Mathematical methods and its applications
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Lecture – 27
Properties of Laplace transforms – II

Hello everyone, so in the last lecture we were studying about properties of Laplace transform. Some of the properties we have seen like shifting property first shifting or first translation or shifting property, then we see second translation or shifting property some problems based on this that we have seen in the last lecture. So, now, we will see some more prosperities of Laplace transforms in this lecture.

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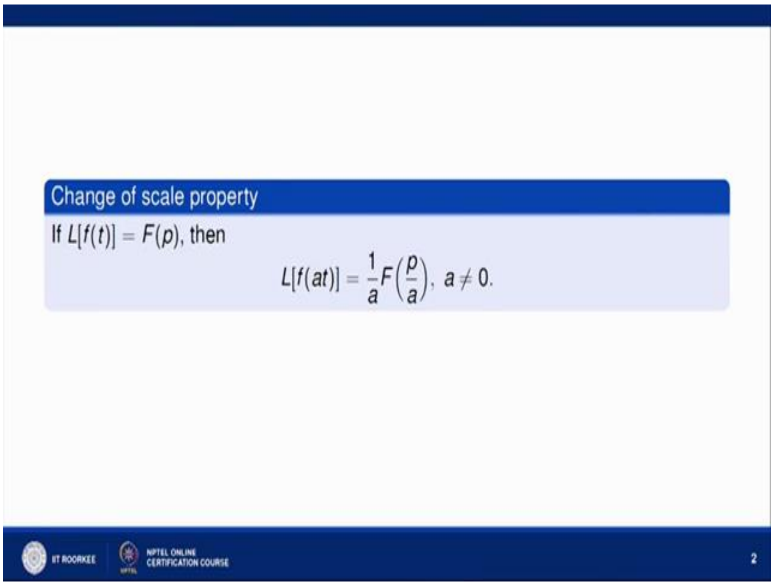

So, the next property is change of scale property. Now, what does it state?

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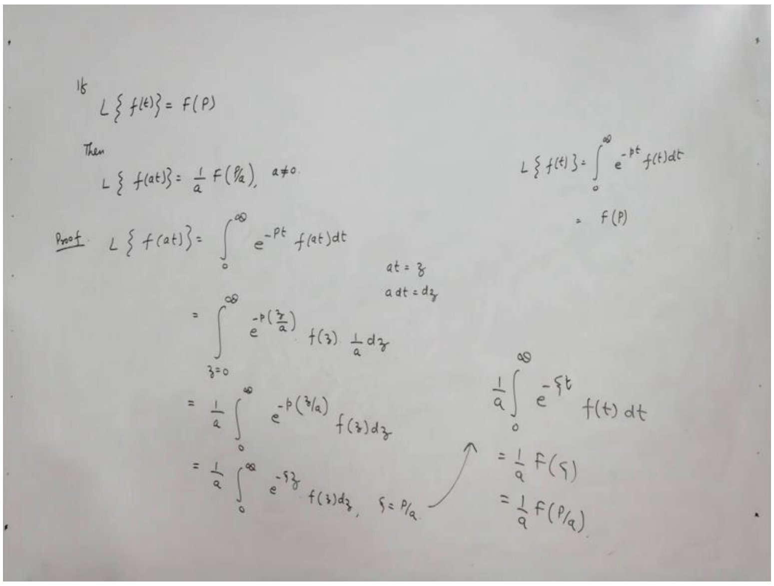

It states that if Laplace transform of some f t is F p then Laplace transform of f a t is nothing but 1 by a F p by a of course, a should not equal to 0. So, this is called change of scale property. Now, how we will prove it. So, the proof is very simple see simply based on the definition of Laplace transforms. So, what is Laplace transform of f a t? It is nothing but 0 to infinity e k power minus p t f a t.

Now, you can substitute a t equal to some new variable z, so a dt will be equals to dz. So, where you substitute this thing over here, so when t is 0, z is 0, so it will be 0. When t tend into infinity z will tend to infinity, so it is infinity e k power, now t is nothing but z by a. So, it is minus p into z upon a into f of z and d t is nothing but d z upon a, so that is 1 upon a times d z. So, this quantity is nothing but 1 by a 0 to infinity e k power minus p z upon a f z into dz or we can write it like this 1 upon a integer 0 to infinity e k power minus some zeta times p into f z dz where zeta is nothing but p upon a.

Now, when you compare this with Laplace transform of f t. So, Laplace transform of f t is given by a 0 to infinity e k power minus p t f t dt, and this we are calling as F p. This we are calling as F p, because p is a parameter, so it is a function of F p. Now, here the same thing what we have in with this expression, so this can also be written as this expression can also be written as 0 to infinity 1 upon a times e k power minus zeta t p into f t dt. So, that is nothing but 1 upon a f zeta because instead of it is z because variable is z, so it is z. So, now, z is here. So, instead of it is t, now t is here. So, instead of p we have psi here. So, it is will be f psi. So, what is psi? It is 1 by a f of p upon a, so that is a proof of change of scale property that is Laplace of f a t is nothing but 1 by a f of p by a.

