Mathematical methods and its applications

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Lecture – 25

Existence theorem for Laplace Transforms

Hello everyone, we have already discussed what Laplace transforms are, and basically what are the Laplace transforms of some standard functions like t k power n or e k power a t or sin a t, cos a t that we have already discussed. Now, we will see what is the existence theorem for Laplace transforms that means, what are the condition the function f x or f t must hold, so that we can say that Laplace of f t exist. How can we guaranty that the function posses, if the function posses these properties then the Laplace transform of f a function where exist.

So, in this topic, in this lecture, we will deal with existence theorem that what are the conditions so function should have, so that we can say the Laplace transform of function exist. So, before defining existence theorem for Laplace transforms, we will deal with two terms; one is piecewise continuous function, and second is exponential order.

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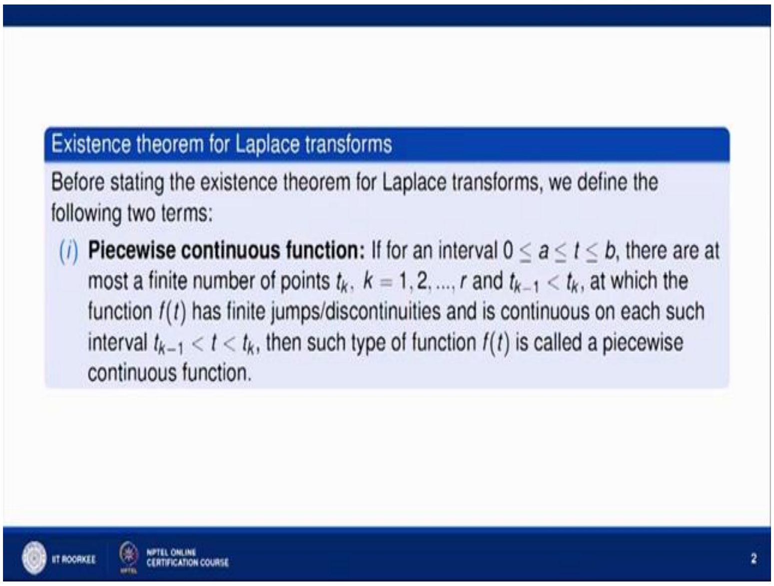

So, what is piecewise continuous function? Now, we already know what are continuous function is continuous function means if you are talking about continuous function that means, that the limit of the function at that point exist and that must be equal to value of the function at that point.

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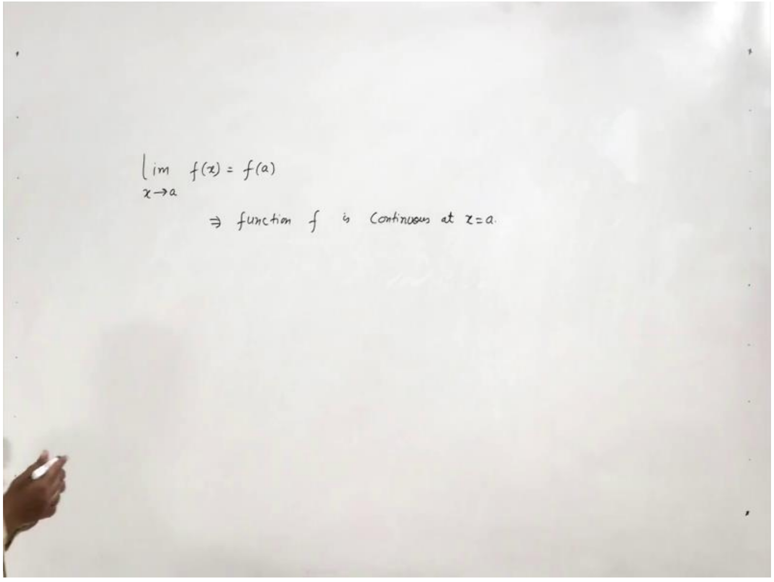

If you are talking about continuity, so continuity means limit x tending to a f x must be equals to f a. So, this means function is function f is continuous at x equal to a, that means if this limit exist that means, left hand limit is equal to right hand limit and that is equals to the value of the function at that point. So, we say there is a function is continuous at that point.

