. So, this gives you m 1 equal to minus a plus under root a square minus 4 b y 2 and m 2 equal to; now whether m 1 and m 2 are distinct, or they are complex conjugate or they are equal to dependents on the discriminant. And discriminant of the given second order equation m square plus a m plus b equal to 0. So, there arise 3 cases. We shall get 2 distinct real roots provided the discriminant a square minus 4 b is greater than 0. 2 complex conjugate roots provided a square minus b 4 b is less than 0 and a real double root when a square minus 4 b equal to 0. So, we shall discuss these 3 cases one by one.

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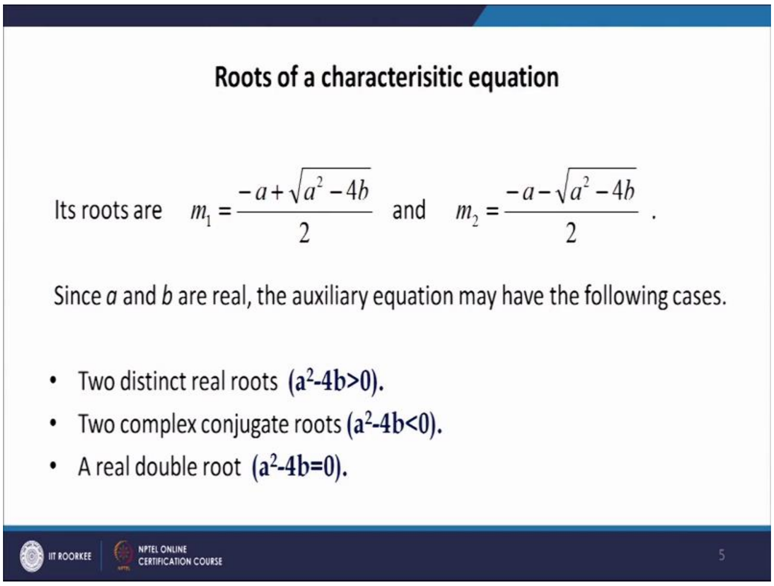

In the case of 2 distinct real roots, we find 2 solutions corresponding to the 2 values of m 1 and m. So, y 1 equal to e to the power m 1 x and y 2 equal to e to the power m 2 x.

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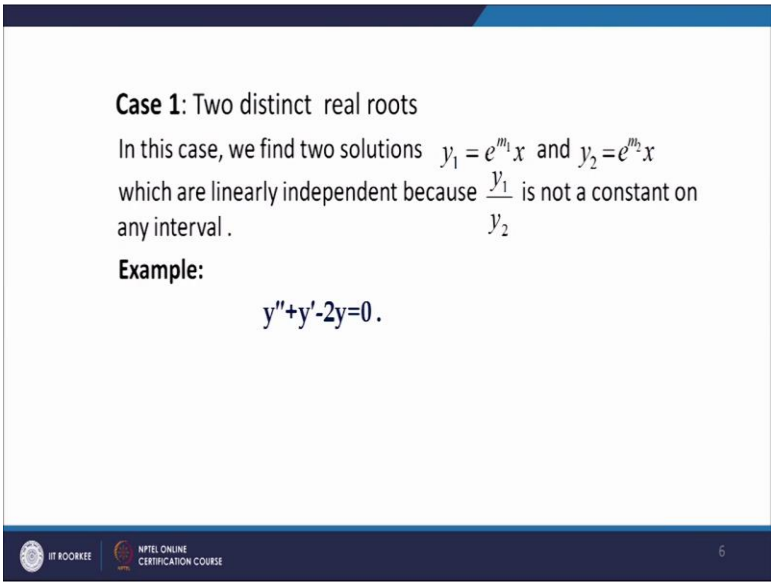

Now, y 1 and y 2 are linearly independent because y 1 upon y 2 is equal to the e to the power m 1 minus m 2 into x. Since m 1 is not equal to m 2 e to the power m 1 minus m 2 into x is not a constant and therefore, y 1 and y 2 are linearly independent.