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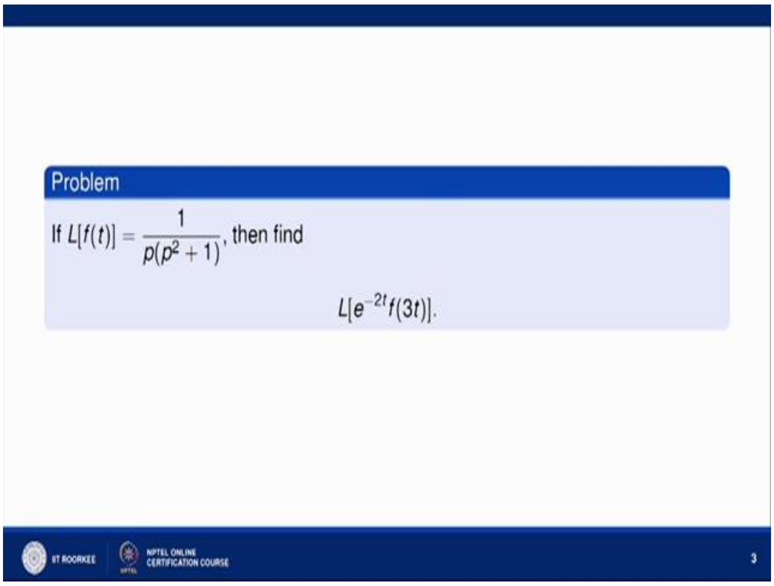

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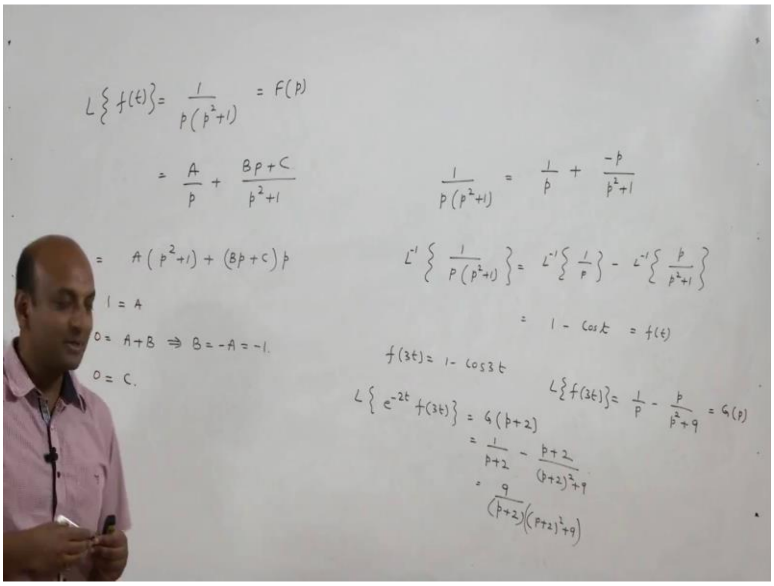

Now, we will see some problems based on this. Suppose, it is given to us that Laplace transform of f t is nothing but 1 by p p square plus 1 and we have to find Laplace transform of e k power minus 2 t into f of 3 t. Now, there are two ways to solve this problem the one way is you find the Laplace inverse of this F p. This is F p you find the

Laplace inverse of this F p that will be nothing but f t. Find f of 3 t and then f of 3 t will be find and then using shifting property, you can find Laplace transform of e k power minus 3 t f 3 t. And second way is you apply change of scale property and then the shifting property to directly find out the Laplace transform of this expression.

Now, how you can find a Laplace inverse of this again using partial fractions. This is nothing but some A upon P plus B P plus C upon P square plus 1. So, 1 is equals to A into P square plus 1 plus B P plus C into P. So, when you take P equal to 0, this implies 1 is equals to A. So, A is nothing but 1. Now you compare the coefficient of p square here P square is 0, here p square is A plus B. So, this implies B is minus A, and hence is equals to minus 1, because A is 1. Now, for C, you can compare the coefficient of P here P coefficient is only this which is C and here there is no coefficient involving P, so 0 equal to C. So, C must be 0.

So, what are the partial fraction of this expression 1 by P into P square plus 1 the partial fraction will be nothing but A is 1. So, it is 1 by P plus B is minus 1 and C is 0. So, it is minus P upon P square plus 1. So, this is partial fraction of this F p. So, what will be Laplace inverse of this? So, by the linearity property of Laplace inverse, this is nothing but Laplace inverse of 1 by p minus Laplace inverse of p upon p square plus 1, and this is nothing but 1 minus cos t. So, this is f t, Laplace inverse of F p is nothing but f t.