Now, if it holds over line that domain of the function then we say the function is a continuous over the entire domain d suppose. Now, what do we mean by piecewise continuous function? Suppose, we have an interval and in that interval function have some discontinuity, but if we break the function into finite number of subintervals and in each subintervals function is continuous, then we say that function is piecewise continuous. That means, suppose you have a interval 0 to b suppose you have some interval 0 to b, you divide that interval into finite number of subintervals in each subinterval function is continuous, but at the end points function as some discontinuity and that two of jump discontinuity then we say that the function is piecewise continuous.

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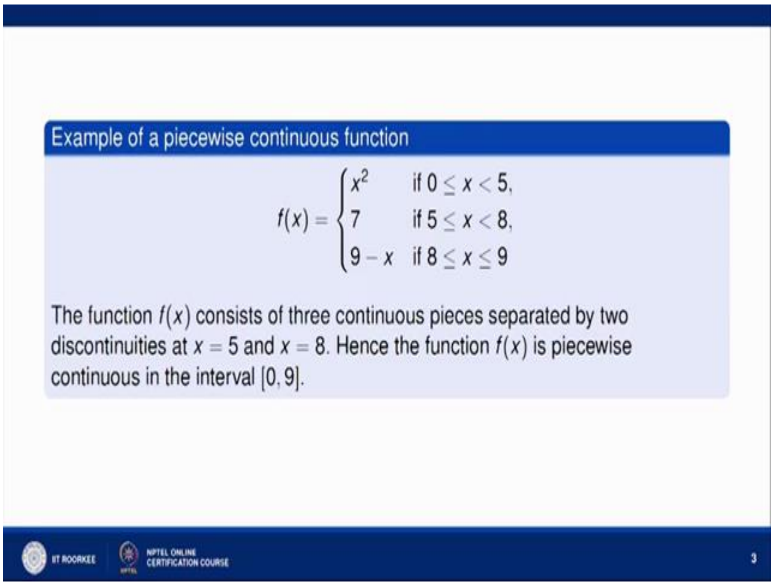

We will illustrate this definition by an example.

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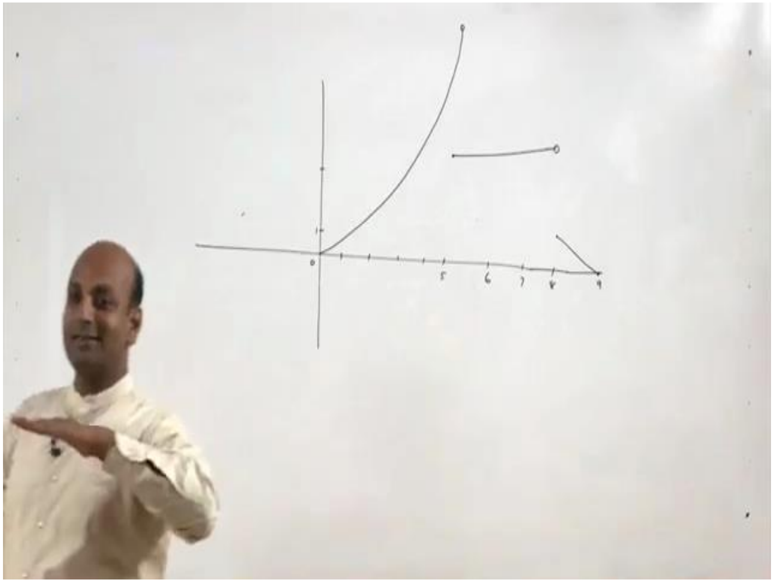

Now, suppose f x is equal to x square when x in between 0 to 5 and so f x equal to x square. Now, it is 0 1, 2, 3, 4, 5; between 0 to 5 this is x square that is something like this, this is 5. And from 5 to 8, the function has value 7. So, this is some 7 point here suppose, so function is 7 here. From 5 to 8 function is it is 6, 7, 8 function is 8. At f equal to 8, function is though we have a hollow here, we have a circle point bold point here. And at x equal to 5, this we have a bold here. Now, from x equal to 8 to 9, we have a function 9 minus x. So, at x equal to 8, function will be function value of when we put x equal to 8 here, so 9 minus 8 is 1. So, function as a value 1. So, suppose one is here. So, suppose function has a value one here when x is 8. And 8 to 9 function is 9 at 9 is 0, at 9 it is 0 function as straight line positive function here.

