Mathematical methods and its applications
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Lecture – 26
Properties of Laplace Transforms-I

Hello everyone, so in the last lecture, we have seen that what is the existence theorem for Laplace transforms. We have seen that if a function is a piecewise continuous and is of exponential order alpha then its Laplace transform always exists; otherwise if function is not piecewise continuous or is not of exponential order then Laplace may or may not exist that also we have seen. So, now in this topic, we will see what are the properties of Laplace transforms.

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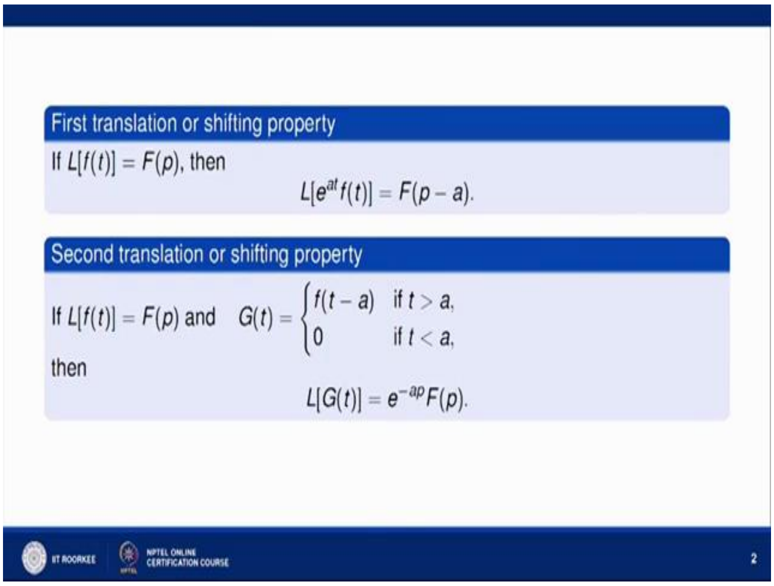

So, we will start with the first property that we call as shifting property. Now, shifting property is first we have first translation and then we have second translation of the shifting property.

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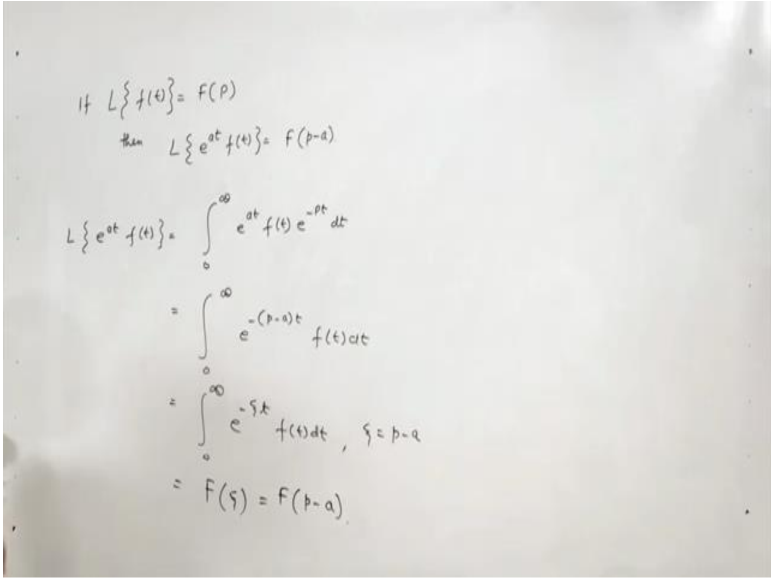

What is first translation? First translation means if Laplace transform of f t is say F p then if this holds then Laplace transform of e k power a t into f t is nothing but F of p minus a. That means if we know the Laplace transform f t which is given as F p if you know this then Laplace transform of e k a t into f t is nothing but you simply replace p by p minus a. So, then we will get the Laplace transform of this function e k power at into f t. Now, what are the proof of this result, how we will obtain this result? Now, what is Laplace transform of e k power a t into f t by the definition? This is nothing but 0 to infinity e k power a t f t into e k power minus p t dt, this is by the definition of Laplace transform.

So, this is equal to 0 to infinity e k power minus p minus a times t into f t dt. So, this is nothing but 0 to infinity, suppose p minus a is some psi of f t where psi is nothing but p minus a. Then this quantity is nothing but f of psi because what is F p, F p is Laplace of f t which is 0 to infinity e k power minus p t f t dt. So, instead of p, we have psi in the same expression. So, we replace f as psi and what is psi? Psi is p minus a, so it is p minus a. So, hence Laplace of e k power a t into f t is nothing but F of p minus a. So, simply replace p by p minus a.

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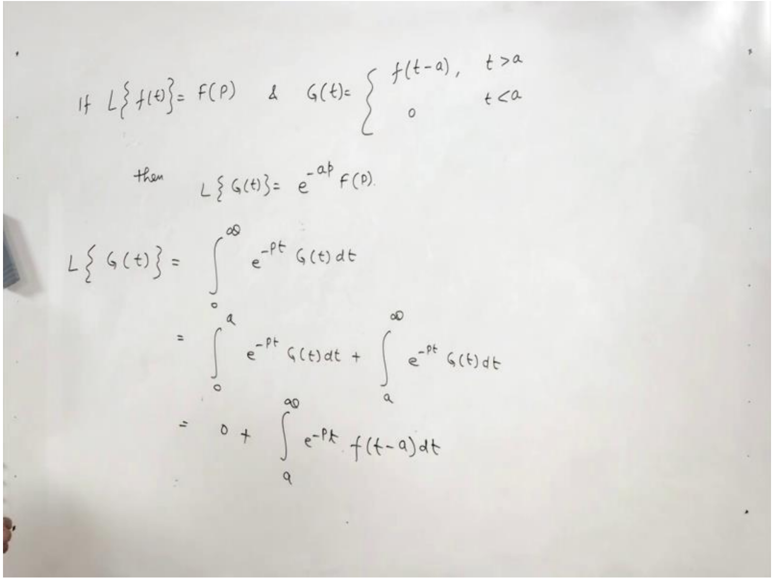

Now, second translation or the shifting property is if Laplace transform of f t is F p, and G t is given as f of t minus a, when t greater than a; and 0 when t less than a then Laplace transform of G t is nothing but e k power minus a p times F p. Again we can prove this using the definition of Laplace transform, Laplace transform of G t is nothing but we have to prove this result. So, this is equal to 0 to infinity e k power minus p t G t dt by the definition of Laplace transforms. This is equal to 0 to a e k power minus p t G t dt plus a to infinity e k power minus p t G t dt now 0 to a G t is 0, because G t is 0 and t less than a, so it is 0, so this quantity 0 plus because g t is 0. So, integration value will be 0 plus. So, this is a to infinity e k power minus p t. When t is greater than a, so G t is f t minus a. So, you simply take f of t minus a here dt.