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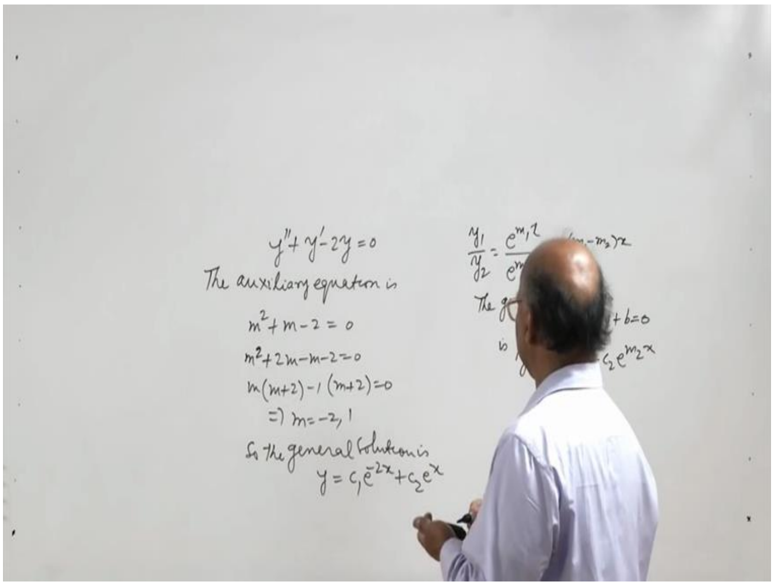

So, in this case the general solutions of the equation is, y equal to c 1 e to the power m 1 x plus c 2 e to the power m 2 x. For example, let us consider y double dash plus y dash minus 2 y equal to 0. So, the auxiliary equation will be, m square plus m minus 2 equal to 0. And we can then see that m equal to 2 m square plus m minus 2 equal to 0. So, we can write m square plus 2 m minus m minus 2 equal to 0. This will give you the roots m equal to minus 2 and 1, which are distinct roots. And so the general solution of the given equation will be y equal to c 1 e to the power minus 2 x plus c 2 e to the power x.

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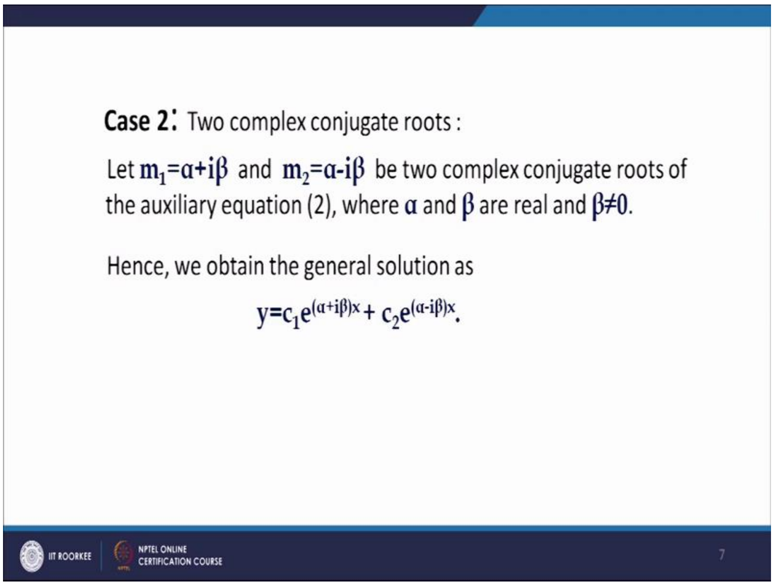

Now, let us consider the case of 2 complex conjugate roots. So, let us say m 1 is equal to alpha plus i beta and m 2 equal to alpha minus i beta be 2 complex conjugate roots of the auxiliary equation m square plus a m plus b equal to 0. Where alpha and beta are real numbers and beta is not equal to 0. Then the general solution will be, since in applications with real solutions therefore, instead of complex exponentials here.

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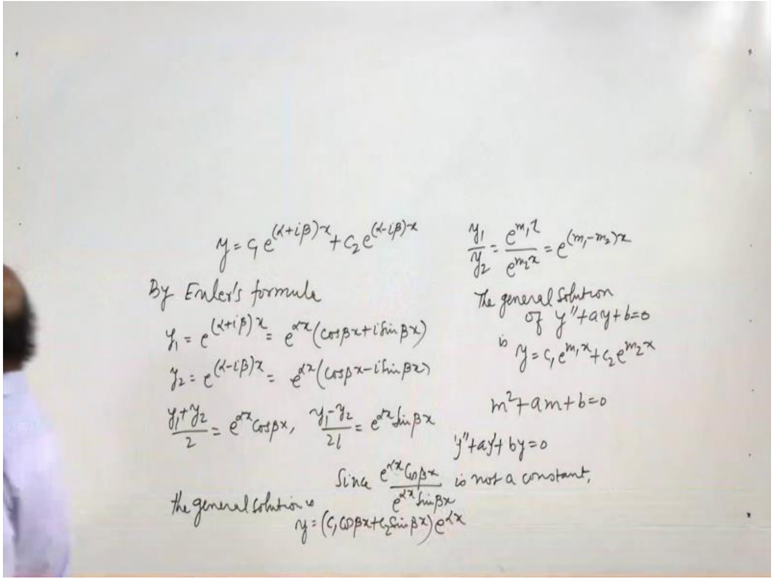