So, this is how this function looks like geometrically. From 0 to 5 function is x square; from 5 to 8 function is a straight line having the value 7; from 8 to 9 again it is a straight line the given by 9 minus x. Now, if we take, if we see this function from 0 to 9, 0 to 9 function discontinuous; at which point function at discontinuous - two points we have x equal to 5 and x equal to 8, where function have discontinuity. And the nature of discontinuity is jump type, jump type means it have discontinuity, but value is some, but it is not going to infinity.

You see that x equal to 5, if we approach from left hand side the value is x square or 25; and where the approach from right hand side the value is 7. At x equal to 8, if we approach from the left hand side, the value is 7; and if we approach from right hand side the value is 1. So, function have two discontinuity at x equal to 5 and x equal to 8. So, if we see from 0 to 5, 0 to 5 the function is continuous; from 5 to 8, function is continuous; from 8 to 9, function is continuous. So, if we break this interval 0 to 9 into three subintervals 0 to 5, 5 to 8, 8 to 9 in each three subintervals function is continuous. And it is having two discontinuity x equal to 5 and x equal to 8. So, this function we say as piecewise continuous function, because as a whole function is not continuous, but if we divide this function into finite number of subintervals in each subinterval function is continuous. So, we say the function is piecewise continuous function.

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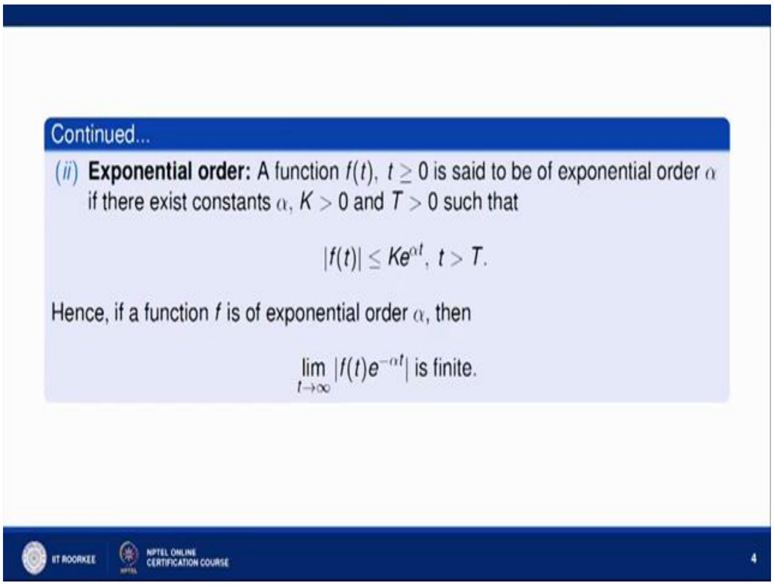

Now, the next is, the next point is the next definition is exponential order. Now, what do we mean by exponential order? Now, let us see this also. Now, a function f t, t greater than equal to 0 is set to of exponential order alpha.

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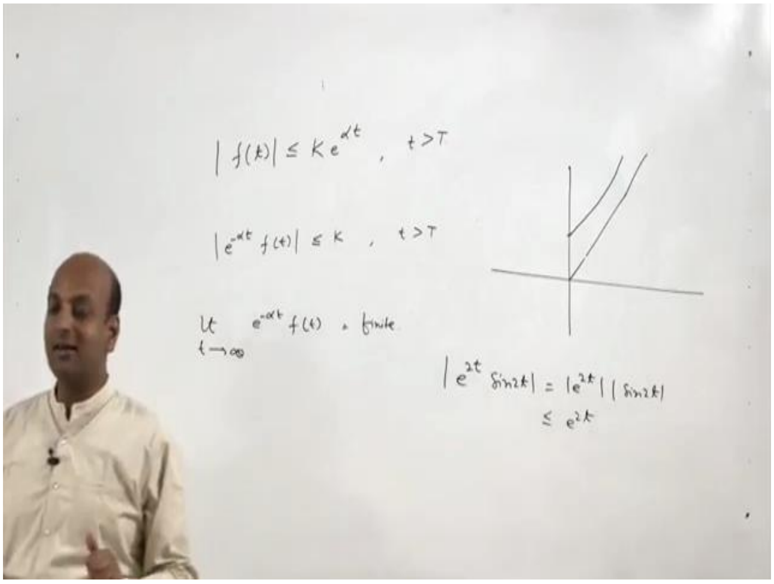

If there exist alpha k and t greater than 0, such that mod of f t is less than equals to k times e k power alpha t for t greater than t. So, if for a function f t, this condition hold there exist some alpha and k such that this condition hold then we say that the function is of exponential order alpha. Now, what does mean geometrically? It means that the absolute value of the function of f does not grow faster than the exponential function. This value is this mod of this value or less than equal to this value that means, the absolute value of the function f does not grow faster than exponential order that is e k power alpha t, for any t, for all t. If this holds for all t, though we say that the function is a exponential order alpha.

