Mathematical methods and its applications
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Lecture – 29
Properties of Laplace transforms-IV

Hello everyone, we will be studying about properties of Laplace transforms. In the last three lectures we have seen some of the properties of Laplace transform and the questions related to that. Now this is the last lecture on properties of Laplace transform. So, the next property is, if Laplace transform of f t is F p, then Laplace transform of t k power n f t is nothing but minus 1 k power n, nth derivative of F p, so how can we prove this property.

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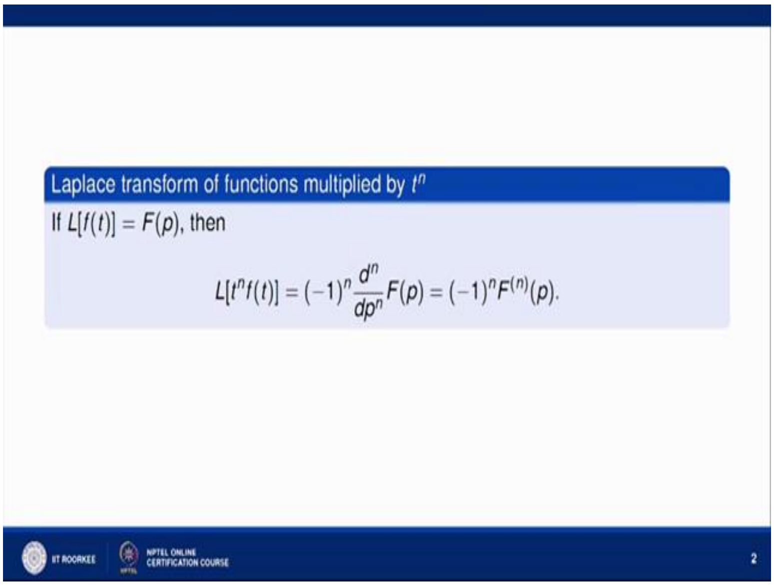

So, Laplace transform of t k power and f t is nothing, but minus 1 k power n nth derivative of F p. So, what is F p here? Laplace of f t.

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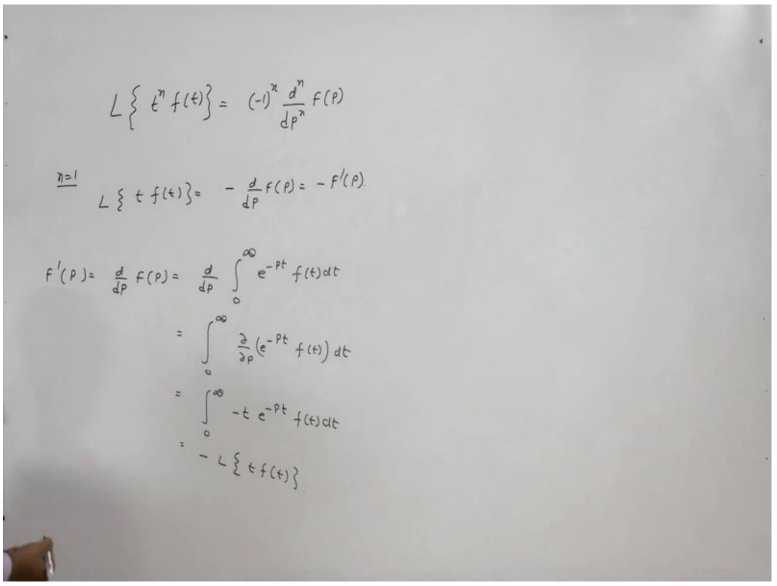

So, again we will use the method of induction to prove this result. So, let us prove for n equal to 1 first, for n equal to 1. So, Laplace transform of t f t is nothing, but negative of d by dp of F p, or is we call it as minus F dash p. So, how to prove this? So, what is F dash p? F dash p is nothing, but d by dp of F p and F p is nothing, but Laplace transform of f t which is 0 to infinity e k power minus p t f t dt. this is because F p is nothing, but Laplace transform of f t. So, by Leibniz rules, we can easily write this as 0 to infinity del by del p of e k power minus p t f t dt.

So, this is 0 to infinity, partial derivative with respect to p of this expression will be nothing, but minus t e k power minus p t f t dt, and it is nothing, but negative of Laplace of t f t, because Laplace of t f t is what, 0 to infinity t f t e k power minus p t which is this expression. So, hence Laplace of t f t is nothing, but negative of F dash p. So, we obtain this result for n equal to 1. Now assume that result hold for n equal to say k, and we will try to prove that is result also hold for n equal to k plus 1. So, assume that this result holds for n equal to k.

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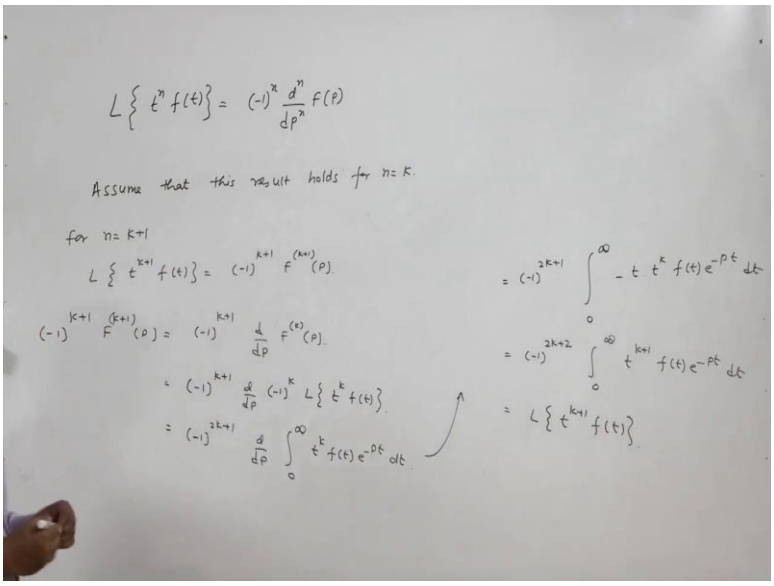

