Mathematical methods and its applications
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Lecture – 28
Properties of Laplace transforms-III

So, we were studying some of the properties of Laplace transforms. We have already seen that what is shifting property and how can we solve the questions related to shifting property and change of scale property also we have seen in the last lecture, and some problems based on change of scale property also we have seen.

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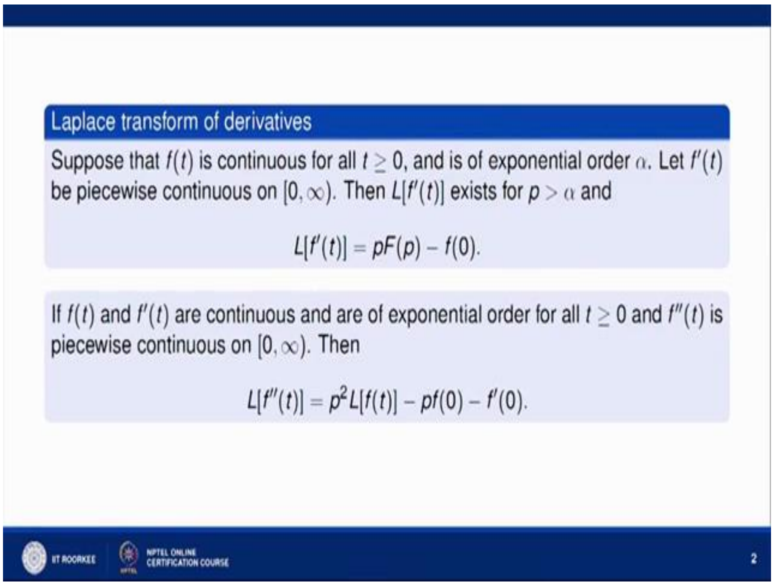

Now, let us see some more properties of Laplace transforms. So, this is Laplace transform of derivatives. So, if function f t is continuous for all t greater than and equal to 0 and is of exponential order say alpha and f dash t is piecewise continuous on 0 to infinity. Then Laplace of f dash t exist for p greater than alpha and Laplace of f dash t which is a derivative of f respect to t is given by p F p minus f 0, where F p is nothing but Laplace of f t.

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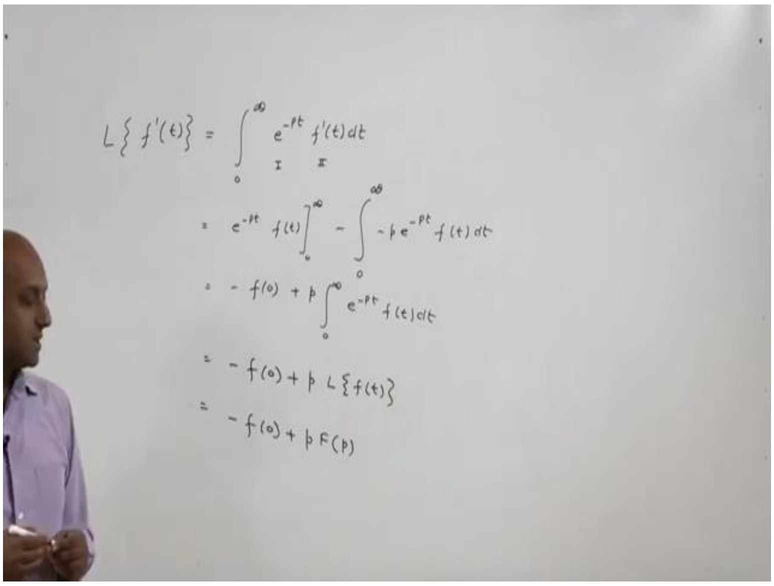

So, how can we proof this? So, Laplace of f dash t. So, what is Laplace of f dash t? It is 0 to infinity e k power minus p t f dash t dt. Now, you can apply integration by a parts you can use integration by a parts to find out the integral of this. So, first this, second this let us suppose. So, first as it is integral of second which is f t from 0 to infinity minus integral derivative of this, integral of second dt and it is from 0 to infinity. Now, when t is tending to infinity, this value tends to 0, why because f t is of exponential order. Since, f t is an exponential order, so there will exists some k and alpha such that mod of f t will be less than or equals to k e k power n alpha t. So, mod of e k power minus p t into f t will be less than equals to k into e k power alpha t into e k power minus p t, because it is a non negative quantity we can multiply both the sides, it will not change the inequality.

So, this is equal to k into e k power minus p minus alpha times t and as t tending to infinity, so since p is greater than alpha we are assuming, so this value will tend to 0. So, hence the limit, hence this quantity will tends to 0 as t tends to infinity. And this is because of the property of exponential order. So, this will tends to 0. So, as t tend to infinity, this will tends to 0; and as t tends to 0, this is nothing but minus f 0 because e k power 0 is 1. So, this is minus minus plus p times 0 to infinity e k power minus p t f t dt.

And what is the integral? The integral is nothing but Laplace of f t. So, it is minus f 0 plus p into Laplace of f t. And Laplace of f t, we are calling as f p though it is minus f 0 plus p into F p. So, this is how we can obtain this result that Laplace of f dash dt will be nothing but p F p minus f 0. Now, on the same lines if f t and f dash t continuous and are of exponential order, and f double dash t is piecewise continuous, then Laplace of f double dash t is given by this expression. So, this also we can prove, it is very easy to prove this, once we obtain for f dash t.