Now, we can also see that this is minus e k power alpha t f t modules will be less than equal to k as t tending t that means, if becomes very large tending to infinity then this value is always finite because it is less than equal to k. So, that means that limit t tending to infinity e k power minus alpha t f t must be finite value. It is a finite value. So, if a function is a exponential order alpha then for that alpha this quantity as t tend to infinity must tends to a finite quantity. So, this is the result of the exponential order alpha from the definition we get this result.

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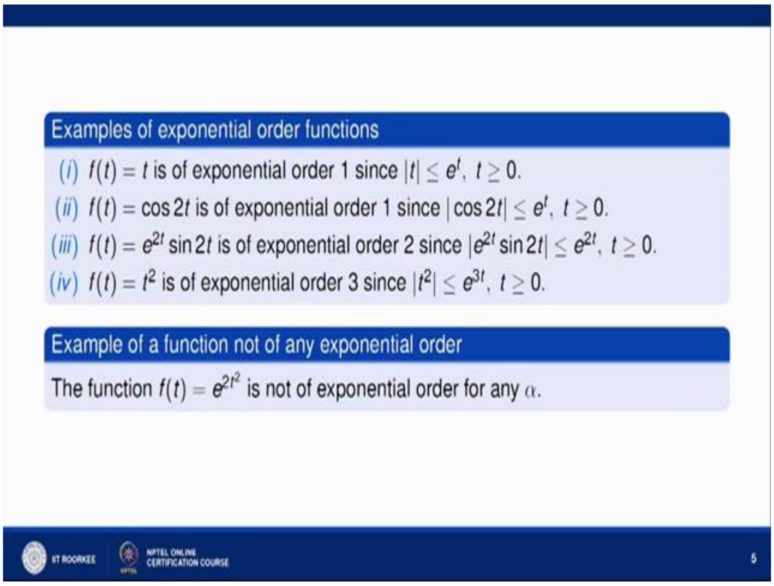

Now, we have some functions examples of exponential order alpha. Now, if we see f t equal to t the first function f t equal to t, it is clearly of exponential order alpha exponential order 1, where alpha is 1, since mod of t is always less than e k power t. You can visualize graphically also. If we see the graph of t when t is greater than equal to 0 is straight line is a straight line as t is a straight line, and e k power t graph of e k power t is something like this at t equal to 0 m and this is it is something like this. So, e k power t is always greater than equals to mod of t when t is greater than equal to 0 that means, there are exist some k which is 1 here, and there are exist alpha which is 1 for which this definition holds. So, we say that t function is equal to f t equal to t is a exponential order 1.

Similarly f t equal to cos 2 t is also exponential order one since mod of cos 2 t is less than equal to e k power t. Again we visualize the same definition graphically f t equal to e k power 2 t sin 2 t is of exponential order 2. Since if you take mod of this quantity mod of e k power 2 t sin 2 t, so it is nothing but it is equals to mod of e k power 2 t into mod of sin 2 t. And mod of sin 2 t is always less than equals to 1 for any t, so it always less than equals to e k power 2 t. So, here if we compare with this definition there exist k equal to 1 and alpha equal to 2, so that means, this definition holds. And since alpha is 2, so we say that this function e k power 2 t sin 2 t is of exponential order 2. Similarly, f t equal to t square is also of exponential order 3, since mod of t square is less than equals to e k power 3 t. This also we can prove also and we can visualize geometrical also that this inequality hold.

Now, we have also an example which is not of any exponential order I mean is not of exponential order of any alpha, for any alpha. So, what does it means, how can we say this? Now, we have seen that if a function of exponential order alpha, then limit t tending to infinity e k power minus alpha t into f t minus tends to a finite quantity. Now, let us check whether this function f t equal to e k power 2 t square holds that property or not it is that property or not satisfied that property or not.