So, we have to show that this result also hold for n equal to k plus 1. So, for n equal to k plus 1, Laplace of n k plus 1 f t is equals to minus 1 k power k plus 1 and k plus 1th derivative of p F p, this to show. So, let us take the right hand side, right hand side is minus 1 k power 1 k plus 1 k plus 1th derivative of F p, which is nothing, but minus 1 k power k plus 1 d by dp of f of kth derivative with respect to p of F p, and this we have already assumed that the result hold for n equal to k. So, since the result hold for n equals to k. So, this value is nothing, but minus 1 k power k plus 1 d by dp of, from here you substitute n equal to k. So, this will be nothing, but minus 1 k power k Laplace of t k power k f k t.

So, this is minus 1 k power 2 k plus 1, and this is d by dp of 0 to infinity t k power k f t e k power minus p t dt, because Laplace of t k power k f t is nothing, but t k power k f t e k power minus p t dt integral 0 to infinity. Again use Leibniz rules. So, when this go inside this will be del by del p. So, this expression will be nothing, but is equal to minus 1 k power 2 k plus 1, and del by del p of this expression will be 0 to infinity minus d times into t k power k f t e k power minus p t dt. So, this is nothing, but minus 1 k power 2 k plus 2 0 to infinity t k power k plus 1 f t e k power minus p t dt, and this minus having the even power, this will be plus 1 and this is nothing, but Laplace transform of this expression, Laplace transform of t k power k plus 1 f t.

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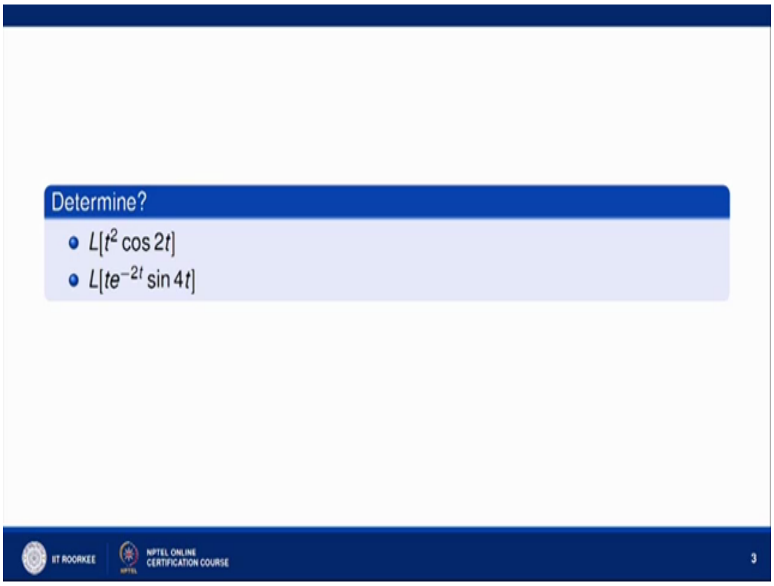

So, hence this result also hold for n equal to k plus 1. So, hence hold for all n. So, hence this result, hence we have this result. Now let us try to solve some problems based on this result, whenever we have in some problems involving multiplication with t, and we have to find Laplace of that. So, we will always use this property of Laplace transforms, whenever we have multiplication of t with f t, of any positive integer power we will use Laplace of t k power in f t, I mean this expression.

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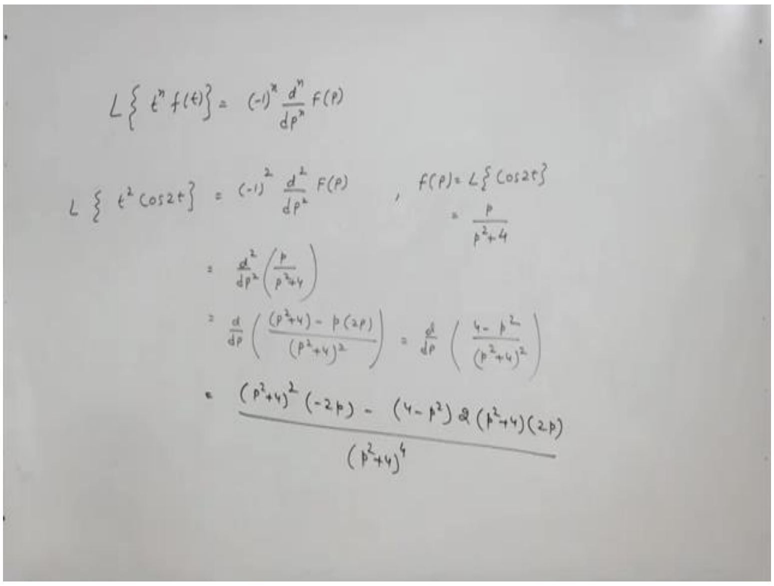

So, suppose you want to find Laplace transform of t square cos 2 t. So, what is f t here, it is cos 2 t what is n, n is 2. So, this is this will be nothing, but minus 1 k power 2, second derivative of F p and what is F p. F p is Laplace transform of f t, f t here is cos 2 t.