Now, we have to find out first f 3 t. So, what will be f 3 t? It is nothing but 1 minus cos 3 t. And Laplace of e k power minus 2 t into f 3 t will be nothing but first we will find out Laplace transform of f 3 t that will be new F p; and in that new F p, we will apply shifting property. So, what is Laplace transform of f 3 t which is nothing but 1 by p minus 3 upon p minus p upon sorry p upon p square plus 9, where cos a t is nothing but Laplace of cos a t is p upon p square plus a square. So, this we are taking as suppose G p function of p, suppose it is G p. So, Laplace of f 3 t is G p then Laplace of this expression will be nothing but g you replace p by a p minus a, here a is minus 2. So, it is nothing but p plus 2, so that is nothing but 1 upon p plus 2 minus p plus 2 upon p plus 2 whole square plus 9. So, if you take LCM and simplify it is clearly visible that it is nothing but p plus 2 into p plus 2 whole square plus 9, so that will be the Laplace of e k power minus 3 t into f 3 t, e k power minus 2 t into f 3 t so this is the one way.

Now finding Laplace inverse of given F p this F p is very simple. So, we have taken the partial fraction and simply computed first computed Laplace inverse of this F p let it suppose it is f t and then using shifting property we have find the Laplace transform of e k power minus 2 t into f 3 t. Now, if you have complicated expression, so finding Laplace inverse is a difficult job I mean is a time consuming job. So, for this particular type of problems, we can solve directly also using first change of scale property and then using shifting property.

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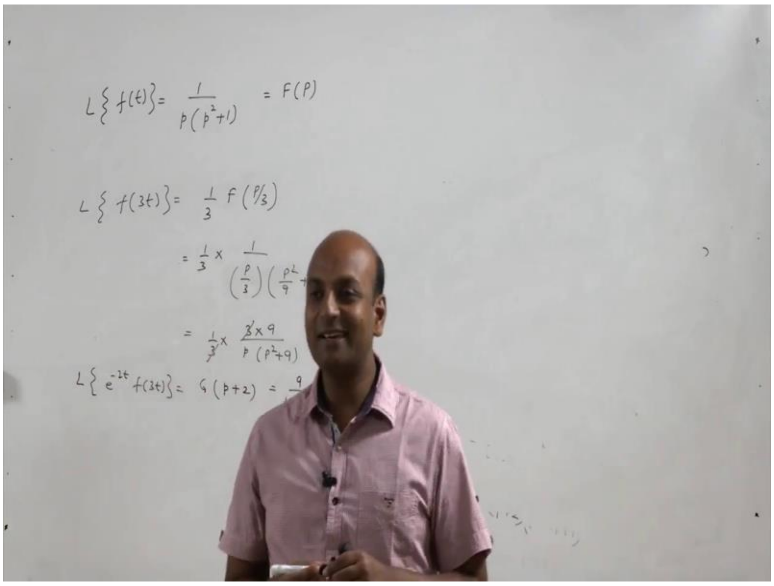

How, now let us see. So, this is suppose F p, this is suppose. Now, is Laplace of f 3 t it is nothing but 1 by a, f a here is 3, so 1 by 3 f of p by 3. So, it is nothing but 1 by 3 1 upon p by 3 into p square by 9 plus 1, so that is nothing but 3 3 cancels out this is p by 3 and this is 1 by 3 this is 1 by 3. So, this is 1 by 3 into. So, this is 3 into 9 upon p into p square plus 9. So, this 3, 3 cancels out. Now, what is Laplace of e k power minus 2 t f 3 t.

So, suppose this is some G p function of p suppose this is G p. So, Laplace of e k power a t or in some f t will be nothing but by a shifting property, it is F of p minus a here Laplace of this function is G p, so we have to apply shifting property for this function. So, this will be nothing but by the shifting property it is p plus 2, because here a is minus 2. So, it is nothing but 9 upon p plus 2 into p plus 2 whole square plus 9, so that is how using change of scale property and shifting property, we can find the Laplace transform of this particular problem very easily.

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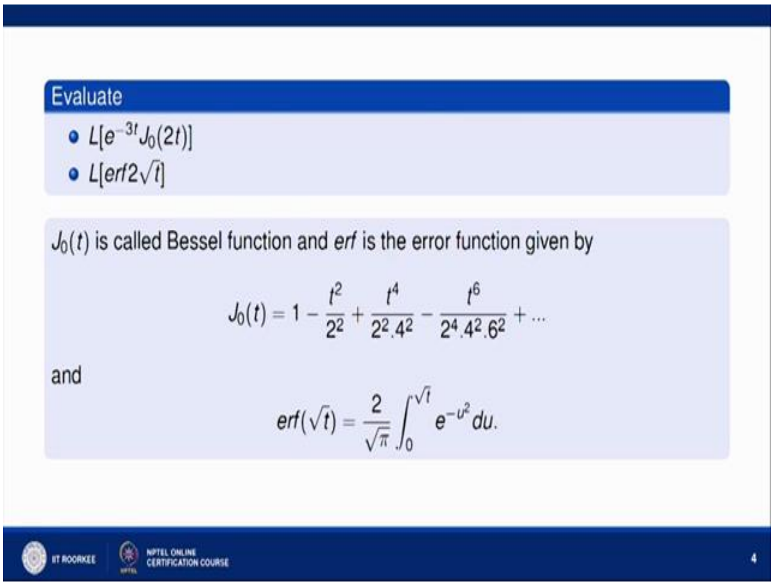

Now, let us find out Laplace transforms of some special functions special functions like Bessel function. So, I am not going into much detail of special functions. Simply assume that J naught x is given by this series and error function all is given by this particular expression. So, how to find Laplace transform of J naught t, which we call as Bessel function, and how to find Laplace transform of error function. So, let us try to find out this thing and then once we find Laplace transform of Bessel's function which is J naught t or error function, we can solve those two problems using shifting property and change of scale property.