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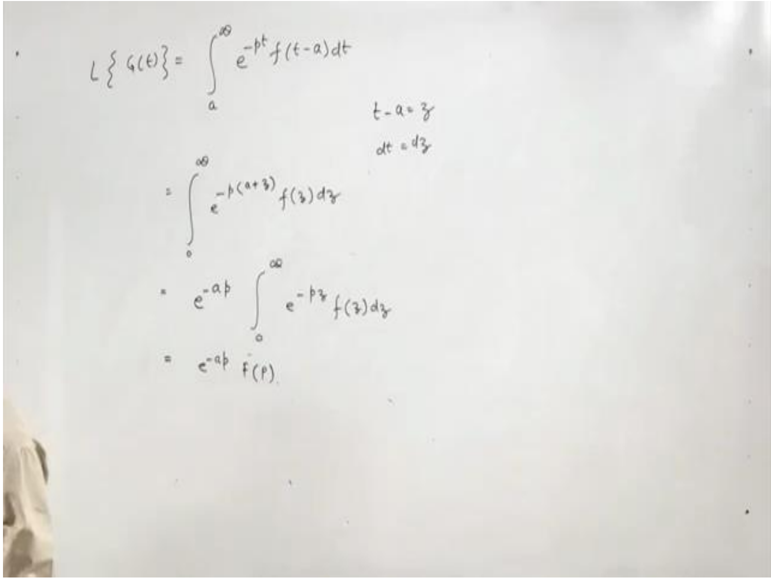

Now, this expression, so therefore, therefore, what is Laplace of G t. Therefore Laplace of G t will be nothing but from here we got a into infinity e k power minus p t f of t minus a into dt. Now, you can take t minus a equal to some new variable say z, though dt will be dz. So, this will be equals to when t is a, so z will be 0; when t is tend into infinity, z tend into infinity e k power minus p t from here a is a plus z. So, it is a plus z times f z into dz. So, this is nothing but since e k power minus p a is free from z, we are integrating it with respect to z. So, we can take it outside the integration e k power minus a p 0 to infinity e k power minus p z f z dz, which is same as e k power minus a p. So, this is nothing but Laplace transform of f t or f z, so this is F p. This is nothing but Laplace transform of f t. So, e k power minus a p into f p. So, hence Laplace of G t is nothing but e k power minus a p into F p. So, this is called second translation or shifting property.

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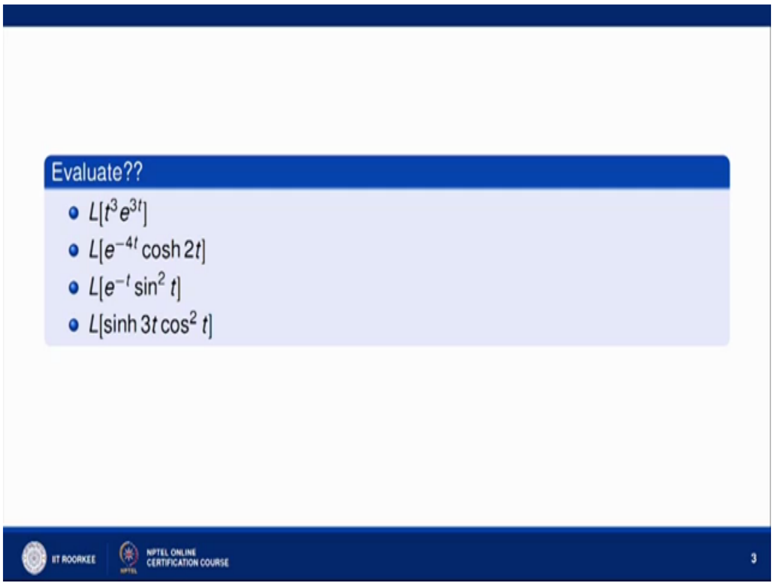

Now, how we can solve problems based on this. Let us solve few examples on this. The Laplace transform of t cube e k power 3 t.

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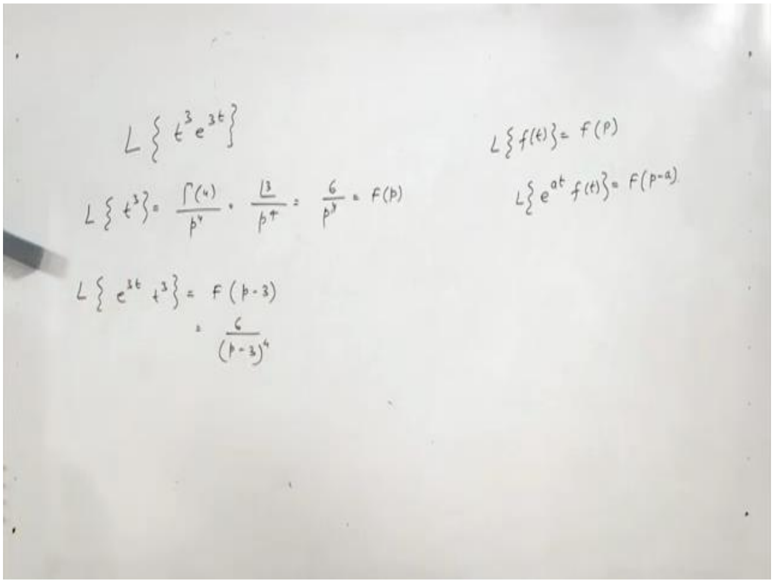

So, we have just discussed that if Laplace transform of f t is F p then Laplace transform of e k power a t into f t is nothing but F of p minus a. So, here f t is t cube; if you compare this with this, so here f t is t cube. So, what is Laplace transform of t cube? It is gamma 4 upon p k power 4 this is nothing but factorial 3 upon p k power 4 which is nothing but 6 upon p k 4. So, 6 upon p k power 4. So, this is F p. Now, you have to compute Laplace transform of e k power 3 into t cube. So, you simply replace F p in F p p by p minus a and a here is 3, so that is nothing but F of p minus 3. So, that is nothing but 6 upon p minus 3 k power 4, so that will be the Laplace transform of t cube e k power 3 t.