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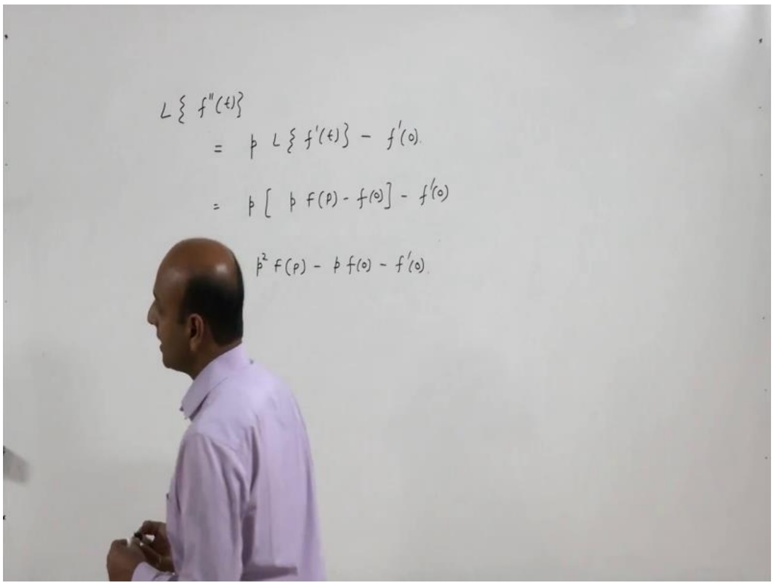

Now, what is Laplace of f double dash t? So, the first expression which we have just obtained, you simply replace f by f dash in that expression, you simply replace f by f dash. So, what we will obtained, this is p into Laplace of f dash t minus f dash 0, because you see that this result the first result holds for every p I mean every f. So, this will hold for f dash also. So, we replace f by f dash in the first expression. So, we will we get this result this expression. Now, this is equals to p into what is Laplace of f dash t. Now, you again apply the result, it is this minus f 0 minus f dash 0. So, this is nothing but p square f p minus p f 0 minus f dash 0. So, hence we obtain the next result of Laplace of f double dash t.

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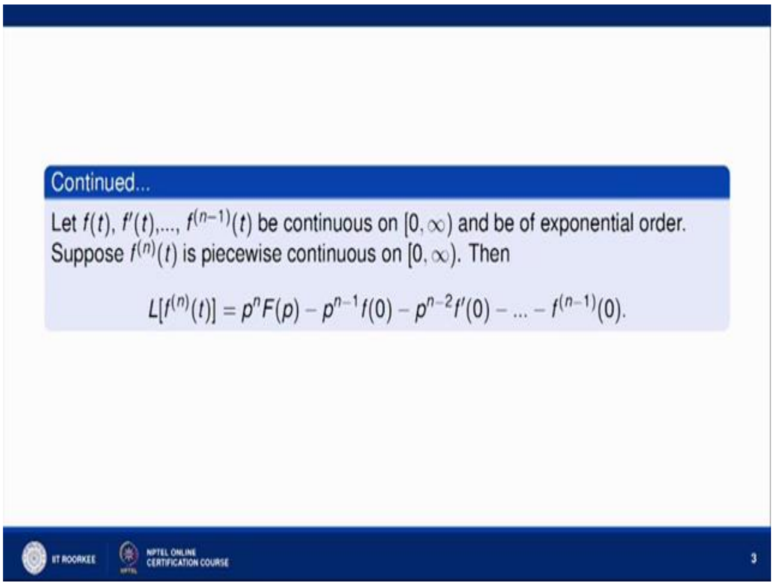

Now, we can generalize this result for nth order derivative for Laplace of f of I mean n th or derivative of f respect to t will be given by this expression. This can be proved very easily, because in order to prove this, we can use mathematical induction. When n equal to 1 that is Laplace of f dash t, we have just proved that Laplace of f dash t is p F p minus f 0 when you substitute n equal to 1 in this expression. So, we will get by the same expression that is p F p minus f dash 0, p F p minus f 0 sorry.

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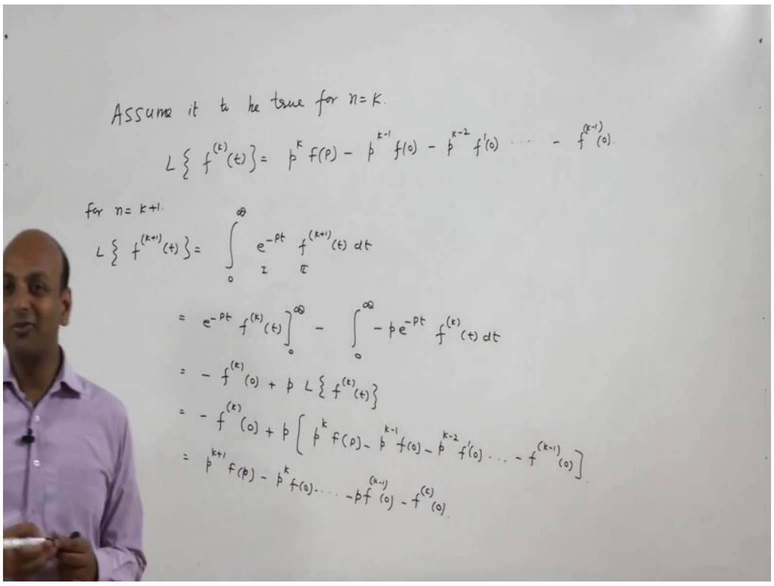

Now, let us assume it to be true for n equal to k. So, assume, it to be true for n equal to k. So, what we have assumed, we have assumed that Laplace of f k th derivative is k th derivative with respect to t is nothing but you can simply replace n by k. So, it is p k power k F p minus p k minus 1 f 0 minus p k minus 2 f dash 0 and so on minus k minus 1 derivative of at t equal to 0. Now, we have to show the same result for n equal to k plus 1, this is mathematical induction. We first prove it for n equal to 1; assume it to be true for n equal to k, and try to prove the result for n equal to k plus 1. If you obtain the result for k equal to n equal to k plus 1, this means the result hold for all n this is the concept of mathematical induction.