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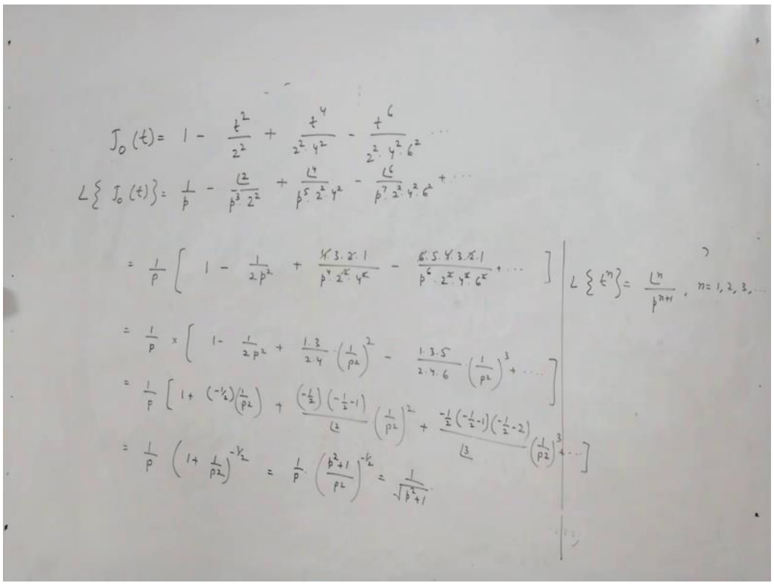

So, let us find first J naught t. So, J naught t is nothing but 1 minus t square by 2 square plus t key power 4 upon 2 square into 4 square minus t key power 6 upon 2 square 4 square into 6 square and so on as given in the problem. So, what is Laplace transform of J naught t? Now Laplace transform one is 1 by p. Laplace transform of t square is factorial 2 upon p square into 2 square. Laplace transform p q, it is because Laplace transform of t k power n is what is nothing but factorial n upon p k power n plus 1, if n is 1, 2, 3 and so on for positive integers. So, again it is factorial 4 upon p k power 4 2 square into 4 square minus factorial 6 upon p k power seven 2 square 4 square 6 square and so on. So, it is nothing but 1 by p take 1 by p common. So, it is 1 minus 1 2 2 cancels out. So, it is 1 by 2 p square then it is nothing but 4 into 3 into 2 into 1 upon p k power 4 into 2 square into 4 square minus it is nothing but 6 into 5 into 4 into 3 into 2 into 1 upon p k power 6 because one p is common. So, and it is 2 square 4 square and 6 square and so on. Now, one 2 2 cancel out one 4 4 cancel out.

So, we have observed that numerator always contain odd terms, and denominator always contain even terms that is it is nothing but 1 by p minus 1 minus 1 by 2 p square plus 1 into 3 upon 2 into 4 into 1 by p square k whole square. It is nothing but minus 1 into 3 into 5 upon 2 into 4 into 6 into 1 by p square k whole cube and so on. So, this is nothing but 1 by p into m this is 1 by p into 1 by p into, now this is 1 minus, this is nothing but you can say plus minus 1 by 2 minus 1 by 2 into p 1 by p square. Now, this is nothing but minus 1 by 2 into minus 1 by 2 minus 1 upon factorial 2 into 1 by p square. When you simplify this expression, so what we get it is minus 1 into minus 3, when you taken LCM, so 1 into 3 minus minus plus and 2 cube, so 2 cube in the denominator.

Next is, it is plus minus 1 by 2 minus 1 by 2 minus 1 into minus 1 by 2 minus 2 upon factorial 3 into 1 by p square k whole cube. So, again it is nothing but when you simplify, so you will get back the same expression. It is 1 into 3 into 5 upon two cube into factorial 3 which is nothing but 2 into 4 into 6. So, what this expression basically this is nothing but 1 minus 1 by p square 1 plus p square power minus half, this is by the binomial theorem 1 plus m x plus m minus upon factorial to into x square and so on. So, now, this is nothing but when you simplify it this is 1 by p into p square plus 1 upon p square whole power minus half which is nothing but 1 upon p square plus 1 under root. So, this is the Laplace transform of J naught t, which we are calling as Bessel function.

So, you simply open the series, this is the series is given to you. You take the Laplace transform both the side simplify. So, this will be reduced to some binomial expansion of 1 plus 1 by p square whole power minus half. And when you simplify, we will get 1 upon under root p square plus 1 at the Laplace of Laplace transform of J naught t.