Next problem, suppose the next problem is e k power minus 4 t cos hyperbolic 2 t. So, again here if you compare with this, so f t is cos hyperbolic 2 t. So, first find Laplace of cos hyperbolic 2 t, let us put equal to F p and then replace p by p minus a, here a is minus 4. So, replace p by p plus 4. So, what is Laplace transform of cos hyperbolic 2 t? We already know that Laplace transform of cos hyperbolic a t is p upon p square minus a square. So, it is p upon p square minus 4. So, Laplace transform of e k power. So, this if F p, this is F p. So, Laplace transform of e k power minus 4 t into cos hyperbolic 2 t will be nothing but F of p, you replace p by p minus a, a is minus 4. So, p plus 4, so that is nothing but p plus 4 upon p plus 4 whole square minus 4. So, this will be the Laplace transform of the e k power minus 4 t cos hyperbolic 2 t.

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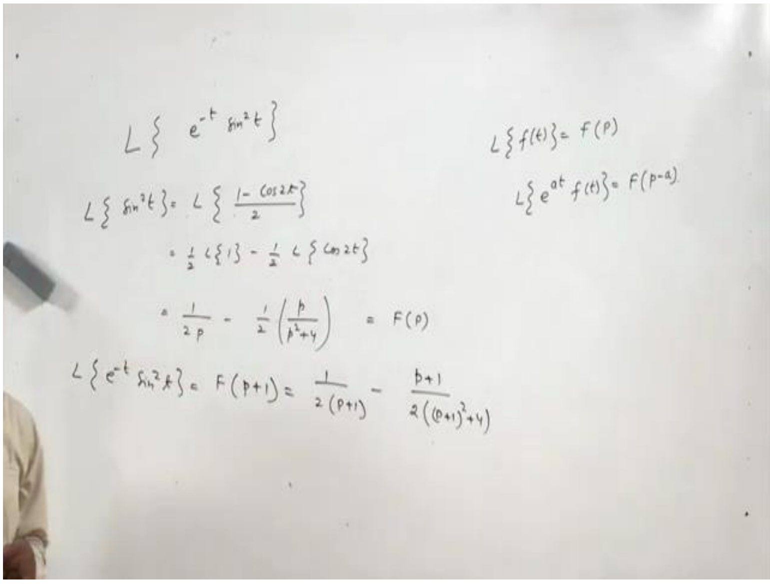

Now, the third problem, suppose you want to solve the third problem it is e k power minus t sin square t. So, first find Laplace transform of sin square t put it equal to F p and replace p by p minus a, and here is a minus 1. So, replace p by p plus 1. So, first find Laplace transform of sin square t. So, sin square t is nothing but 1 minus cos 2 t by 2. So, this is nothing but 1 by 2 Laplace transform of 1 minus 1 by 2 Laplace transform of cos 2

t. So, Laplace transform of 1 is 1 by p minus 1 by 2, Laplace transform of cos 2 t is p upon p square plus a square, so p upon p square plus 4. So, this whole is F p. Now Laplace transform of e k power minus t square t will be nothing but F of you replace p by p minus a, a is minus 1 that is p plus 1. So, that is nothing but 1 upon 2 p plus 1 minus p plus 1 upon 2 into p plus 1 whole square plus 4, so that will be the Laplace transform of e k power minus t sin square t.

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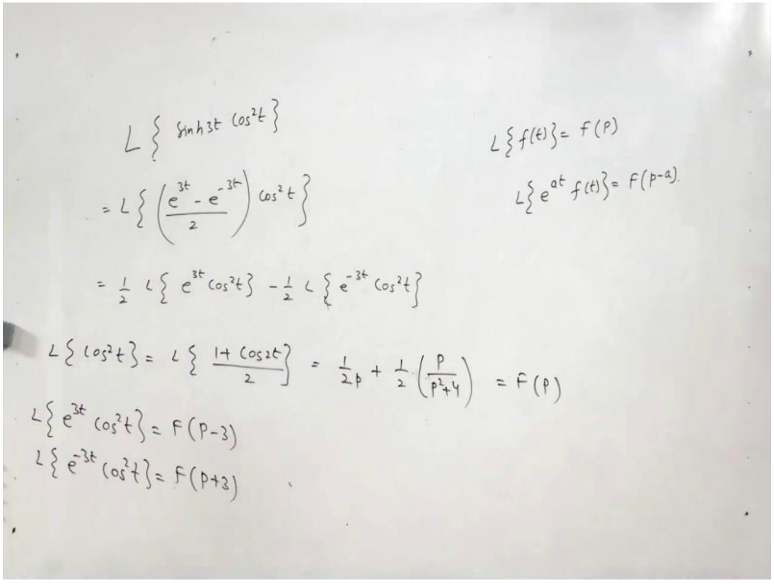

Again in the last problem, we can use shifting property, what is it? Is sin hyperbolic 3 t into cos square t, it is nothing but Laplace of e k power 3 t minus e k power 3 t by 2 cos square t. So, it is 1 by 2 Laplace transform of e k power 3 t cos square t minus 1 by 2 Laplace transform of e k power minus 3 t into cos square t. So, we will find Laplace transform of cos square t, put it equal to F p, and then in the first part replace p by p minus 3 and for the second part replace p by p plus 3.