So, f t will be Laplace transform of cos 2 t which is nothing, but p upon p square plus 4. So, it is d 2 by dp 2, I mean second derivative of p upon p square plus 4. So, it is d by dp of. First derivative is numerator derivative a denominator minus denominator derivative numerator upon denominator whole square, this is by quotient property of derivatives. So, this is d by dp of this is 4 minus p square upon p square plus whole square. So, again we will find derivative of this F p. So, what is derivative of this F p? So, derivative of this F p will be nothing, but again the denominator, derivative of numerator minus numerator derivative of denominator and denominator of whole square.

So, we will simplify this and we will got the final answer for the Laplace transform of t square cos 2 t. Now let us try to solve the second problem, in which we have to Laplace transform of t e k power minus 2 t sin 4 t. So, here we have only t that is a 1 power of t. So, we have to differentiate only 1 time.

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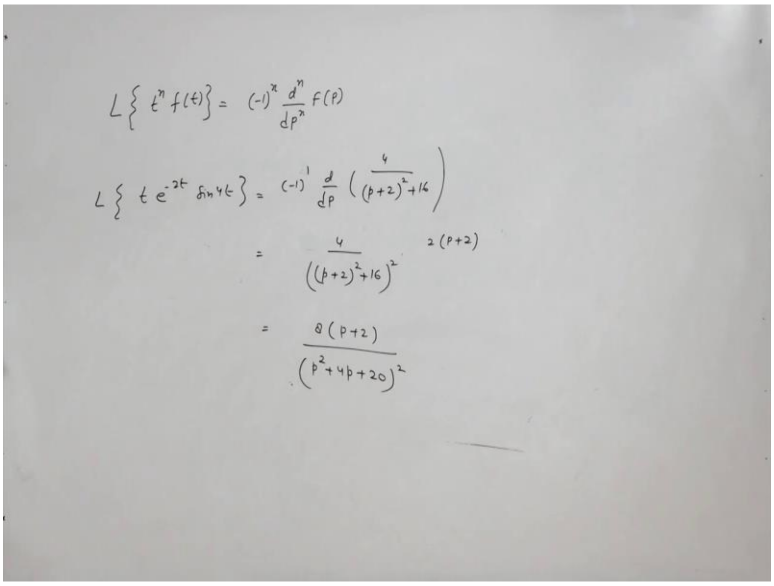

So, that is Laplace transform of t e k power minus 2 t sin 4 t. This will be nothing, but minus 1 k power 1 d by dp of. Now Laplace transform of this expression this is f t. So, Laplace transform of sin 4 t is 4 upon p square plus 16, and using shifting property we replace p by p plus 2. So, it will be nothing, but 4 upon p plus 2 whole square plus 16. This is f t, and Laplace transform of f t is F p.

So, Laplace transform of this f t is this. So, you now you can differentiate here. So, it is 1 by t square and t again that is 2 p plus 2. So, it is nothing, but 8 p plus 2 upon p square plus 4 p plus 20 whole square. So, this will be the Laplace inverse of this expression.

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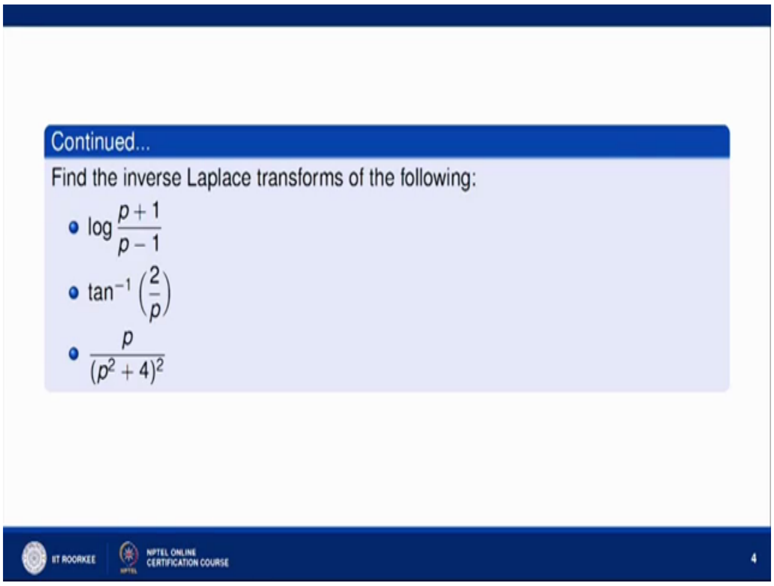

Now, let us try to find out Laplace inverse of these expression using this property. So, whenever we have log tan inverse cot inverse, we always use Laplace transform of t f t equal to minus d by dp of F p property, this property. So, let us try to solve few problems based on this.