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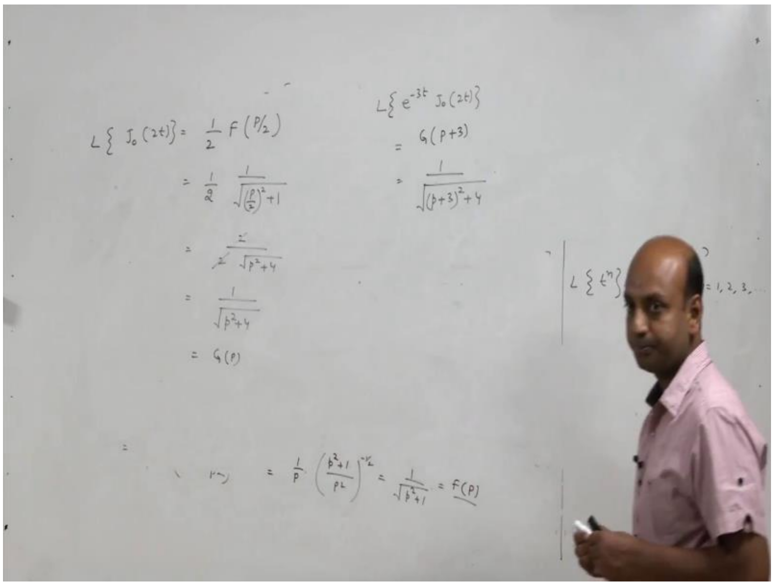

Now, we can solve first problem since now we know Laplace transform of J naught t, so what will be the Laplace transform of J naught 2 t first. So, this is using change of scale property because if Laplace transform f t is some F p, then Laplace transform f a t is nothing but is nothing but 1 by a F p by a. So, here this is F p, and this is Laplace transform of J naught t. So, what will be Laplace transform of this? Here it will be 1 by 2 F p by 2. So, this nothing but 1 by 2 f of 1 by under root p by 2 whole square plus 1 which is nothing but 2 upon 2 under root p square plus 1 p square plus 4. So, it is nothing but 2 2 cancel out. So, it is 1 by under root p square plus 4.

So, what should be Laplace transform of e k power minus 3 t of this, J naught 2 t this will be equal to. Now, let us suppose this is G p the Laplace transform of J naught 2 t there a suppose it is G p. So, Laplace transform of e k power minus 3 t into this is nothing but by using shifting property it is G of p plus 3, because in the Laplace transform this J naught 2 t, you replace p by p plus 3. So, Laplace transform of this is G p. So, you replace p by p minus a, and a is minus 3. So, it is nothing but 1 upon under root p plus 3 whole square plus 4. So, this will be the final answer. You can simplify and that should be the Laplace transforms of e k power minus 3 t J naught 2 t. Now let us find out Laplace transform of error function under root t. So, then using change of scale property, we can solve that problem.

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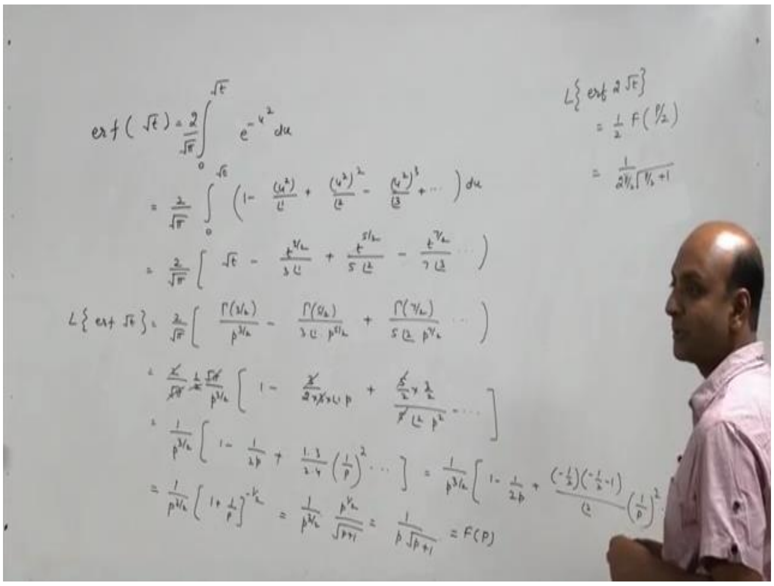

So, what is Laplace transform of error function let us see. So, error function what is error function of under root t, how you define it, it is nothing but 0 to under root t 2 upon pi times 0 to under root t e u square e k power minus u square du. So, now, it is equal to 2 upon under root pi 0 to under root t. Now, we know the expression of e k power minus x we can use this expression here it is 1 minus u square upon factorial 1 plus u square whole square upon factorial 2 minus u square whole square upon factorial 3 and so on into du. So, this is nothing but 2 upon under root pi. So, integral of 1 will be u, and when you apply a upper limit upper limit is sorry it is under root t.