So, what is Laplace transform of cos square t? That again we can find out cos square t will be nothing but Laplace transform of 1 plus cos 2 t by 2 that is nothing but 1 by 2. Laplace transform of 1 is 1 by p, so that is 1 by p plus 1 by 2. Laplace transform of cos 2 t is p upon p square plus a square, so p upon p square plus 4. So, this is F p. Now, what is Laplace transform of this e k power 3 t cos square t? This is nothing but F of p minus a. And what is Laplace transform of e k power minus 3 t cos square t, this is nothing but f of p plus 3. So, substituting these two values, you replace p by p minus 3 for the first part, and you replace p by p plus 3 for the second part and you substitute these values here. So, we will get the Laplace transform of sin hyperbolic 3 t into cos square t.

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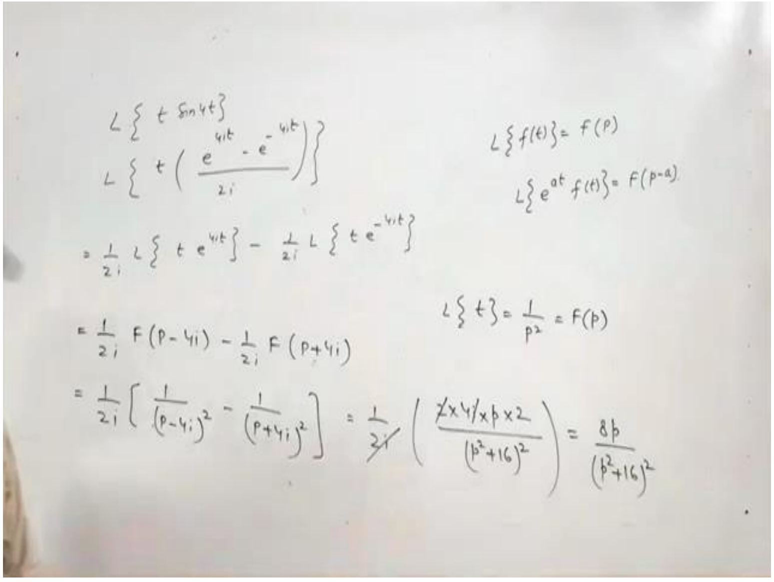

Now, let us find Laplace transform of t sin 4 t. So, sin 4 t does not involve as such does not involve e k power a t, but we can break it. So, Laplace of t sin 4 t, so Laplace of t. Now, sin 4 t can be written as e k power 4 iota t minus e k power minus 4 iota t upon 2 iota, this is how we can replace sin 4 t. Now, this is nothing but 1 by 2 i Laplace transform of t into e k power 4 iota t minus 1 by 2 i Laplace transform of t into e k power minus 4 iota t. So, again here a is 4 iota, if you compare with this; and here a is minus 4 iota. So, you find Laplace of t first let it put it equal to F p, and for the first expression replace p by p minus 4 iota, and for second expression replace p by p plus 4 iota.

So, what is Laplace transform of t, what is Laplace transform of t? Laplace of t is nothing but 1 by p square, it is equals to F p. And for this part, it is nothing but 1 upon 2 iota f of p minus 4 iota minus 1 by 2 iota F of p plus 4 iota by this shifting property. So, this is nothing but 1 upon 2 iota. What is F of p minus 4 iota? 1 upon p minus 4 iota whole square minus 1 upon p plus 4 iota whole square. So, the simplify it 1 upon 2 iota into, so this will go here this will go here simplify it, so this is nothing but 2 a b 2 into 4 iota into p 2 a b into 2 again 2 1 2 again comes. Then it is nothing but p square plus 16 whole square, this 2 iota will cancel with 2 iota and this is nothing but 8 p upon p square plus 16 whole square. So, this will be the Laplace transformation of t sin 4 t.

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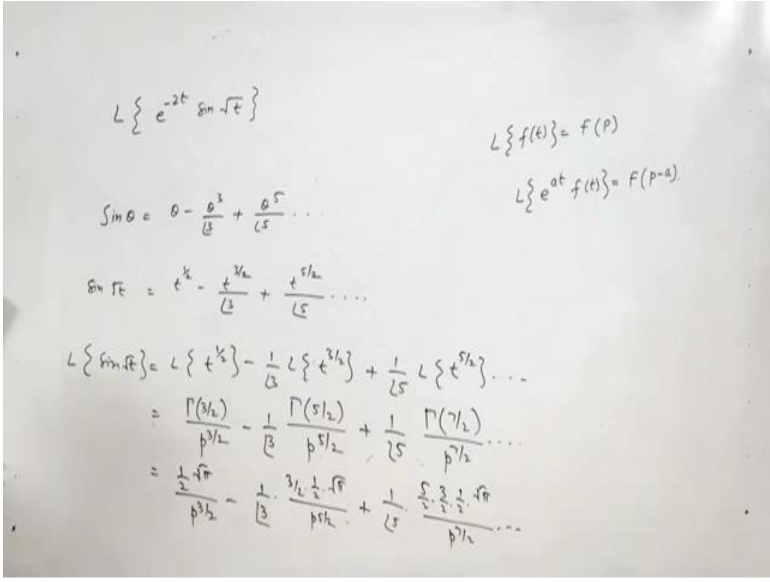

Now, this problem on shifting property; Laplace transform of e k power minus 2 t sin under root t, again we first we find out Laplace transform of sin under root t and using shifting property, we will replace p by p plus 2 because here a is minus two. So, what is Laplace transform of sin under root t let us see. So, what is sin theta? Sin theta is theta minus theta cube upon factorial 3 plus theta k 5 upon factorial 5 and so on. What is sin under root t sin under root will be t k power half minus t k power 3 by 2 upon factorial 3 plus t k power 5 by 2 upon factorial 3 factorial 5 and so on.

Now, it is Laplace of sin under root t will be nothing but Laplace transform of this entire expression, and by the linearity property we can split in term wise. So, this is nothing but Laplace transform of t k power half minus 1 by factorial 3 Laplace transform of t k power 3 by 2 plus 1 by factorial 5 Laplace transform f t k power 5 by 2. So, this is t k power half is gamma 3 by 2 upon p k power 3 by 2. Using Laplace transform of t k power n minus 1 by factorial 3, it is gamma 5 by 2 upon p k power 5 by 2 plus 1 by factorial 5 gamma 7 by 2 upon p k power 7 by 2 and so on. So, this is gamma 3 by 2 is because gamma n plus 1 is gamma n. So, gamma 3 by 2 will be half gamma half and gamma half is under root pi. So, it is 1 by 2 under root pi p k power 3 by 2 minus 1 by factorial 3. It is nothing but 3 by 2 1 by 2 under root pi. Again the property of gamma function p k power 5 by 2 plus 1 by factorial 5, it is 5 by 2 3 by 2 1 by 2 under root pi p k power 7 by 2.