Now, for n equal to k plus 1, Laplace of f k plus 1 t is nothing but it is 0 to infinity e k power minus p t k plus 1th derivative respect to t into dt. Again by part first second, so it is nothing but first as it is integral of second, 0 to infinity minus integral derivative of first it is minus p e k power minus p t, integral of second it is kth derivative t dt. So, again since all the derivatives up to n minus 1 that is up to k here are of exponential order some exponential order. So, this will tends to 0 as t tends to infinity, this can be proved in the same lines. And when t equal to 0 it is nothing but minus kth derivative at t equal to 0 minus minus plus p, and it is nothing but Laplace of kth derivative of t this. So, this will be the expression.

Now, the Laplace of this we have already assumed this equal to this expression. Now, this is minus f kth derivative at t equal to 0 plus p into Laplace of kth derivative of t we have assumed equal to this. So, it is p k F p minus p k minus 1 f 0 minus p k minus 2 f dash 0 and so on minus k minus 1 derivative at t equal to 0. So, this is nothing but p k plus 1 from this expression F p minus p k f 0 and so on minus f k th derivative at 0. And this is at infinity, it is 0; and at 0, it is minus k. And when you multiply by p, it is p k plus 1 into f k it is k, so p k p times yeah it is p times sorry. So, p times f k minus 1 and minus f k derivative at 0. So, what we have obtained that if we replace in this expression and by k plus 1. So, the result holds this we have shown that if we replace in this expression and by k plus 1, so we hold this the result hold hence the results holds for all n. So, hence the hence proved this result.

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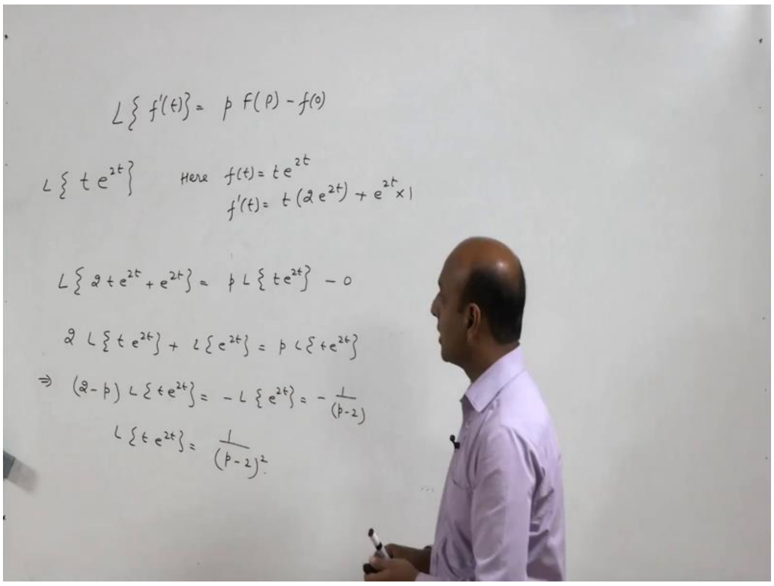

Now, the next is how can we solve some problems using derivatives using this concept. So, just for illustration, let us try to find out Laplace of these simple problems using derivatives, Laplace of derivatives. So, we already know that Laplace of f dash t is nothing but p F p minus f 0. So, what is the F p here? Fp is nothing but Laplace of f t. So, we have to find Laplace of t e k power 2 t.

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So, here f t is t e k power 2 t. So, we if we apply this result though Laplace of f dash t. So, first we find f dash t, what is derivative of f respect to t, it is first as it is, derivatives of second plus second as it is derivative of first. So, Laplace of use this result Laplace of f dash t that is 2 t e k power 2 t plus e k power 2 t is nothing but p F p, p F p is Laplace of f t e t e k power 2 t, F p is nothing but Laplace of f t minus f 0. So, what is f 0? It is 0. So, it is nothing but 2 times Laplace of t e k power 2 t plus Laplace of e k power 2 t equal to p Laplace of p e k power 2 t. So, it is nothing but this implies 2 minus p Laplace of t e k power 2 t is equals to minus of Laplace of e k power 2 t which is nothing but minus of 1 upon p minus 2. So, Laplace of p e k power 2 t will be nothing but this will go in the denominator and negative if we adjust though this is nothing but 1 upon p minus 2 square. So, this will be the Laplace of this expression.

So, this we can directly find of course, because we know the Laplace of t that is 1 upon p square and using shifting property, if we replace p by p minus 2 though we get back of this expression, but it is just for illustration that how can we solve some problems using Laplace of derivatives also. So, similarly, we can solved the second problem also, it is also based on Laplace transform of derivatives that also we can find out. Otherwise, also we can find out the Laplace on sin square by writing sin square as 1 minus cos 2 t whole divided by 2 and then use linearity property of Laplace transform we can find out Laplace transform of sin square t. Now, let us try to solve few problems again of Laplace inverse based on Laplace transform derivatives.