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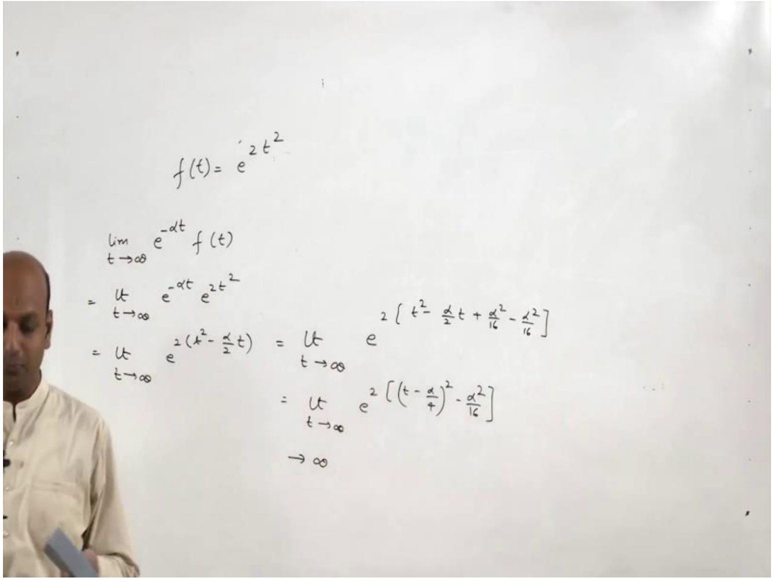

So, what is f t here, what is f t now? f t is e k power 2 t square. So, you find e k power minus alpha t into f t, this limit as t tending to infinity. We try to find this limit, because if it is of exponential order alpha, so this limit must tend to a finite quantity. If it is not tending to a finite quantity this means, it is not a exponential order for any alpha. So, here we are taking an arbitrary alpha, so alpha may be any constant. So, for this particular problem, so t tending to infinity e k power minus alpha t, what is e f t e k power 2 t square? So, this is nothing but limit t tend into infinity e k power 2 t square minus alpha by two times t.

Now, this is nothing but is equal to this is equal to limit t tend into infinity e k power now we will make perfect square here, here is t square, here is minus alpha by 2 times t. So, we will add and subtract alpha square by 4 in this expression. So, it is 2 t square minus alpha by 2 t plus alpha square by 4 minus alpha square by 4. So, let us see limit t tending to infinity, it is e 2, it is t minus alpha by 4 whole square. So, since alpha by 4 square, so we have to subtract alpha square by 16 here add and subtract alpha square by 16. So, it will make it perfect square and minus alpha square by 16.

Now, when you take limit as t tending to infinity, since it has a positive coefficient of t here as e k power positive coefficient of t square, so it will tends to infinity of course, So, it will tends to infinity. So, this means that it is not of exponential order for any alpha because we have taken alpha is a arbitrary constant. So, this if we take any alpha, this always tend to infinity, so we can say that this function e k power 2 t square is not of exponential order for any alpha.

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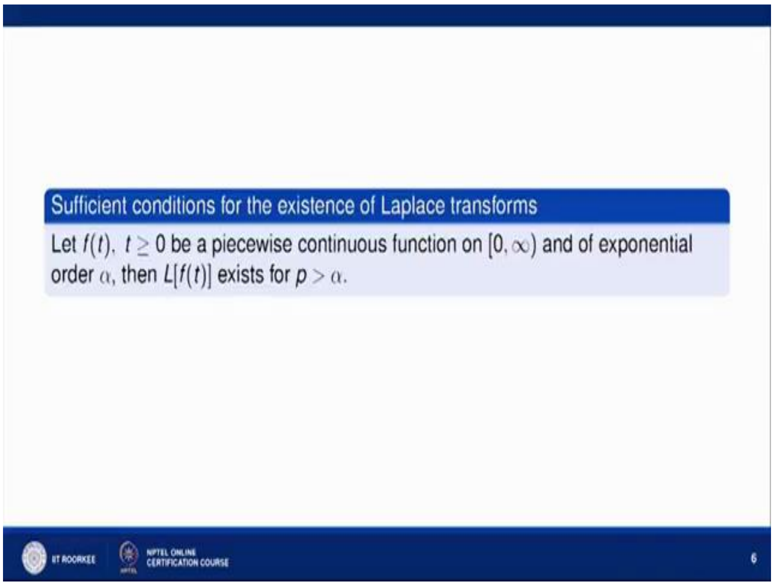

Now, we come to sufficient condition for the existence of the Laplace transforms. So, we have already seen what are piecewise continuous functions. And when we say, there a function of a exponential order alpha. Now, if f t is piecewise continuous function for t greater than equal to 0 and is of exponential order alpha then Laplace transform f t always exists, this is sufficient condition for the existence of Laplace transform. So, for the existence condition two conditions must hold; if function is piecewise continuous and exponential order alpha then Laplace transform of that function always exists, this we can surely say; now what is a proof of this, now how can we say this though proof is very simple.