So, when you apply upper limit minus lower limit, so this will be nothing but under root t minus. Now, integration of u square is u cube upon 3. So, u cube, and when you apply upper limit minus lower limit is nothing but t power 3 by 2 upon 3 into factorial 1 plus u key power 4 integral is u power 5 upon 5. And when you apply the limit is t k power 5 by 2 upon 5 factorial 2. Similarly, here it will be u k power 6 that is t k power 7 by 2 upon seven factorial 3 and so on. So, this would be the expression of error function in the series from.

Now, when you take Laplace transform of e k error function of under root t. So, this will be nothing but 2 upon under root pi you take common. And this is nothing but gamma 3 by 2 upon 3 by 2 upon p k power 3 by 2 minus it is gamma 5 by 2 upon 3 into factorial 1 into p k power 5 by 2 plus gamma 7 by 2 upon 5 into 2 factorial into p k power 7 by 2 and so on. This is nothing but again 2 upon under root pi take gamma 3 by 2 common and p k power 3 by 2 common from the entire expression from the entire series. Now, gamma 3 by 2 is nothing but 1 by 2 under root pi this we know as gamma and plus 1 is n gamma n. So, this is 1 by 2 under root pi, this I have taken common and also i have taken p k power 3 by 2 as common this nothing but 1 minus.

Now, gamma 5 by 2 is 3 by 2 gamma 3 by 2 gamma 3 by 2 already common. So, it is 3 by 2 into 3 into factorial 1 and p k power 3 by 2 is common. So, it is p plus now gamma 7 by 2 is 5 by 2 gamma 5 by 2 and gamma 5 by 2 can be again written as 3 by 2 gamma 3 by 2. So, it is nothing but 5 by 2 into 3 by 2 gamma 3 by 2 which is common and upon 5 into factorial 2 p k power 2 and so on.

Now, again let us simplify this expression 2 2 cancels out, this is cancels out 1 upon p k power 3 by 2. And it is 1 minus this 3 3 cancel out, so it is 1 minus 1 by 2 p plus this 5 5 cancels out, it is nothing but 1 into 3 upon 2 into 4 upon 1 by p ka whole square and so on. So, it is something the same expression which we arrived in the derivation of Bessel function J naught t. So, it is something like that. So, we can also write this as 1 upon p k power 3 by 2, 1 remain 1, 1 minus this. So, this can be written as minus 1 by 2 minus 1 by 2 minus 1 upon factorial 2 into 1 by p square and so on.

Similarly, the other expression can be write in the same way. So, it is nothing but 1 upon p k power 3 by 2 1 plus 1 by p whole power minus half. In that expression we have a binomial series of 1 plus 1 by p square power minus half here we have a binomial series of 1 plus 1 by p k power minus half. So, when you simplify it, so it is nothing but 1 upon p k power 3 by 2 into p k power half upon under root t plus 1. So, it is nothing but 1 upon p under root p plus 1. So, this is the Laplace transform of error function of under root t. This is how we can find out Laplace transform of error function of under root t.

Now, we can solve that problem very easily, because we know the Laplace transform under root t, which is given by this F p. Now, Laplace transform of 2 under root t we can use change of scale property, it will be 1 by 2 F p by 2. If it is F p, so Laplace transform of error function of 2 under root t will be nothing but 1 by 2 F p by 2, this is by the scale property change of scale property. So, we can use this, so it is 1 by 2 p by 2 under root p by 2 plus 1. So, this will be the Laplace transform of that expression.

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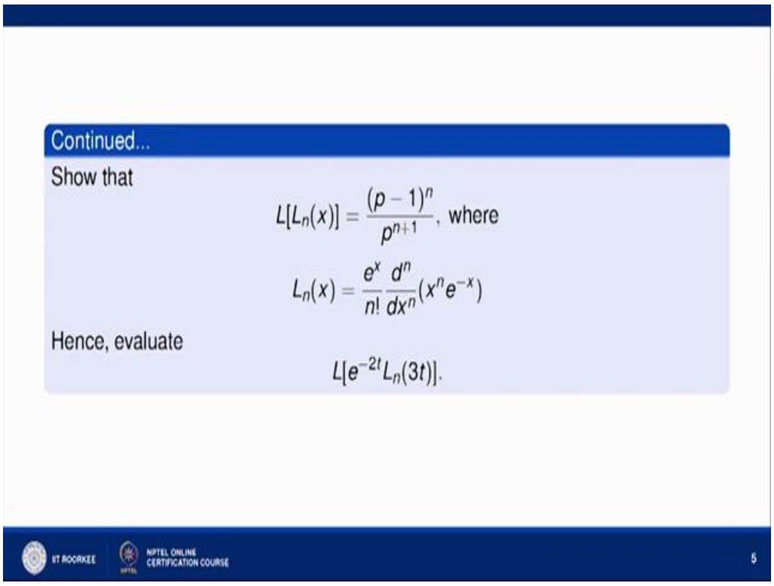

Now, the next problem let us first find Laplace transform L n x, what is L n x? L n x is luxurious polynomial. So, we are not going into the detail of this, but L n x is given by this particular expression, this expression. This expression is called as L n x. Let us try to find out the Laplace transform of this, and again using shifting property or scaling property we can find out the Laplace transform of this expression. So, let us try to find out Laplace transform of L n x. Then the rest of the part, you can easily solve yourself that how can you find out Laplace transform of e k power 2 minus 2 t l n 3 t using shifting or scaling property.