Let us try to find the real solutions of the homogenous second order linear differential equation. So, let us recall the Euler's formula. The Euler's formula tells us that e to the power i theta is cos theta plus i sin theta. So, e to the power alpha plus i beta into x can be written as, let it be y 1. And y 2 be e to the power alpha minus i beta into x. Then we can write it as, so adding y 1 and y 2 we get and also y 1 minus y 2.

Now, we have already seen that in the case of a homogenous linear differential equation of nth order, if y 1 y 2 y k are solutions of that equation then c 1 y 1 plus c 2 y 2 and so on c k y k is also a solution. So, y 1 plus y 2 by 2 is a solution of the equation y double dash plus a y dash plus b y equal to 0. This is our second order differential equation. So, if y 1 and y 2 are solutions of this then y 1 plus y 2 by 2 is also a solution and y 1 minus y 2 by 2 i is also a solution. So, e to the power alpha x cos beta x is a solution of this differential equation. And e power alpha x into sin beta x is also a solution of this differential equation. Furthermore, we note that e to the power alpha x cos beta x divided by e to the power alpha x sin beta x is not a constant. So, e to the power alpha x cos beta x and e to the power alpha x sin beta x are 2 linearly independent solutions of this differential equation.

Since, e to the power alpha x cos beta x upon e to the power alpha x sin beta x is not a constant. The general solution in this case will be y equal to c 1 cos beta x plus c 2 plus sin beta x, into e to the power alpha x. So, in this case of 2 complex conjugated gate roots m 1 equal to alpha plus i beta and m 2 equal to alpha minus i beta. The general solution of the differential equation y double dash plus a y dash plus b y equal to 0, will be written as e to the power alpha x into c 1 cos beta x plus c 2 sin beta x.

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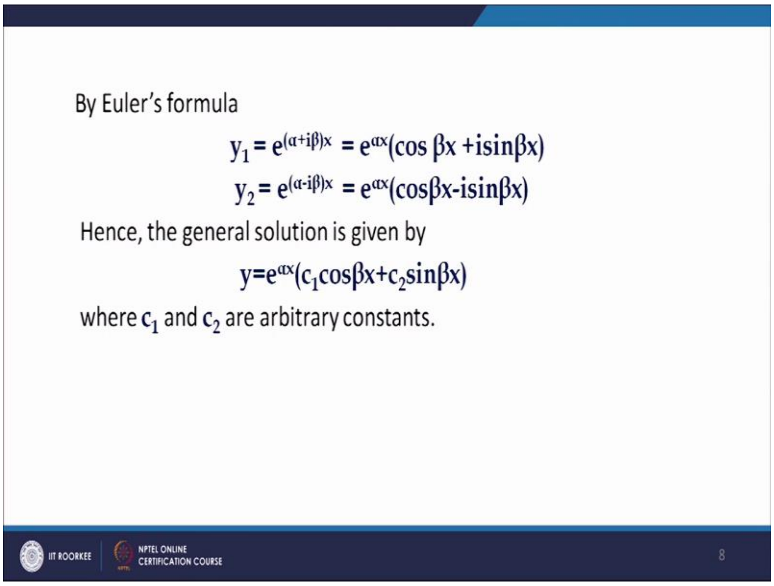

Now, we go over to an example on this. So, this is Euler's formula the general solution is given by this one y equal to which we have already seen c 1 and c 2 are 2 arbitrary constants.