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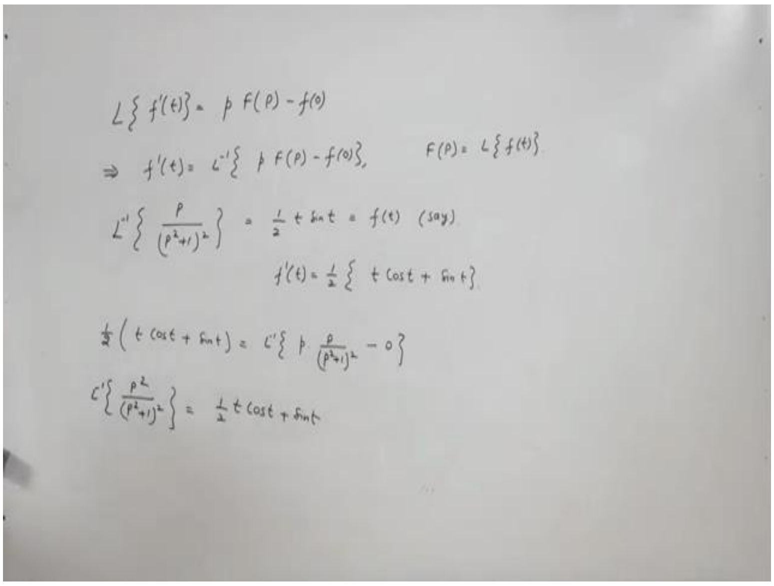

So, we get back the same result again. Now, this is Laplace transform of derivatives. So, what does it imply? This implies f dash t is nothing but Laplace inverse of p F p minus f 0. And here F p is nothing but Laplace of f t. So, this expression we can derive from Laplace transform of derivatives. Now, it is given towards at Laplace inverse of p upon p square plus 1 whole square is equal to 1 by 2 t sin t, it is given to us in the problem. So, suppose it is f t say.

So, if it is f t, so what is f dash t, 1 by 2 t cos t plus sin t. So, if I use this expression, so f dash t is 1 by 2 t cos t plus sin t will be equal to Laplace inverse of p into F p, F p is Laplace transform of f t. And that is nothing but p into p upon p square plus on whole square. If Laplace inverse of this expression is f t, so Laplace of f t will be p upon p square plus on whole square. And minus f 0, f 0 from here is nothing but 0. So, hence we get back to Laplace inverse of p square upon p square plus 1 whole square which is nothing but 1 by 2 t cos t plus sin t. So, in this way we have solved first part of this problem.

Now, the second part can be solved. Now, what is Laplace inverse of p square upon p square plus 1 whole square, we have just obtained that it is equals to 1 by 2 t cos t plus sin t. It is this expression we have just obtained half of whole expression.

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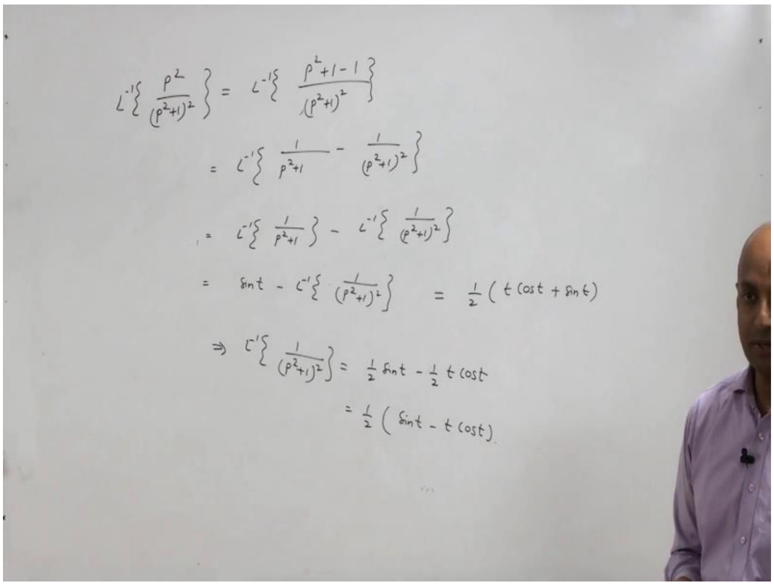

Now, this we have obtained in the first part. In order to obtain the second part, Laplace inverse of p square upon p square plus 1 whole square can be rewritten as Laplace inverse of p square plus 1 minus 1 upon p square plus on whole square where does subtract to 1 in the numerator. This is nothing but Laplace inverse of 1 upon p square plus 1 minus 1 upon p square plus 1 whole square. Now, this is nothing but Laplace inverse of 1 upon p square plus 1 minus Laplace inverse of 1 upon p square plus 1 whole square. And this is nothing but sin t minus Laplace inverse of 1 upon p square plus 1 on whole square, but Laplace inverse of this is equal to this. This we have obtained in the first part of this problem.

So, this is equal to 1 by 2 t cos t plus sin t. So, what will be the Laplace inverse of 1 upon p square plus 1 whole square? So, this implies Laplace inverse of 1 upon p square plus 1 whole square is nothing but you can put this in the right hand side and this here, this will be nothing but half of sin t minus 1 by t 2 cos t. So, that is nothing but 1 by 2 sin t minus t cos t. So, this will be the Laplace inverse of 1 upon p square plus 1 whole square. So, hence we can say that the Laplace transform derivatives can also be used to find out the Laplace inverse of some problems.