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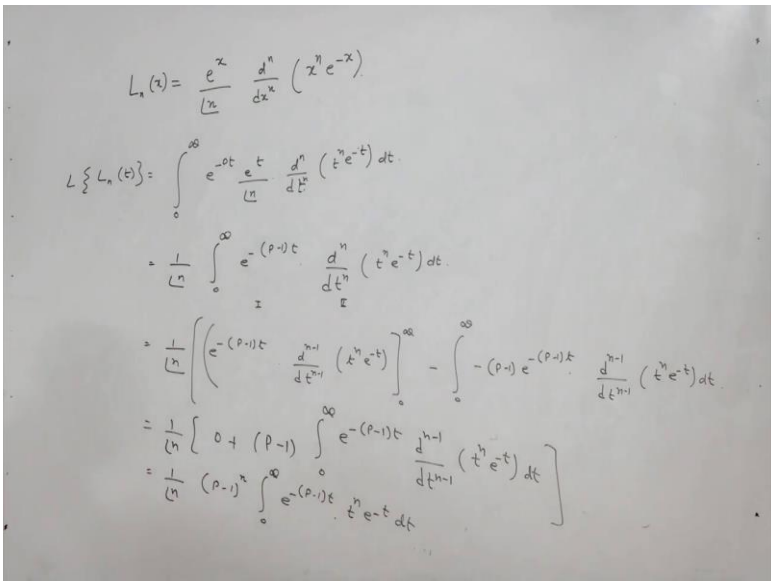

So, what is L n x? L n x is given by it is e k power x upon factorial n nth derivative of x key power n e k power minus x. So, Laplace transform of L n x by the definition of Laplace transform is nothing but 0 to infinity e k power minus p t e k power minus p t into e k power x upon factorial n nth derivative of this into x k power n e k power minus x. So, we have to use one symbol because here t is involved. So, we have to define this in terms of t l n t, because Laplace transform f t we are defining. So, we have to express l n t and terms of t. So, this is the expression for L n x. So, we can reduce in terms of t replace x by t.

So, how can you find out Laplace transform of this function? So, this is nothing but 1 by factorial n can be taken out. So, it is 0 to infinity e k power minus p minus 1 times t into nth derivative respect to t t k power n e k power minus t into dt. Now, you can make use of integration by parts. So, let us suppose this first function and the second function. So, first as it is integration of second it is n minus 1 derivative now, integration 0 to infinity of this expression minus integration derivative of first integration of second because integration is anti-derivative, so derivative powers reduce.

Now, when t tends to infinity, because this have negative powers, and we are assuming p greater than one we can assume. So, this will tends to 0. And hence entire term will tend to 0. And when t tend to 0 this is one. And since it has n minus 1 derivative of this function, all the terms involved at least one power of t because here the highest power of t is t k power n and when we differentiate n minus 1 times of this part of function, so all the terms will involve at least t k power 1. So, when you take t equal to 0 that value will be 0. So, this is 1 by factorial n into 0 minus, now minus minus plus. So, it is p minus 1 times integral 0 to infinity e k power minus p minus 1 times t n minus 1 derivative of this expression into t key power n e k power minus t into dt. Now, t k yeah, it is dt sorry it is dt because we are differencing with respect to t now, so it is dt.

So, so we have observed that in the integration of this p minus 1 comes out when we integrate first time one times and n minus 1 derivative will come here. So, if we repeat this process n times. So, then we will be having if we integrate again for second time this is the first function this is the second function again we will take one power of p minus 1 one more power of p minus 1 outside the bracket. So, that is p minus 1 whole square will be here. Now, when we integrate this expression n times, so this will be nothing but power n and it will be involve 0 to infinity e k power minus p minus 1 times t and there will be no derivative of this expression t k power e k power minus t into dt.

So, this will be nothing but is equals to 1 by factorial n this will be nothing but now I write here. So, this expression will be nothing but 1 by factorial n p minus n k power n into now e k power t and t cancels out. So, this will reduce to Laplace transform of t k power n. And though this is equals to 1 by factorial n p minus 1 k power n. And Laplace on this is nothing but factorial n upon p k power n plus 1 because n is a integer. So, these two cancels out, so hence we have shown that Laplace transforms L n x is t minus n k power n upon p k power n plus 1. Now, using scale property and shifting property, we can solve the remaining part.