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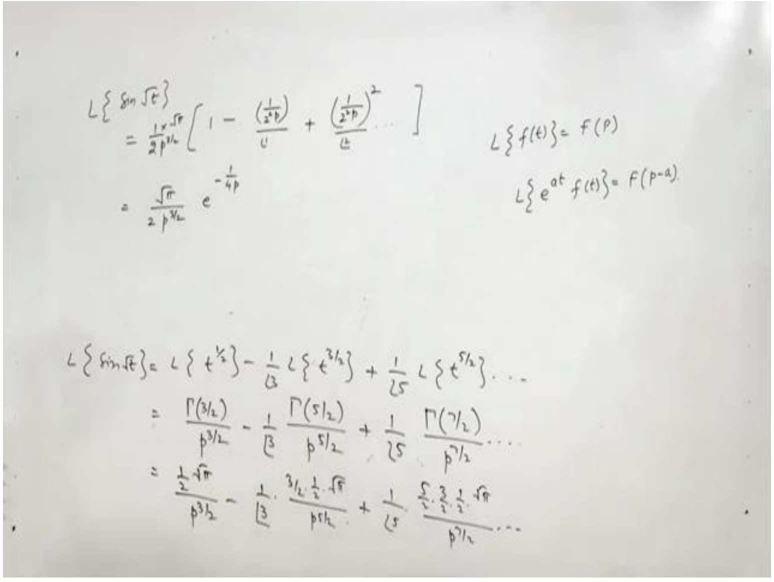

So, what finally, we get. So, Laplace transform of sin under root t, this will be nothing but now you can take 1 by 2 under root pi upon this quantity common form of all the terms. So, this is 1 by 2 p k i power 3 by 2 into under root pi, inside bracket we get 1 minus. So, this is factorial 3 is 3 into 2 into 1, 3, 3 cancels out, under root 2 this we have taken common one. So, this is nothing but 1 by 2 square into p whole power upon 1 factorial because 1 2 comes from here 1 2 from here 1 2 is common, so it is 2 square. From here 5, into 5 cancels out, so it is nothing but plus 1 by 2 square p whole square upon factorial two this comes. When you simplify 5, 5 cancels out, 3 also cancels. So, we will get two square whole square upon factorial two and this is under root 2 p k power 3 by 2 and it is nothing but e k power minus 1 by 4 p, so that is the Laplace transform of sin under root t. So, this is F p.

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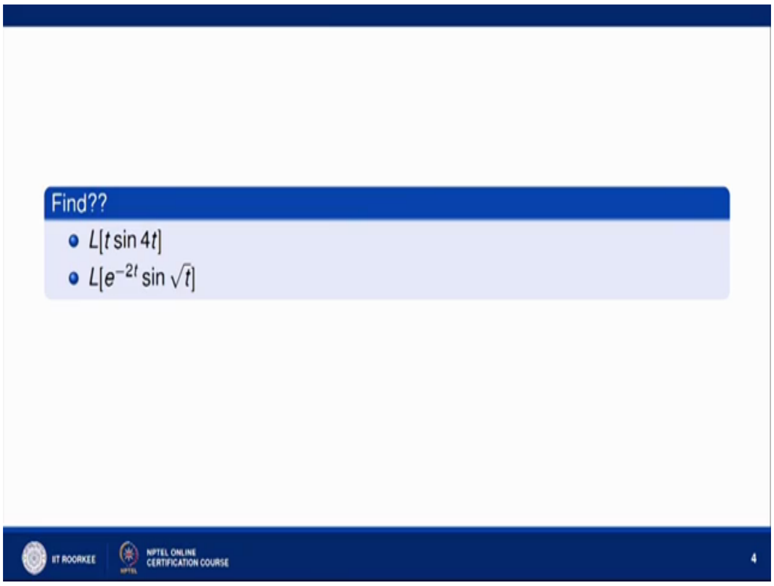

And for to find Laplace transform of e k power minus 2 t you simply replace p by a p plus 2 in this expression. This is F p, you replace p by p plus 2 you will get Laplace transform of this quantity.

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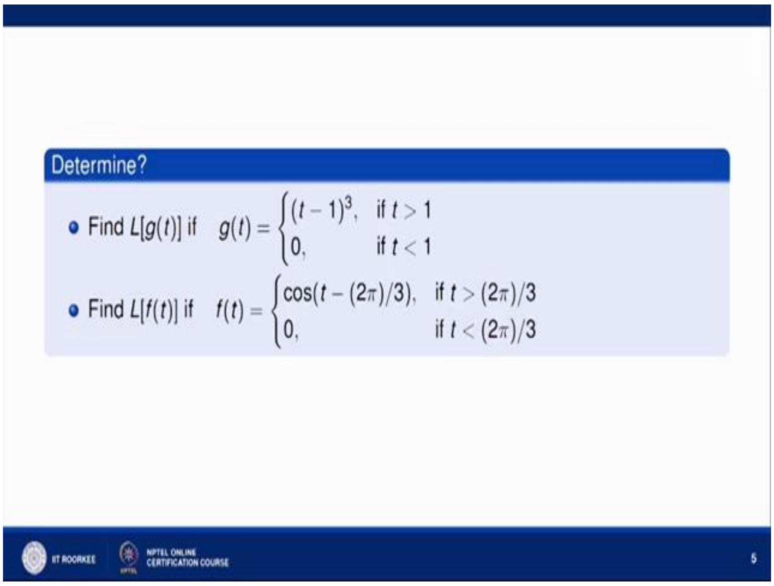

Again using second translation property, we can easily find out Laplace transform of G t because if you compare with second shifting property, second translation, so Laplace transform of f t is F p then Laplace transform G t is e k power minus a p times F p by second translation property.