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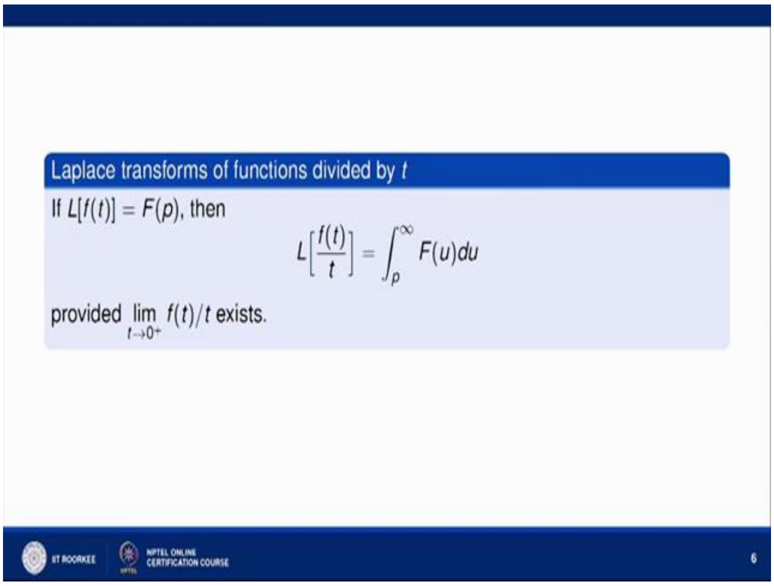

Now, the next property if Laplace transform f t is F p then Laplace transform of f t by t is nothing but integral p to infinity F u du, where this is F u is nothing but again Laplace transform of f t. So, provided this limit exist, because function must be piecewise continuous for existence of Laplace transform. So, f t by t will be piecewise continuous, if limit t tend in to 0 plus f t by t exist, so that is required in this problem. So, what is how to find this? So, take this side integral p to infinity F u du the take the right hand side and we will obtain the left hand side.

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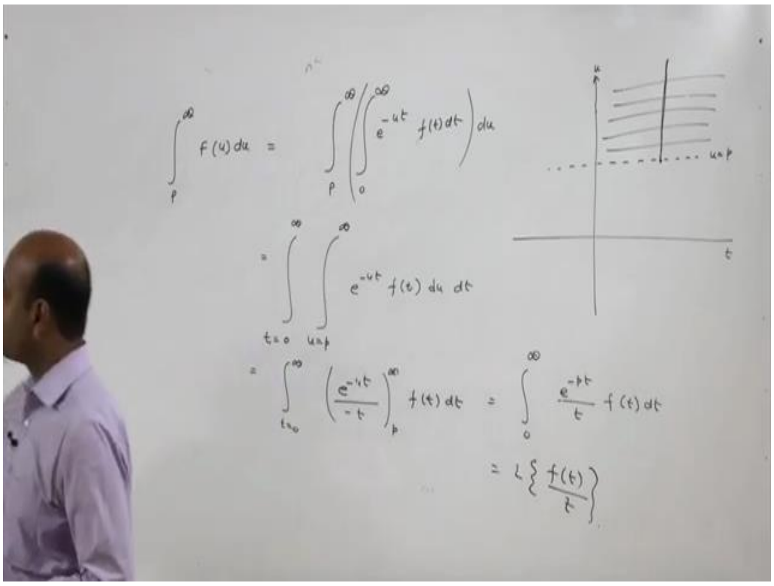

So, what is F u, what is capital F u? It is nothing but Laplace transform of f t. So, it is p to infinity e k power minus u t instead of p we have u, f t dt, of course, integral from 0 to infinity it is dt and whole multiplied by du. So, this we are having in this problem. Now, this is the double integral. So, now, we will make use of change of order of integration. How can we proceed for changing the order of integration? Suppose this is line of t, this is u. Now, t is varying from 0 to infinity, this is the limit for t, t is varying from 0 to infinity. So, this is first and forth quadrant, t is varying from 0 to infinity, and u is varying from p to infinity. So, suppose this is u equal to p line, and u is varying from p to infinity that is from this to this side. So, this is the region, this is the required region, this is the required region of integration.

Now, we have to change the order of integration that means, it is e k power minus u t f t and d u and then dt. So, first you want the limits for u and then you want the limit for u t. So, we have to make a strip parallel to u-axis because you want du first. So, we make a strip parallel to u-axis, so u-axis this way the strip here. Now, here u is p and goes up to infinity. So, u is varying from p to infinity and t is varying from 0 to infinity. So, e is varying from 0 to infinity, so that is how we can change this order of integration.

So, now this is equals to t equal to 0 to infinity. Now, we have the u term only in this expression. So, we can integrate this respect to u. So, it is nothing but e k power minus u t upon minus t, because their integration respect to u. So, this is this, p to infinity f t dt. And when u tending to infinity, it is tending to 0. And when u tending to p, it is nothing but 0 to infinity e k power minus p t upon t f t dt because when t tend to infinity tend into 0 I mean sorry it is limit for u. So, u tend to infinity tend to 0, but when u is p, so it is e k power minus p t upon t. So, it is nothing but Laplace transform of f t by t. So, it is nothing but Laplace transform of f t by t. So, hence we obtain this result that Laplace transform f t by t is nothing but p to infinity f u du. Now, how this result can be used to find out Laplace transform of some function or Laplace inverse.

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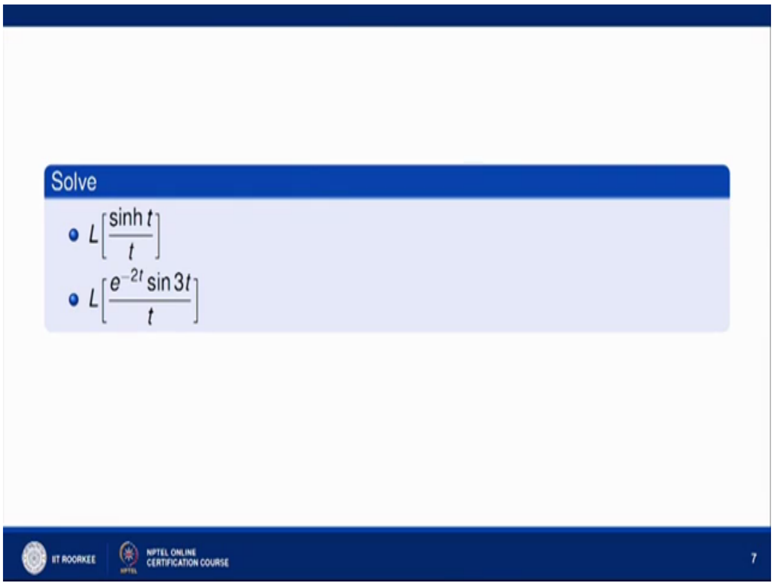

Let us see by solving few problems based on this. So, let us find out Laplace transform of sin hyperbolic t upon t, very simple problem.