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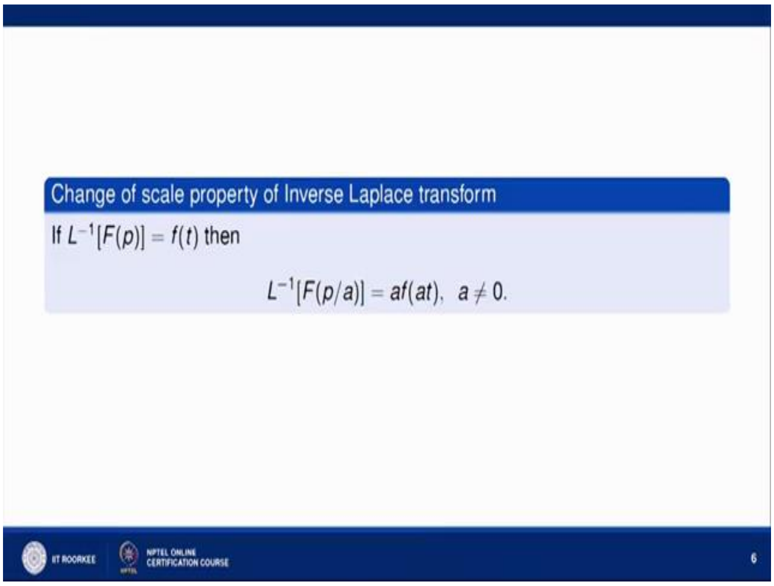

Now Laplace same property that is change of scale property, we can read in terms of inverse also. If Laplace involves of F p of f t, then Laplace inverse of p F p by a is nothing but a f a t, if a is not equal to 0.

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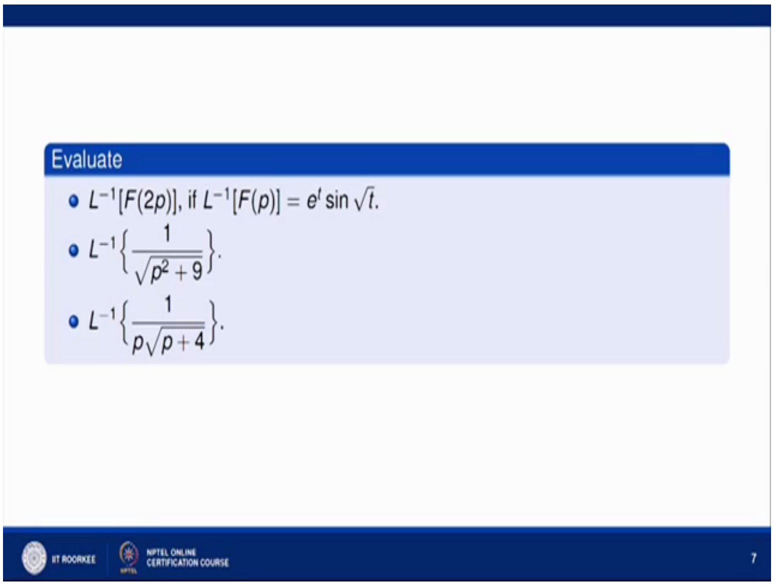

Now, these problems we can easily solve using this property. Suppose, first problem we want to solve. So, Laplace inverse of F p is given to us is e k power t sin under root t and you want to find out Laplace inverse of f 2 p. So, when you compare with this a is nothing but 1 by 2, a is nothing but 1 by 2. So, you simply replace a by 1 by 2. So, that means, Laplace inverse of f 2 p will be nothing but 1 by 2 f 2 by 2 because a is half and this expression is f t. So, what should be the Laplace inverse of this expression?

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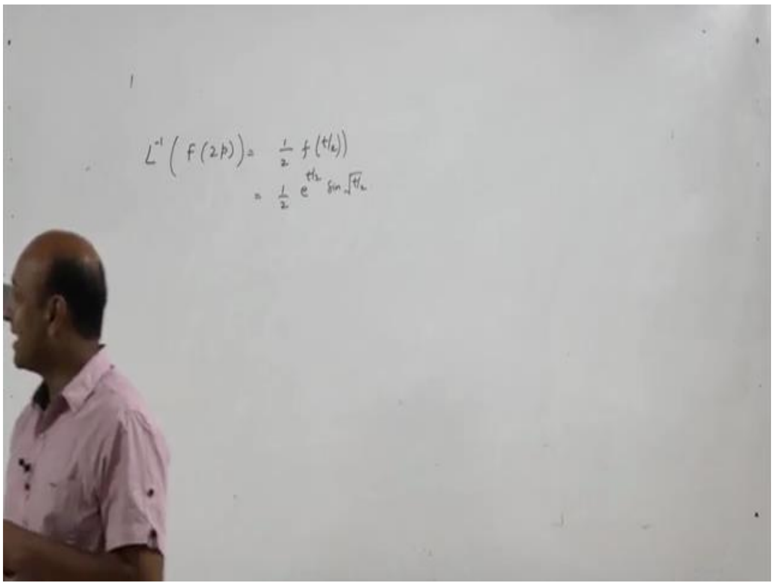

So, Laplace inverse of this expression will be nothing but here a is 1 by 2. So, it is 1 by 2 f t by 2. So, it is 1 by 2, f t is this, so it is e k power t by 2 and sin under root t by 2. So, using the same property, we can solve the other two problems also, it is very easy you take 9 common or 4 common, and then you can use scaling property to solve these two problems also.

Thank you very much.