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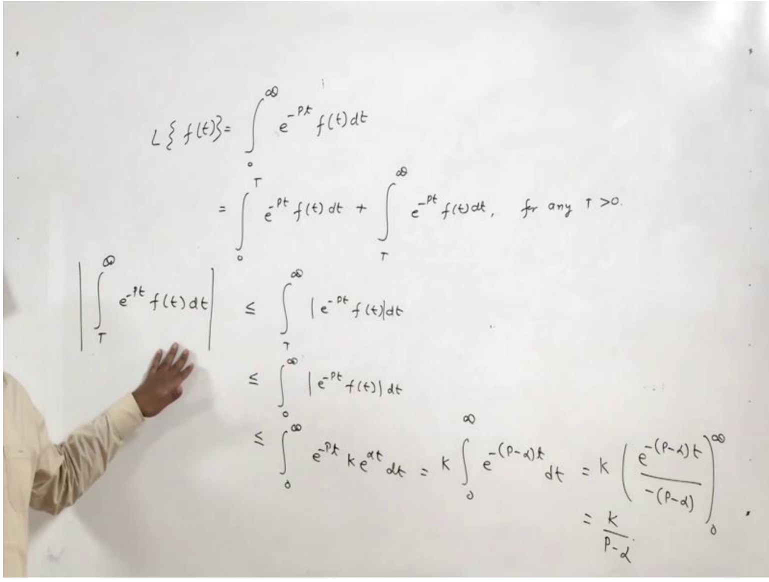

Now, what is Laplace transform of f t, it is nothing but 0 to infinity e k power minus p t f t dt, this is how we define Laplace transform. Now, this can be written as 0 to t e k power minus p t f t dt plus t to infinity e k power minus p t f t dt, this is for any t. For any t greater than 0 for any t greater than 0, we can always break this 0 to infinity as 0 to t plus t to infinity. Now, since f t is piecewise continuous, it is given f t is piecewise continuous that means, it has a discontinuity and that also jump type and interval is finite 0 to t. So, interval is finite and f t is piecewise continuous. So, this value always exist, existence means this values always have some finite value finite quantity, this value have some finite value. This expression has some finite value.

Now, what about this, how can we say that this also have a finite value. So, to see this, now what this expression is this is t to infinity e k power minus p t f t dt. So, if you take the mod of this, so this is less than or equals to t to infinity mod of e k power minus p t f t dt. This is by the result of integration. Now, this can be less than equals to 0 to infinity mod of e k power minus p t f t dt, because if we increase the size of the interval here then the value will increase definitely, because this quantity is will always positive. Here we have t to infinity here we are increasing the interval from 0 to infinity, so value of this expression will be always more than the value of this expression.

Now, this now since we know that f t is a exponential order alpha. So, by the definition exponential order alpha, this quantity is always less than equals to 0 to infinity e k power minus p t k e k power alpha t dt, these all are positive quantity. So, mod can be removed. Now, this is equal to k, we can take common 0 to infinity e k power minus p minus alpha times t into dt. So, this integration is nothing but k times e k power minus p minus alpha times t upon minus p minus alpha, from 0 to infinity. And since p is greater than alpha, so for greater than alpha this quantities p minus alpha is positive.

Now, we have an negative outside as t tending to infinity this will tend to 0. And at t equal to 0, this will tends to k upon p minus alpha. So, we have shown that this quantity is always less than equal to this quantity which is a finite number, finite real number. So, if this expression is less than equal to this expression this means this expression is also have some finite value, and finite value plus finite value is again a finite value that means, Laplace of f t will exist, because for existence this must we have some finite value. So, we have shown that Laplace transform of f t exist if it is a piecewise continuous function and exponential order alpha.

So, we break this function into two parts, since this function is piecewise continuous limit is finite. So, this value will always exist, exist means having some finite value and this because f t is of exponential order alpha. So, by this calculation, we have shown that this value is always less than equal to some finite value that means, that expression also exist. So, hence the Laplace transform is f t exist.