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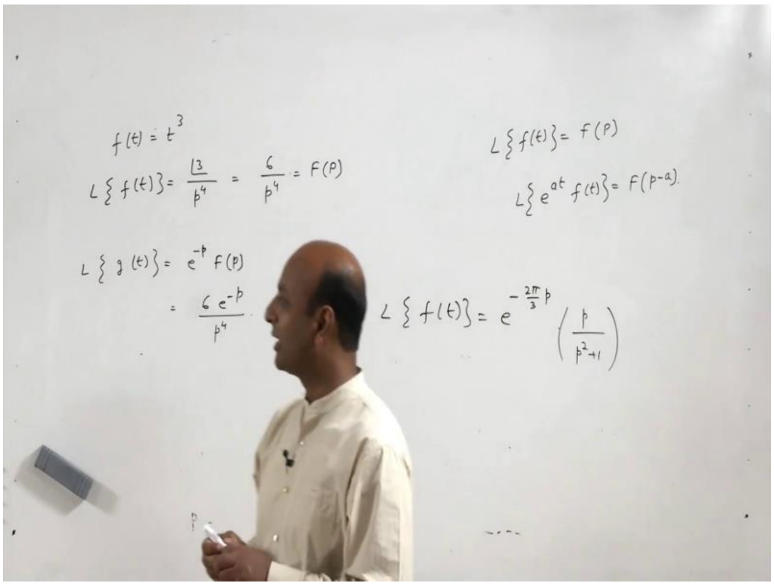

So, if we compare with this, here in the first problem, what is G t, what is f t minus a. Now, F of p minus a, for the first problem f t will be t cube then only it is t minus a, a is 1 whole cube. So, what is Laplace transform of f t? It is nothing but factorial 3 upon p k power 4 that is 6 upon p k power 4. Now, the Laplace transform this G t will be nothing but e k power minus a s e k power minus a p, a is 1 here, a p into F p, and F p is the this quantity. So, this is nothing but e k power minus p e k power 6 into e k power minus p upon p k power 4. So, this will be the Laplace transform of the first problem by the second translation property. Again for the second part you see that here f t is this quantity, if you compare with this property, here f t is the f t given here in this term f t here is cos t. So, the Laplace of cos t is p upon p square plus 1.

So, Laplace of this f t will be nothing but using second translation property, Laplace of ft will be nothing but e k power minus a p, a is 2 pi by 3 e k power minus 2 pi by 3 p into p into Laplace of cos t. Laplace of cos t is nothing but p upon p square plus 1. So, this is how we can find out Laplace transform of this problems using second translation property. Now, the same shifting property also hold for inverse Laplace transforms inverse Laplace.

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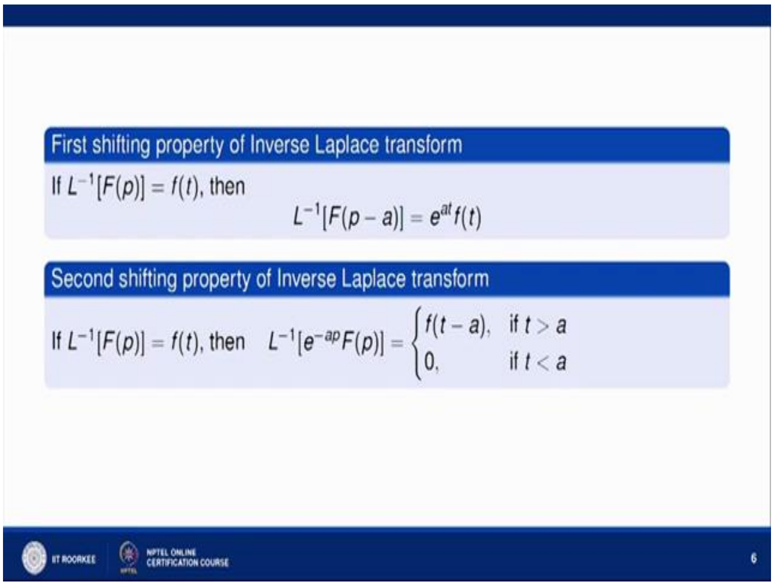

If inverse Laplace of f t is F p, then inverse Laplace transform of p minus a is e k power a t into f t. And the second shifting property states that if Laplace inverse of F p is f t then Laplace inverse of e k power minus a p into F p is nothing but f of t minus a when t is greater than 0m when t less than a, this is by the shifting property itself.

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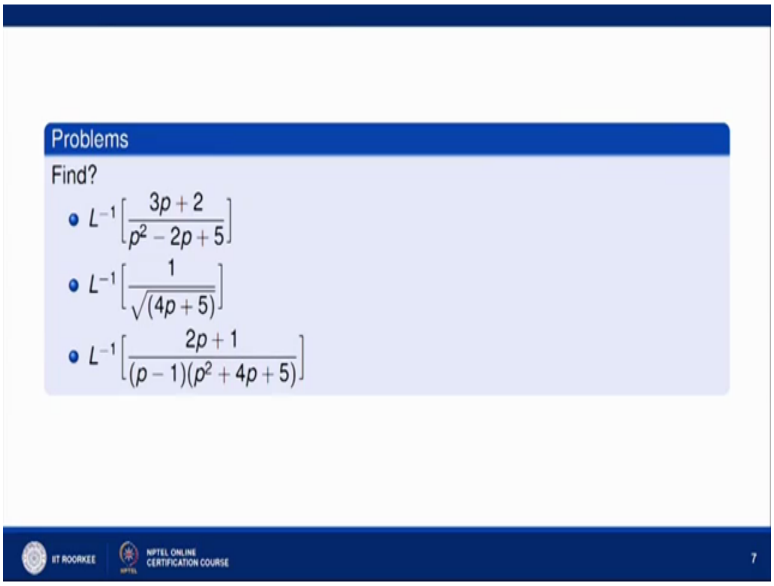

Now, based on this, let us try to solve these problems.