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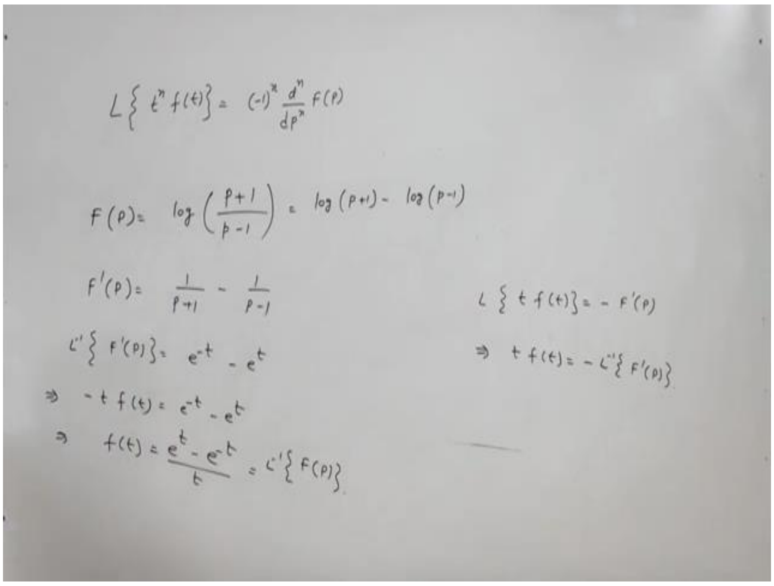

So, this is F p, F p is log of p plus 1 upon p minus 1. So, if we have to find out Laplace inverse of this F p. So, what is F dash p? F dash p will be this is nothing, but log of p plus 1 minus log of p minus 1. So, this is 1 upon p plus 1 minus 1 upon p minus 1. Now take Laplace inverse both the sides. So, Laplace inverse of F dash p is nothing, but Laplace inverse of 1 upon p plus 1, it is e k power minus t minus e k power t. Now Laplace inverse of t f t is what, is minus F dash p minus from this property if you substitute n equal to 1. Here F p is Laplace inverse of this f t.

So,. So, from here we got t f t is nothing, but negative of Laplace inverse of F dash p. So, Laplace inverse of F dash p will be nothing, but minus t f t from that expression is equal to e k power minus t minus e k power t and this f t is nothing, but Laplace inverse of F p. So, therefore, this implies f t will be nothing, but e k power t minus e k power minus t upon t and this is nothing, but Laplace inverse of f t and F p is this. So, hence the Laplace inverse of log p plus 1 upon p minus 1 is e k power t minus e k power minus t by 2.

Now again the second problem. Now F p here is what? F p here the second problem is tan inverse 2 by p.

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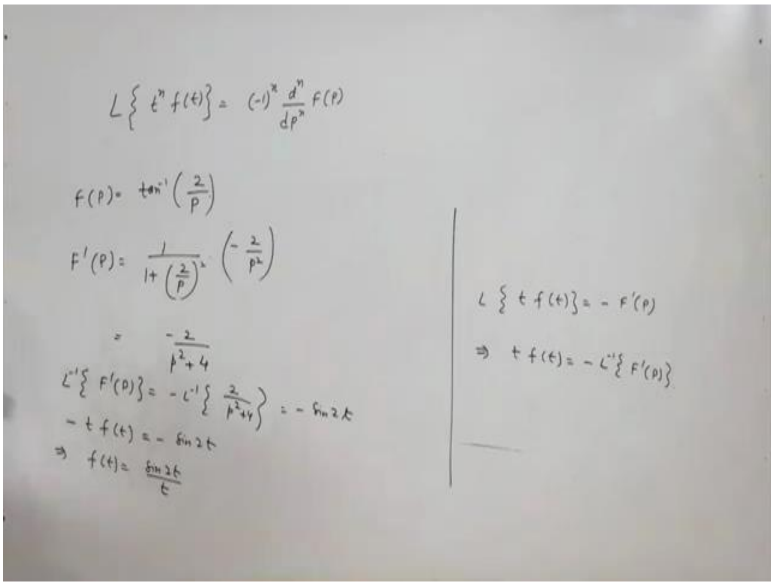

So, what is F dash p 1 upon 1 plus 2 by p whole square minus 2 by p square tan inverse t 1 upon 1 plus t square t square derivative of t again. So, it is nothing, but minus 2 upon p square plus 4. Now take Laplace inverse both sides. So, Laplace inverse of F dash p will be nothing, but minus Laplace inverse of 2 upon p square plus 4, and it is nothing, but minus sin 2 t. and Laplace inverse of F dash p from that expression is nothing, but minus t f t which is equals to minus sin 2 t. So, this implies f t is nothing, but sin 2 t by t and this f t is nothing, but Laplace inverse of this F p. So, these type of problems can be solved using this expression, this expression.

Now the third problem it is F p here is p upon p square plus 4 whole square.

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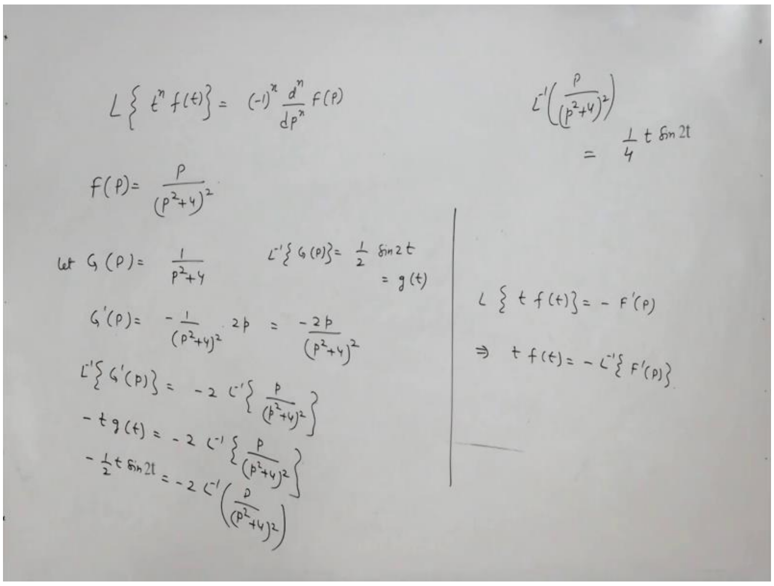

So, let us take, let G p is equals to 1 upon p square plus 4. Of course, Laplace inverse of G p is what? You divide and multiply by 2. So, it is 1 by 2 sin 2 t. Now you differentiate both side G dash p will be nothing, but it is 1 by t minus 1 by d square and derivative of t again 2 p. So, that is nothing, but minus 2 p upon p upon p square plus 4 whole square. So, Laplace inverse, take Laplace inverse both the sides G dash p will be nothing, but minus 2 times Laplace inverse of p upon p square plus 4 whole square.