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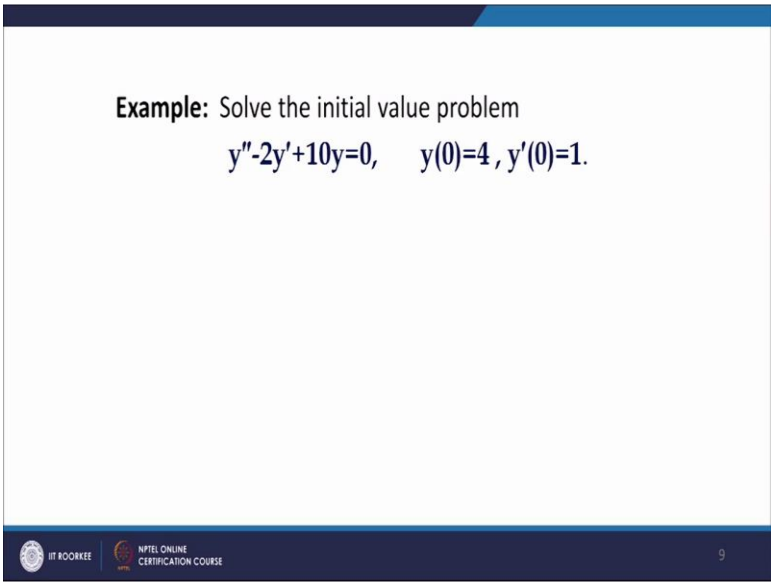

Let us now take up the problem y double dash minus 2 b. This is an initial value problem we are given a homogenous second order linear differential equation with constant coefficients, along with 2 initial conditions. At x equal to 0 y is given as 4 and at x equal to 0, the derivative of y is given as 1. So, y double dash minus 2 y dash plus 10 y equal to 0.

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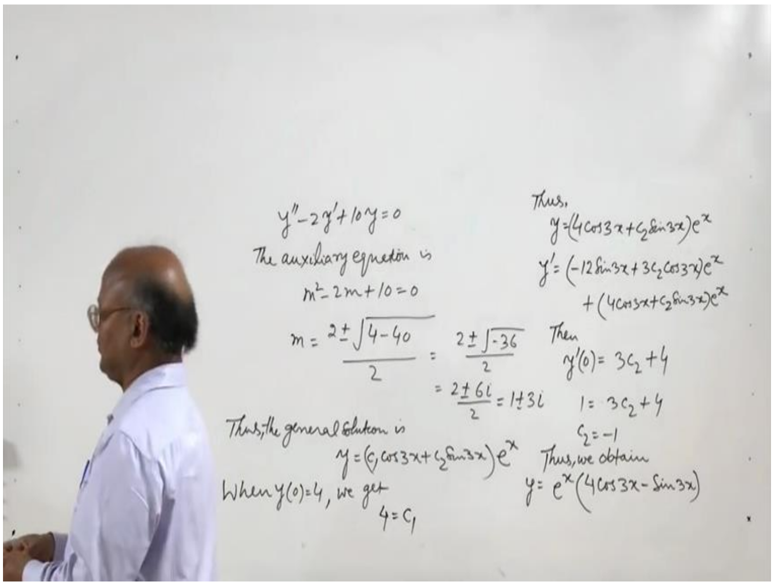

So, these 2 initial conditions will determine a unique solution of the homogenous linear differential equation of second order. So, here auxiliary be auxiliary equation is m square minus 2 m plus 10 equal to 0. So, we can find m. So, this is 2 plus minus 6 i at divided by 2, which is equal to 1 plus minus 3 i.

So, if you compare it with a alpha plus minus i beta alpha is 1 and beta is 3. So, we can write the general solution, y equal to e to the power x into c 1 cos 3 x plus c 2 sin 3 x, where c 1 and c 2 are 2 arbitrary constants. Now with the given initial conditions we shall determine the values of c 1 and c 2. So, let us first take y 0 equal to 4, when y 0 equal to 4, we get putting x equal to 0 here, we get 4 equal to c 1 because when x is equal to 0 cos 3 x is 1 and sin 3 x is equal to 0 and e to the power 0 is equal to 1. So, we have got the value of c 1 now let us use second initial condition to determine the value of c 2. So, we will have to put c 1 equal to 4 here. So, we can say thus y equal to 4 cos 3 x, plus c 2 sin 3 x into e to the power x. Let us differentiate it with respect to x. What do we get?

Y dash equal to minus 12 sin 3 x, plus 3 c 2, the derivative of e to the power x is e to the power.

Now, let us put x equal to 0 in this. So, x equal to 0 when you put sin 3 x will be 0 cos 3 x will be 1. So, we get 3 c 2, this is 1, here when you put x equal to 0, this is 4 this is 0, so 3 c 2 plus 4. Now y dash 0 is given equal to 1. So, we get c 2 equal to minus 1. And thus we have y equal to corresponding to the 2 initial conditions we have got the values of c 2 arbitrary constants c 1 and c 2. So, this is e to the power x times 4 cos 3 x. So, this how we find the solutions of this initial value problem.

Thank you.