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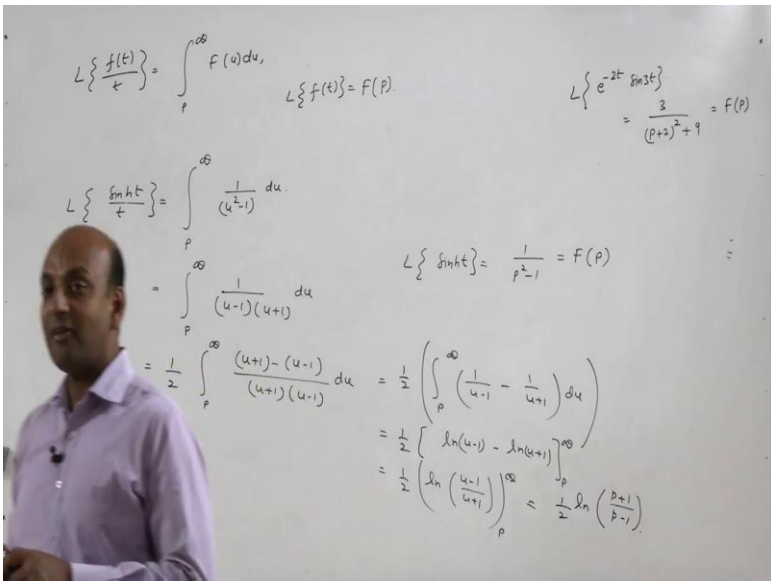

So, we know this is result Laplace transform of f t by t is something p to infinity f u du, we are Laplace transform of f t is F p. So, we know this result. So, using this result, we will try to solve these two problems. So, Laplace transform of sin hyperbolic upon t. So, first we have to find out Laplace transform of numerator quantity. Here numerator is here f t is sin hyperbolic t. So, what a Laplace transform of sin hyperbolic t? This you already know, this is nothing but 1 upon p square minus 1, sin hyperbolic t re a upon p square minus a square. And division by t is here for division by t, we have integral of p to infinity F u du it is integral p to infinity. So, here this is f p.

So, what will be F u 1 by u square minus 1 into du? No, now, we can integrate this using partial fraction or. So, it is u minus 1 into u plus 1 du. The difference of these two difference of this and this is two which is a quotient quantity though you we can multiply a 2 and divide by 2 in the numerator. So, 1 by 2 p to infinity, and 2 can written as u plus 1 minus u minus 1. So, this is nothing but 1 by 2 integral p to infinity 1 upon u minus 1 minus 1 upon u plus 1 du. This is the partial fraction of this. So, this is equals to 1 by 2.

Now, integral of this is ln u minus 1 and integral of this is ln u plus 1 from p to infinity. So, this is 1 by 2 ln u minus 1 upon u plus 1 from p to infinity; at infinity, it is tending to 1, log 1 is 0; and at p it is tending to ln p minus 1 upon p plus 1. So, because of negative sign, integral will change this log u plus 1 upon u minus p plus 1 upon p minus 1 will come because we have upper limit minus lower limit. So, because of that negative these two will interchange So, it is p plus 1 upon p minus 1. So, this will be the Laplace transform of the first problem.

The second problem can be solved on the same lines. First you find Laplace transform of the numerator. You see numerator what is f t here in the numerator, it is e k power minus two t sin 3 t. What a Laplace transform of sin t sin 3 t, it is 3 upon p square plus 9. And for e k power minus 2 t, you simply replace p by p plus 2 that means, the Laplace transform of e k power minus 2 t sin 3 t will be nothing but 3 upon p plus 2 whole square plus 9. So, this will be F p here this will be F p for this problem for the second problem. And we have a division by t. Now, for f t by t, we will apply this property. So, this will be nothing but p to infinity F u du. So, we have to integrate this up from p to infinity, so that will give the Laplace transform of the second problem.

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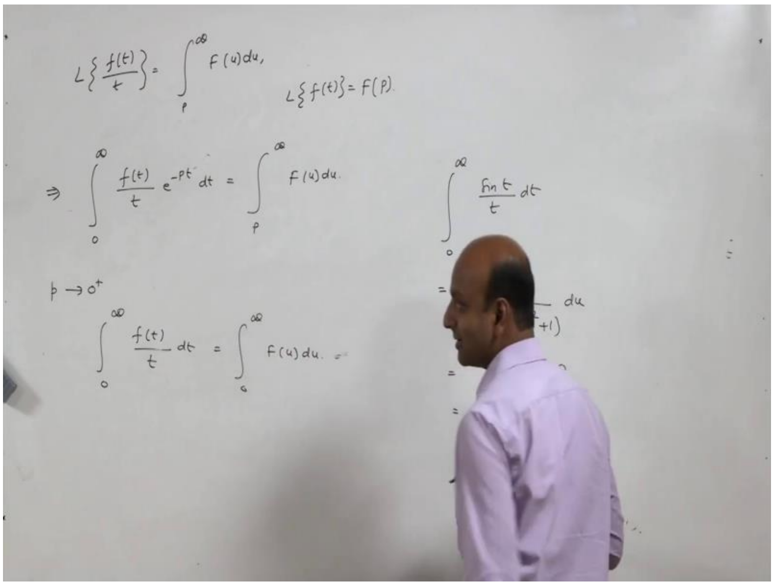

Now, let us try to prove this simple problem. So, from here we can see that Laplace transform of f t is nothing but 0 to infinity f t by t e k power minus p t dt will be equals to p to infinity F u du, this is from this expression. Now, you take the limit p tend into 0 plus both the sides, you take the limit p tend to 0 plus both the side. So, what will obtained, it is 0 to infinity f t by t dt, because e k power 0 is 1, and is equals to 0 to infinity F u du. So, hence we got the first part of this problem. Now, to prove that 0 to infinity sin t by t is integral of sin t by t is pi by 2.