Now, this Laplace inverse is to find out. Now Laplace inverse of G dash p using this property is nothing, but negative of t g t, where g t is Laplace of G p and Laplace of. So, this is g t, Laplace of g t is G p. So, Laplace inverse of G p is g t which is given by this. So, minus 1 by 2 t sin 2 t, is equals to minus 2 Laplace inverse of this expression. So, Laplace inverse of this expression will be nothing, but 1 by 4 t sin t. So, Laplace inverse of p upon p square plus 4 whole square will be nothing, but 1 by 4 t sin 2 t. So, this is how we can find out Laplace inverse of this expression.

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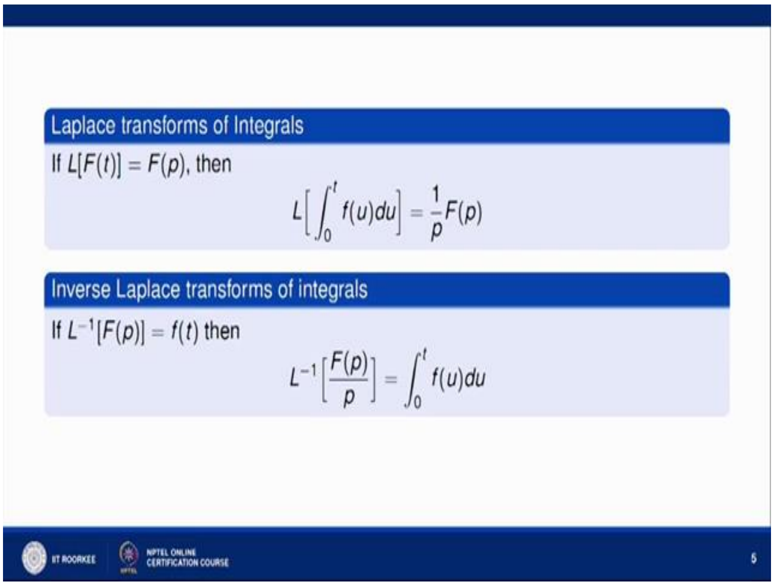

Now, let us see some more properties of Laplace transforms, this is Laplace transform of integers. So, if Laplace transform of f t is F p then Laplace transform of integral 0 to t f u d u is nothing, but F p by p. So, this property is mostly use to find Laplace inverse of F p by p, whenever we have Laplace inverse of F p by p types; that is division by p then this property is more useful, since Laplace inverse of F p by p is given by 0 to t, f u d u.

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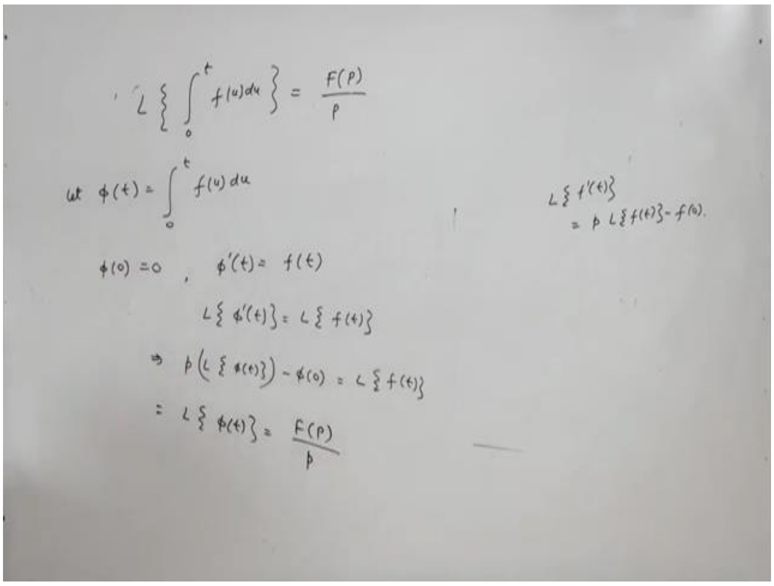

So, how to prove this property, Laplace transform of 0 to t f u d u is nothing but F p by p, and this F p is nothing, but Laplace transform of this f. So, proof is simple you let phi t is equals to 0 to t f u d u. So, what will be phi 0? Is of course, 0, because when you take t equal to 0 value will be 0, and what is phi dash t? So, you have to differentiate with respect to t. So, you have to apply Leibniz theorem here. So, when you apply Leibniz theorem, it is nothing, but f t. now here you take Laplace both the sides. So, Laplace transform of phi dash t will be nothing but Laplace transform of f t, but Laplace transform of phi dash t, Laplace transform of phi dash t is nothing, but p into Laplace transform of phi t minus phi 0, is equals to Laplace transform of phi f t, because Laplace transform of f dash t we have already proved is nothing, but p into Laplace transform of f t minus f 0. So, Laplace transform of phi dash t is nothing but p into Laplace transform of f t minus f 0. So, p into Laplace transform of phi t a minus phi 0 is equals to Laplace of f t because these 2 are equal.