Now, important point to note here is if function is piecewise continuous and exponential order alpha, then Laplace exist; there is no doubt in that, because that is we have shown also that is the sufficient condition. Otherwise, if a function suppose not of exponential order or is not of piecewise continuous, then Laplace transform may or may not exist we cannot say, this is only a sufficient condition.

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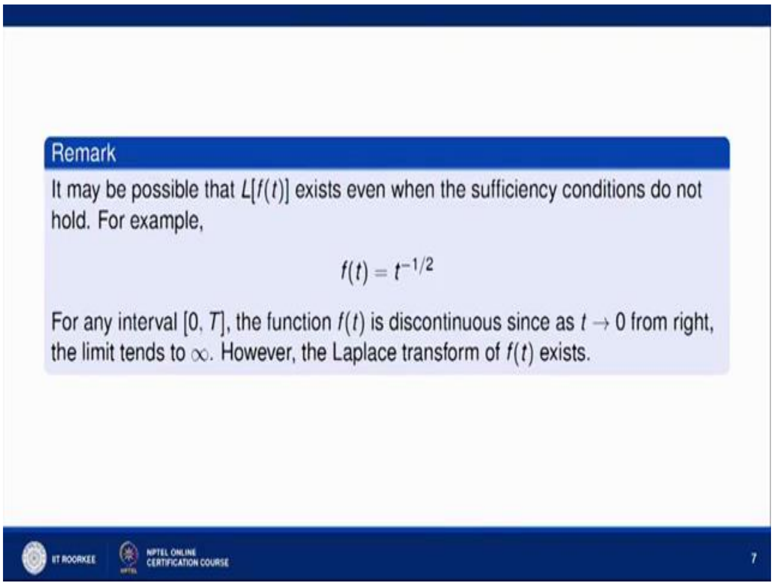

So, we have an example also. Suppose we have example f t equal to t k power minus half, if we see this example, this problem f p equal to t k power minus half.

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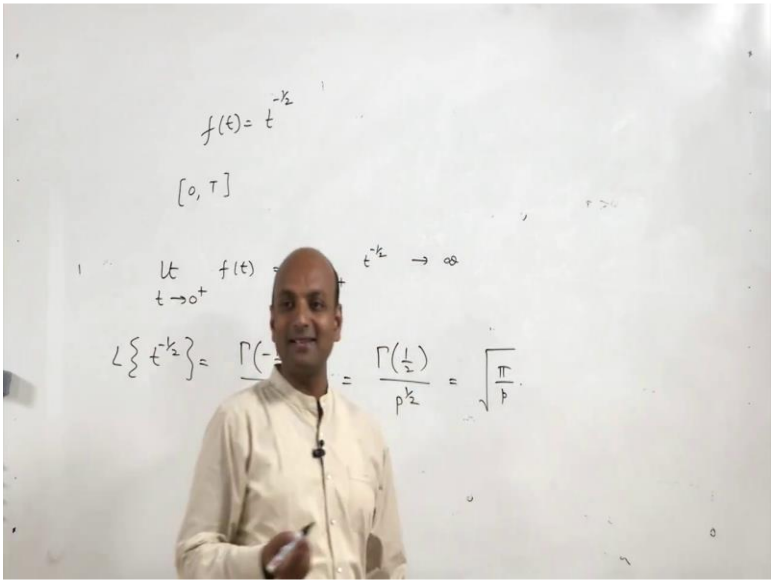

Now, the interval is, now you take any interval from 0 to t, for any t, you take any t no matter how small you are taking or no matter how big you are taking, you take any t 0 is always in the interval. Now, from the right side of the 0, if you take the limit t tending to 0 that means, if you take this limit, limit t tending to 0 plus of f t that is limit t tending into 0 plus of t k power minus half. So, of course, it will tend to infinity from the right hand side so that means, this function is not piecewise continuous because if it is piecewise continuous, so it must have jump discontinuities. So, this limit tending to infinity that means, this function is not piecewise continuous, but its Laplace exist.

How, because Laplace of t k power minus half will be nothing but gamma minus half plus 1 upon p k power minus half plus 1, because Laplace of t k power n is nothing but gamma n plus 1 upon p k power n plus 1 here n is minus half. So, it is nothing but gamma half upon p k power half and gamma half is under root pi. So, it is under root pi by p. So, hence Laplace exist, however, it does it not satisfied sufficient condition, so that means, if Laplace if function is piecewise continuous and in a exponential order alpha then surely Laplace exist otherwise we cannot say, so that is the that is only the sufficient condition.