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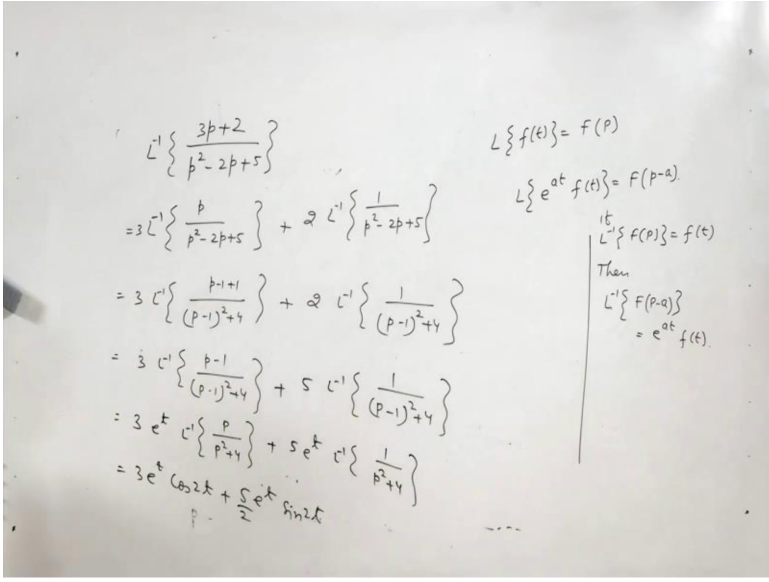

So, the first problem is Laplace of 3 p upon plus 2 upon p square minus 2 p plus 5. So, Laplace of this is 3 times p upon p square minus 2 p plus 5 plus 2 times Laplace inverse it is Laplace inverse yeah. So, it is Laplace inverse of 1 upon p square minus 2 p plus 5. So, it is 3 Laplace inverse of p upon it is p minus 1 whole square plus 4, it is 2 times Laplace inverse of 1 upon p minus 1 whole square plus 4. Now, here p minus 1, so you subtract 1 and add 1 here, so this is 3 Laplace inverse of p minus 1 upon p minus 1 whole square plus 4 and 3 into 1 upon this quantity plus 2 into 1 upon this quantity will be 5 times Laplace inverse of 1 upon p minus 1 whole square plus 4.

Now, we know that Laplace inverse of F p is f t, if this happens then Laplace inverse of F p is minus a will be nothing but e k power a t into f t, by the inverse of property shifting property. Now, here if you use this definition F p minus a, here a is 1. So, it is nothing but 3 e k power a t, a is 1, so 3 into e k power t. And f t and f t will be Laplace inverse of F p. So, it will be Laplace inverse of p upon p square plus 4. You replace you replace p plus 1, so it will be p upon p square plus 4. So, that will be Laplace inverse of this quantity. Again plus you replace p by this property is nothing but 5 into e k power t Laplace inverse of 1 upon p square plus 4. So, this is nothing but 3 into e k power t this is cos 2 t plus 5 into e k power t, you divide multiply into 2. So, it is 2 upon this sin 2 t, so that will be the Laplace inverse of this expression that will be Laplace transform of this expression.

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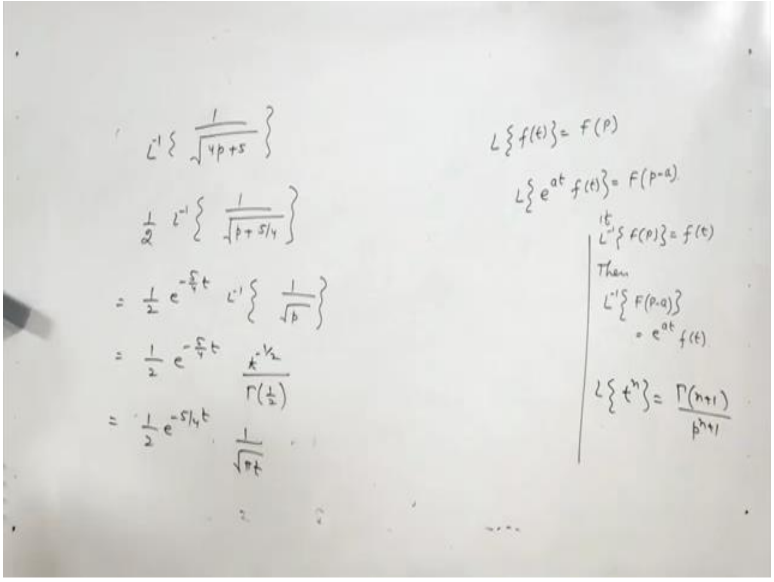

Similarly, when you apply the same thing in second problem Laplace inverse of this, so it is 2 p plus 1, it is 1 upon under root 4 p plus 5, second problem. So, you take 4 common first. So, it is 1 by 2 Laplace inverse of 1 upon under root p plus 5 by 4. Use this shifting property. Now here a is minus 5 by 4, so it will be 1 by 2 e k power minus 5 by 4 t Laplace inverse of one by under root p by the shifting property or for inverse Laplace transforms. And this is again is equal to 1 by 2 e k power minus 5 by 4 t, we know that now for to find Laplace inverse of this expression, we know that Laplace transform of t k power n is nothing but gamma n plus 1 upon p k power n plus 1.

So, Laplace inverse of 1 upon p k power n plus 1 is nothing but t k power n upon gamma n plus 1. So, replace n by n by minus half. So, it is t k power minus half upon gamma half, so that is nothing but 1 by 2 e k power minus 5 by 4 t 5 by 4 into t into 1 by under root pi into t so that will be the Laplace inverse of this expression. Now, to solve the last problem we can use partial fractions. This we use partial fraction to simply this and then we apply the shifting property to find out the Laplace inverse of this expression. The third problem we can solve.