Now phi 0 is 0. So, Laplace transform of phi t is nothing, but Laplace transform of f t. We are taking as F p. So, it is F p and this p will come in the denominator. So, divided by p and phi t is nothing, but this expression. So, Laplace transform of integrals 0 to t f u d u is nothing, but F p by p, where F p is Laplace transform of f t. So, whenever we have division by p in the Laplace inverse, then we mostly use this property.

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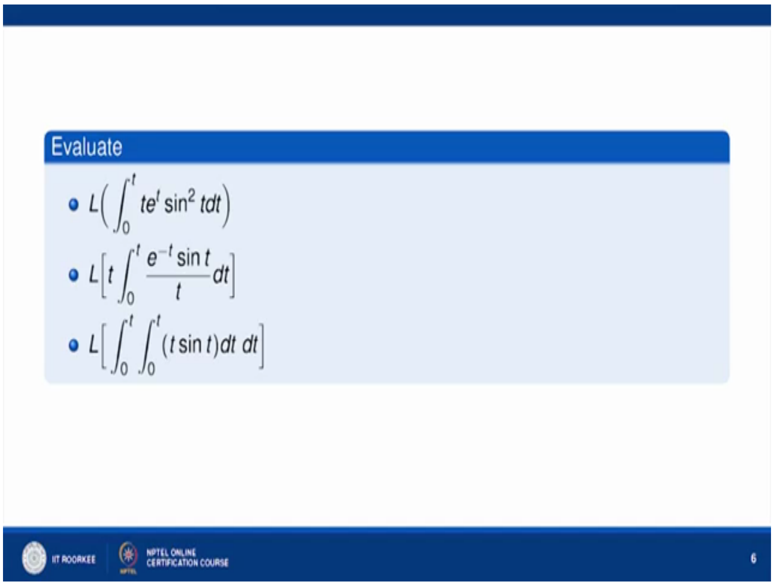

Let us see some problems based on this. Now 0 to t integral 0 to t t e k power sin square t, this simple problem, so 0 to t t e k power t sin square t. Suppose we have to find out Laplace of, we have to find out Laplace of this.

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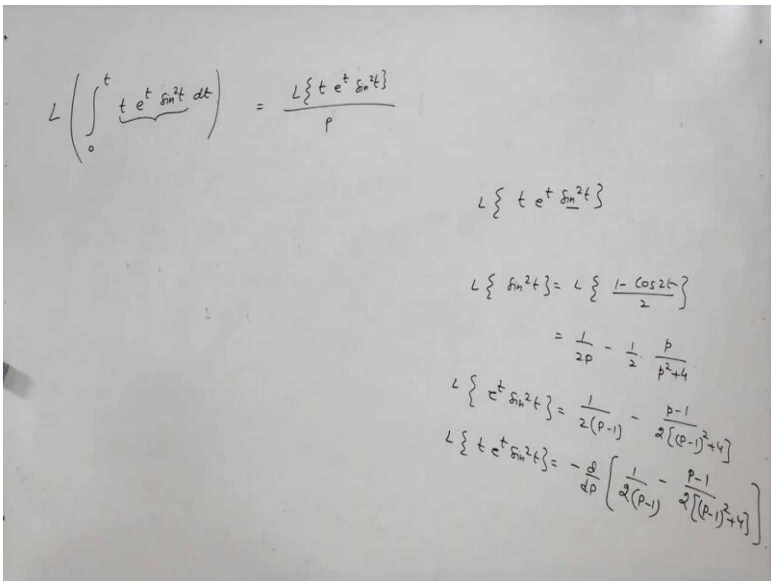

So, we have to find out Laplace of this. So, how to find Laplace of this? Laplace of this, for Laplace of this first find this is f t. So, by this property this is nothing, but F p by p that is Laplace of t e k power t into sin square t by p, because this is f t. So, by using this property, by using this property Laplace of f t will be here divided by p. So, this is why we have p here, and this Laplace we can easily find out, because Laplace of t e k power t sin square t. First find Laplace of t sin t square, apply a shifting property of e k power t, and for t we already know that Laplace of t f t is minus of F dash p. So, it is nothing, but. So, what is Laplace of sin square t first? Laplace of 1 minus cos 2 t by 2. So, it is 1 by 2 p minus 1 by 2, it is p upon p square plus 4, and Laplace of e k power t sin square t will be nothing, but you replace p by p minus 1 using shifting property, so 2 upon p minus 1 minus p minus 1 upon 2 into p minus 1 whole square plus 4.

And for Laplace transform of t e k power t sin square t, you simply take minus of d by dp of this f, this f s, you take d by dp of this f s. So, you can differentiate it respect to p and you can find out the final answer. So, that divided by p will be the Laplace transform of this expression. So, in the first expression it is, here we have a Laplace transform of this. Now, how to find out Laplace transform of this expression? So, that is also we can find.