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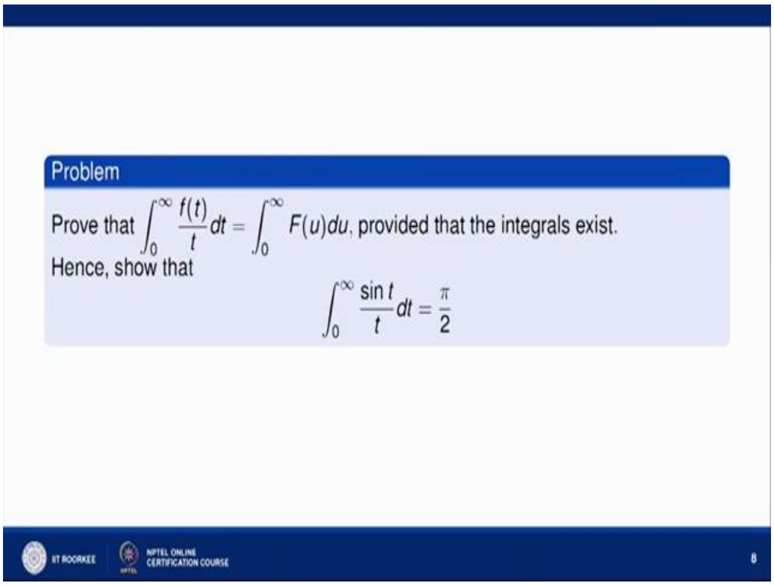

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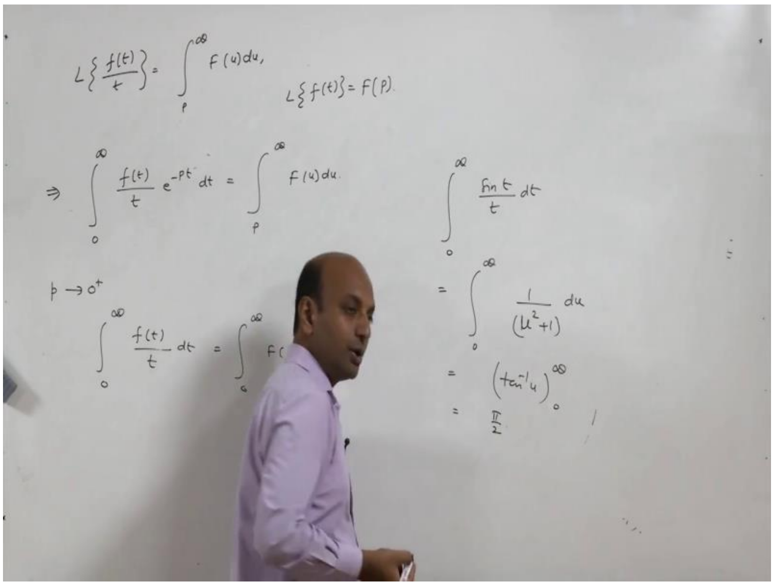

So, what is f t for if we compare with this f t sin t? So, what is this problem now, it is 0 to infinity sin t by t dt. So, it is equals to integral p to infinity. Now, F u is nothing but Laplace of f t; and Laplace of sin t is nothing but 1 upon u square plus 1 du. So, it is tan inverse from p to infinity, and from 0 to infinity sorry because we have 0 to infinity in this result for 0 to infinity. So, at infinity is pi by 2 at 0 is 0. So, we have this value that integral of 0 to infinity sin t by t nothing but pi by 2. So, some integrals having limit lies to 0 to infinity or something can also be solved using Laplace transforms.

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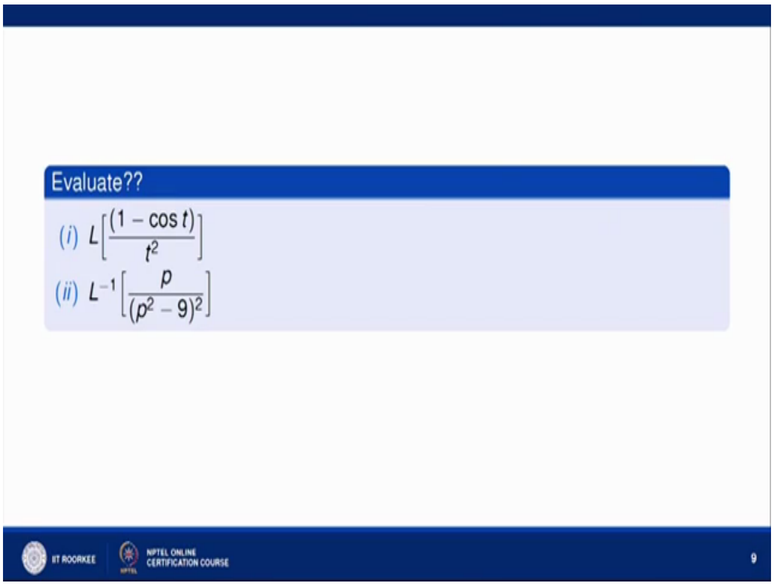

Now, we can also solve these problems based on these properties. So, I am just illustrating how to solve the first part or the second part.