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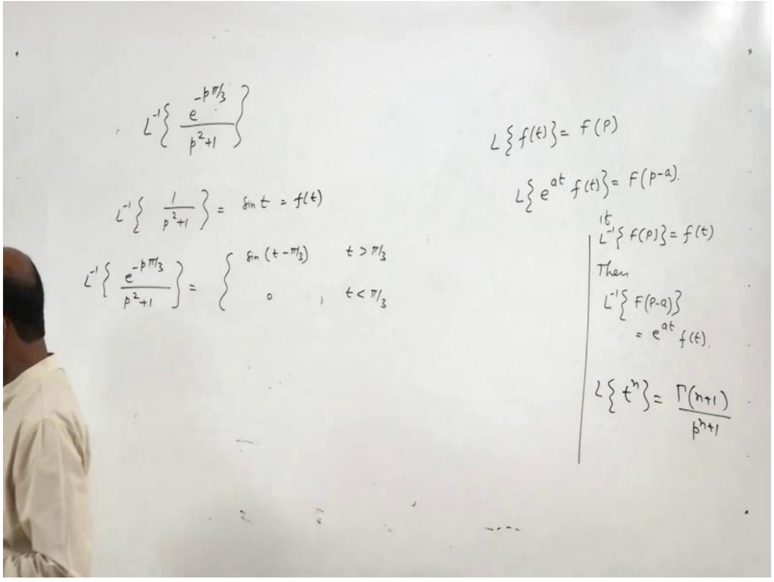

Now to find out Laplace inverse of these two problems again we will use second shifting second translation property for inverse Laplace transforms. First we will find Laplace inverse of here we have problem e k power minus p pi by 3 upon p square plus 1. So, first find Laplace inverse of 1 upon p square plus 1, so that is nothing but sin t. Now, using second translation property Laplace inverse of e k power minus p pi by 3 upon p square plus 1 will be nothing but sin t minus pi by 3, when t greater than pi by 3, and 0 when t is pi by 3. And this is because by the by the second property by the second translation property we have seen that Laplace inverse of this quantity is nothing but f of t minus a when t greater than a 0 t when less than a.

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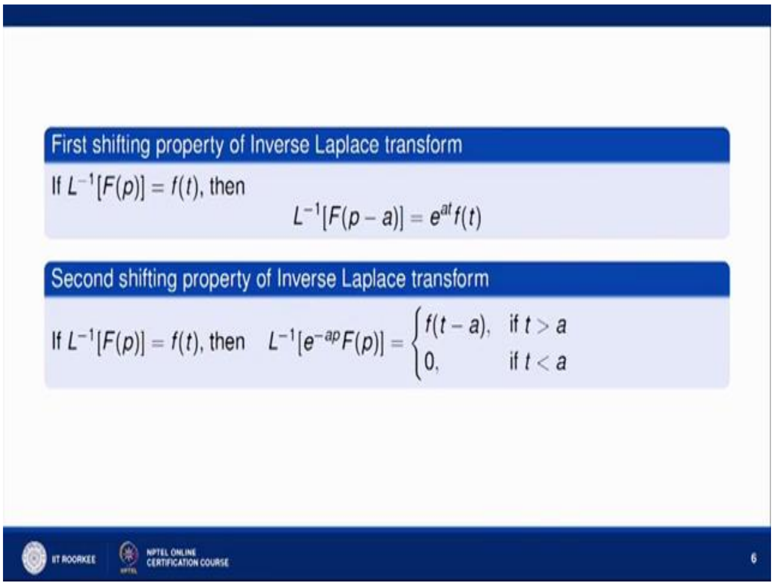

Here a is minus pi by 3. So, you replace t minus pi by 3 in this f t, this is f t, this is f t. Now, for second the last problem again we will first find Laplace inverse of p plus 1 upon 1 p square p plus 1 and then using second shifting property, we will find Laplace inverse of entire expression.

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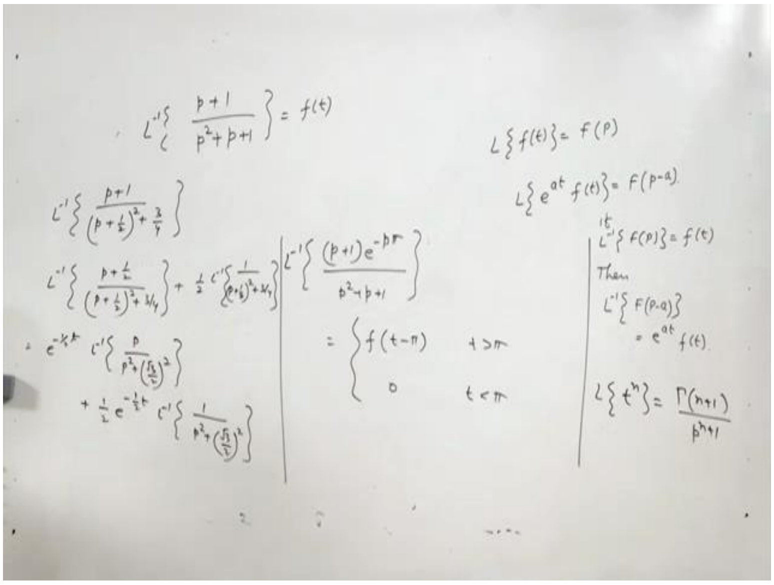

So, what is, so first we will find Laplace inverse of this expression p plus 1 upon p square plus p plus 1. We will call it f t. And then for this entire expression, we replace t by t minus pi the Laplace inverse of this will be nothing but minus p pi, this will be nothing but f of t plus t minus pi when t greater than pi and 0 when t less than pi. So, this we can obtain by using second property. So, Laplace of inverse of this, we can find making perfect square in the denominator.

Let us to make it this is p plus half whole square plus 3 by 4, and this is nothing but Laplace inverse of p plus half upon p plus half whole square plus 3 by 4 plus 1 by 2 Laplace inverse of 1 upon p plus half whole square plus 3 by 4. And it is e k power minus 1 by 2 t by using shifting property p plus half plus half is 1. And it is 1 by 2 square plus yeah it is e k power e k power a t a is minus half minus half is Laplace inverse of p upon p square plus under root 3 by 2 whole square plus 1 by 2 e k power minus 1 by 2 t Laplace inverse of 1 upon p square plus under root by 3 by 2 whole square. Now, this we can find out this is nothing but cos a t, a is under root 3 by 2, you multiply and divide by this quantity here, this will be nothing but sin a t and sin a is under root 3 by 2. So, that will be f t and you replace you substitute f t here, so you will get Laplace inverse of this expression.

So, thank you very much.