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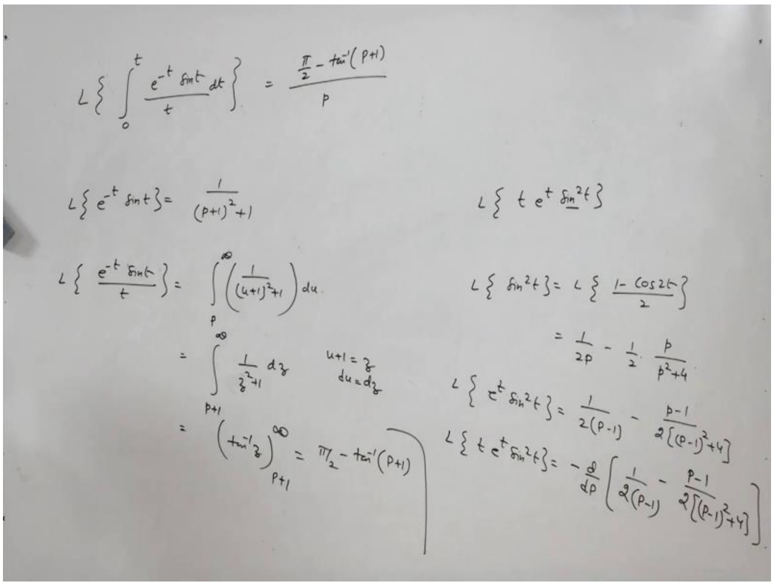

So, first we will find Laplace transform of a integral part and t into f t will be we use minus of F dash p; that property. So, how to find Laplace of integral 0 to t e k power minus t sin t by t, so it involves so many things. First we have to find out Laplace of sin t sin t is 1 upon p square plus 1. For e k power minus t we have to apply shifting property you replace p by p minus, you replace p by a p plus 1 because a here is minus 1.

So, Laplace of e k power minus t sin t will be nothing, but 1 upon p plus 1 whole square plus 1, because Laplace of e k power a t f t is f of p minus a, and here a is minus 1 and for Laplace of e k power minus t sin t by t. So, it is f t by t, and for f t by t we are having for Laplace of f t by t. What we have? We have Laplace of f t by t, it is p to infinity f u d u Laplace of f t by t is nothing, but p to infinity f u d u. So, this we can easily integrate. The integration of this will be you can take u plus 1 as some z. So, this is d u equal to d z. So, this is nothing, but when this is p, it is p plus 1 and when it this is infinity, this is infinity 1 upon t square plus 1 into dt for d z, sorry it is z square it is z square and d z which is nothing, but tan inverse z infinity plus 1 to infinity, and it is pi by 2 minus tan inverse of p plus 1.

So, we have find Laplace of this. Now for 0 to t it is nothing, but F p by p where F p is nothing, but Laplace transform of f t, here f t is a entire expression. So, Laplace transform of f t is this. So, this is a F p this is F p. So, what is Laplace transform of this? This is nothing, but pi by 2 minus tan inverse p plus 1 upon p, this property we use. Now

0 to t f u d u is 1 by p F p. Now for the last problem for the third problem of this slide we have to take, use this property 2 types 0 to t 0 to t sin t. So, first we will find Laplace transform of this expression, and then take this as a f t find out Laplace transform of entire expression. So, basically we have to use this property, this property 2 types. So, you can try this problem. Now we can also find Laplace inverse, Laplace inverse of some problems where we have division by p. So, how can we solve the last problem of this slide? Now here we have integral 2 times 0 to t 0 to t t sin t dt dt, where 2 times integral. So, we have to use this property 2 times. So, how can we use this property 2 times? This is what, this is Laplace transform of 0 to t 0 to t, t sin t dt, dt.

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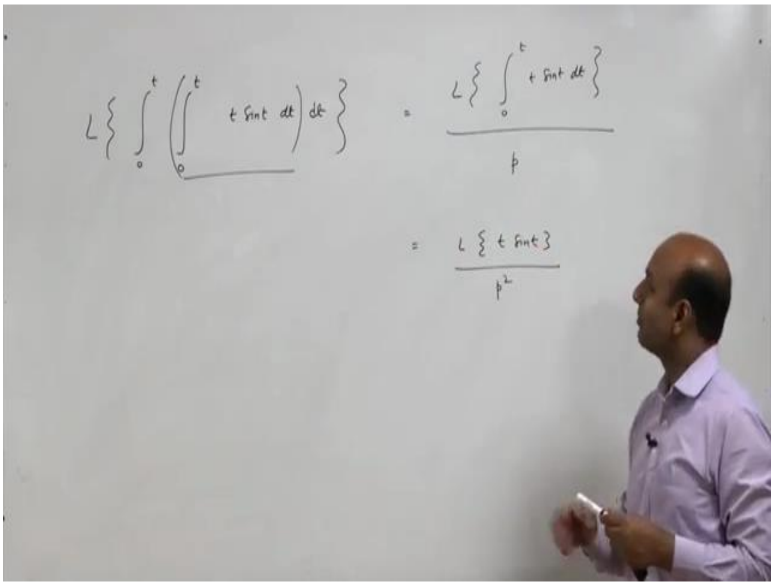

So, first we can find out Laplace transform of this f t, this is nothing, but F p by p where F p will Laplace transform of t sin t. So, for t sin t, you can easily find out, you can first find out Laplace of sin t and using Laplace transform of t f t is minus F dash p. We can find out Laplace transform of t sin t then this is Laplace of this is F p again we will use 0 to t f t dt and the same property F p by p we can find out Laplace transform of this entire expression.

So, basically this expression will be nothing, but Laplace transform of 0 to t t sin t dt upon p the same on to say, and again for Laplace transform of this this is nothing, but Laplace transform of t sin t upon p square. So, this will be the basically the final expression for this problem. Now how can you find out Laplace inverse of some problems, where we have division by p F p by p types?