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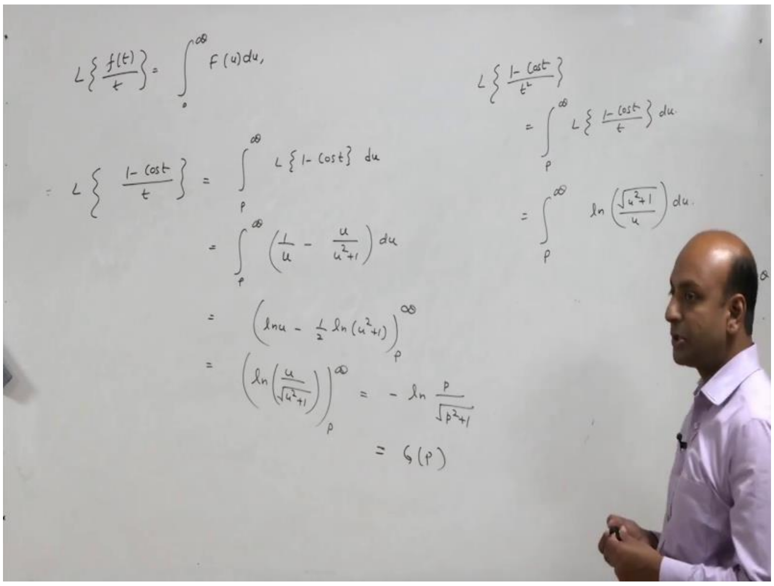

So, it is in this problem 1 minus cos t upon t square, we have to integrate 2 times. What this integral is basically it is 1 minus cos t? First you will find Laplace of this, which is nothing but integral p to infinity Laplace of 2 minus cos t du. If we are taking as F u. Now, Laplace of, so this we can easily find this p to infinity 1 by u minus, it is u on u square plus 1 du. So, it is ln u minus 1 by 2 ln u square plus 1 from p to infinity. So, it is nothing but ln u upon under root u square plus 1 from p to infinity at infinity is tend to 0 at p it is sending to negative of ln p upon under root p square plus. So, this is Laplace transform of this quantity.

For Laplace transform of 1 minus cos t upon t square, we again have to apply the same property, so what is Laplace transform of 1 minus cos t upon t square? It is nothing but p to infinity Laplace transform of 1 minus cos t upon t and du, because this property holds for f t by t. So, if you compare with this. So, what is the f t 1 minus cos t by t? So, it is Laplace transform of one minus cos t by t into d u and that we already calculated. So, it is nothing but p to infinity. So, here this is the f. So, this is new F p u suppose this is G p. So, it is G u. So, it is l n under root u square plus 1 upon u into du. So, this can easily be integrated if you take u as suppose ten theta you substitute here u equal to ten theta proceed your integration. So, this integral can the value of the integral can be find. So, how to find Laplace inverse of p upon p square minus 9 whole square using this property?

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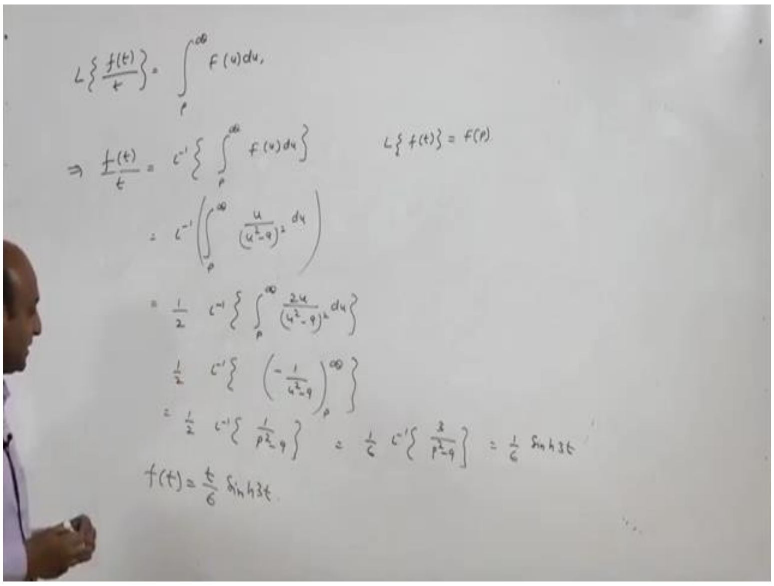

So, from this we got f t by t is nothing but Laplace inverse of p to infinity F u du. Here what is F u? It is Laplace of f t. So, this f t is nothing but Laplace inverse of this. So, if you obtain f t that means, we obtain Laplace inverse of this expression. So, take this as f p. So, Laplace inverse of p to infinity, it is u upon u square minus 9 whole square du. So, this is because I am taking the expression given inside the bracket as F p, and Laplace inverse of this F p to find out which is nothing but this f t, because we know this.

So, now, we divided multiply by 2 and 1 by 2, it is Laplace inverse of the derivatives of denominator u square minus 2 is in the numerator. So, it is a minus 1 upon u square minus 9 from p to infinity and it is 1 by 2 Laplace inverse of at infinity it is tending to 0 and at p it is nothing but 1 by p square minus 9. Now, you divide by a divide multiply by 3. So, it is 1 by 6, Laplace inverse of 3 upon p square minus nine and this is nothing but 1 by 6, it is hyperbolic 3 t. So, what will be f t? Ft will be t by 6 sin hyperbolic 3 t. So, and f t is nothing but Laplace inverse of f p that is this quantity. So, Laplace inverse of this quantity is nothing but this expression. So, hence we can find Laplace inverse also using this property.

Thank you.