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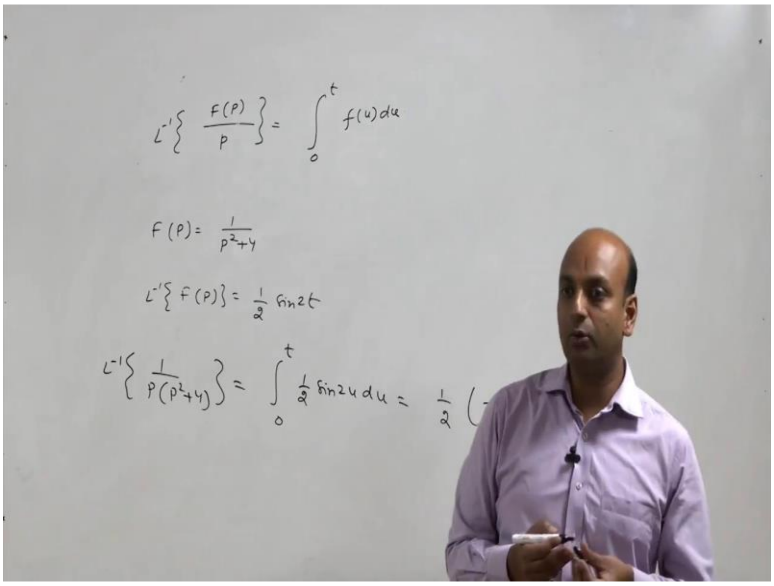

So, how can you find out? This is Laplace transform of, now F p by p is what - it is 0 to t f t dt f u d u, where Laplace transform of f u is F p. So, here we have Laplace inverse of this expression. So, F p is 1 upon p square plus 4.

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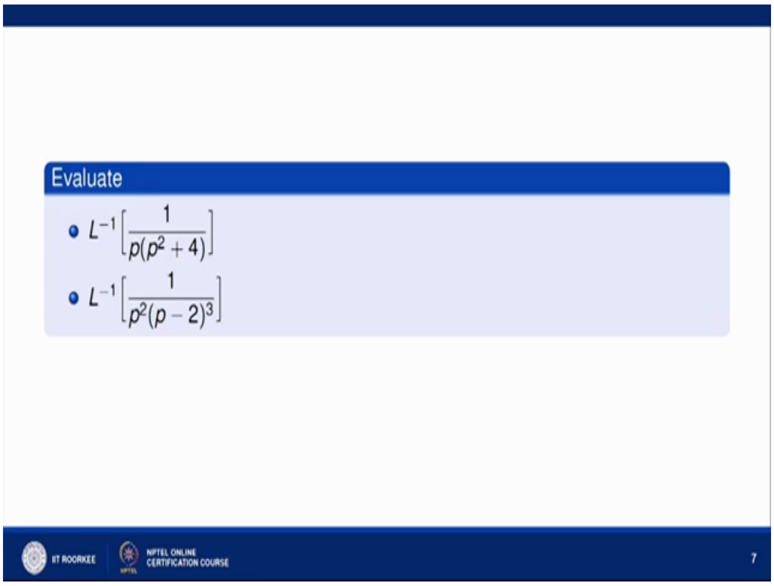

So, Laplace inverse of F p will be nothing, but 1 by 2 sin 2 t. So, if we put it here. So, Laplace inverse of 1 upon p into p square plus 4 will be nothing, but 0 to t. Now this is f t the Laplace inverse of this F p. So, this is nothing, but 1 by 2 sin 2 u d u. So, this we can easily integrate 1 by 2 integral of sin 2 u this minus cos 2 u by 2 minus cos 2 u by 2 from 0 to t. So, solving this we can easily find out Laplace inverse of the first problem. Now, for the second problem, we have to use this property again 2 times, because we have 1 by p square. We have 2 ways we can use partial fraction also but it is better if you use division by p concept, because that is less time consuming, and it is simple to solve also. What is Laplace inverse of 1 by p minus 2 whole cube?

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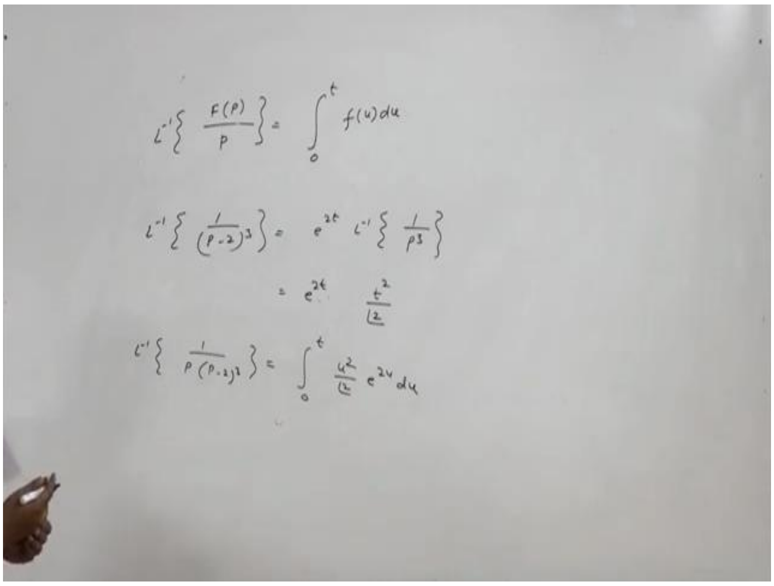

So, that is nothing, but e k power 2 t Laplace inverse of 1 by p q and for 1 by p q. It is factorial it is t k power p n plus 1 and it is 2. So, it is factorial 2 upon factorial 2, and Laplace inverse of first division by p is nothing, but 0 to t f u d u. And again division by p, so we have to integrate once again 0 to t of this entire expression; so that will give us a final answer.

Thank you very much.