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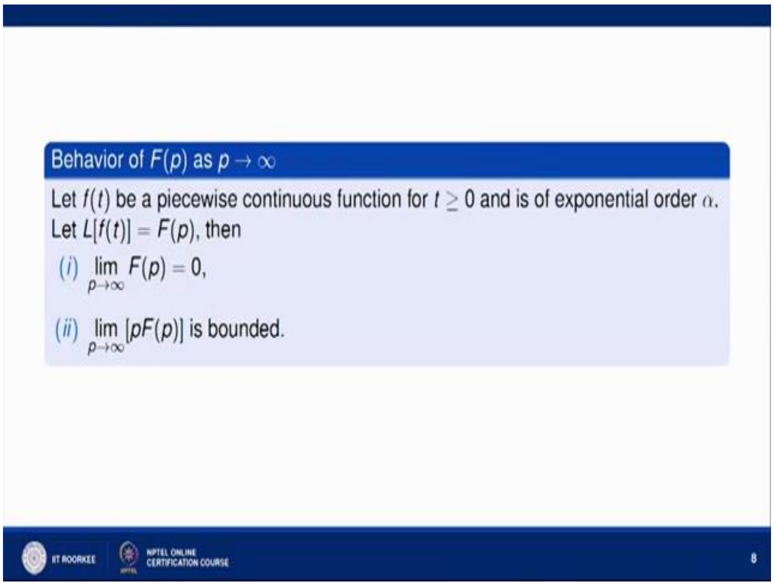

So, next is behavior of p, as p tending to infinity F p. So, let F p be a piecewise continuous function and is a exponential order alpha. Now, while you assume this condition always because by the sufficient condition we know that for the existence of the Laplace transform we must have if we state some theorem, so we need the existence of Laplace theorem. So, for that we are assuming that the function is piecewise continuous and is of exponential order alpha. Now, we have to prove now if Laplace of f t is F p then limit p tend into infinity F p is always 0.

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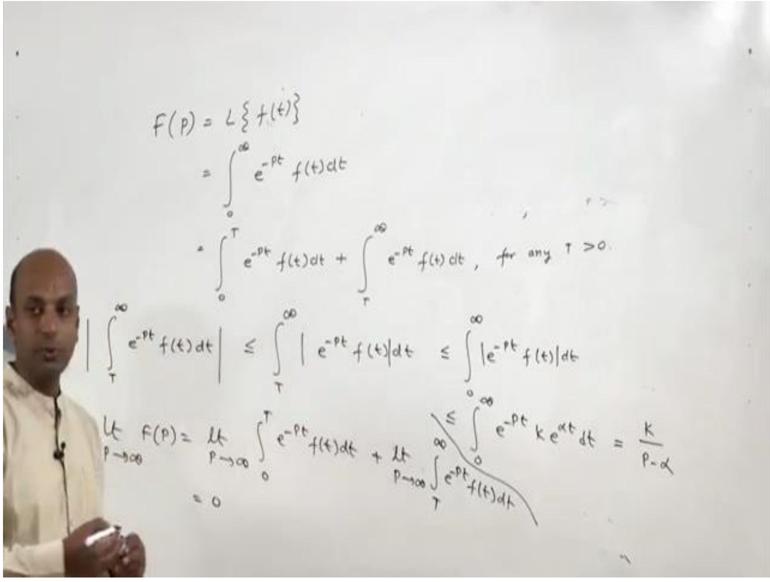

This is; what is F p, F p is nothing but Laplace of f t. What is Laplace of f t? It is 0 to infinity e k power minus p t f t dt. Now, it is again we can write it like this 0 to t e k power minus p t f t dt plus t to infinity e k power minus p t f t dt for any t greater than 0. Now, for this quantity again, if this quantity t to infinity e k power minus p t f t d t if it is this quantity, so this quantity less than equals to integral t to infinity mod of e k power minus p t f t dt. So, this is again less than equals to 0 to infinity e k power minus p t f t d t and since f is of exponential order alpha. So, therefore, this is less than equals to 0 to t e k power minus p t k into e k power alpha t dt 0 to infinity. So, this quantity we have just shown that this quantity is nothing but k upon p minus alpha. When you integrate this and taking the limit 0 to infinity for p greater than alpha this quantity will be this.

Now, as we take limit p tend into infinity limit p tend to infinity F p, so that means, limit p tend to infinity 0 to t e k power minus p t f t dt plus limit p tend to infinity 0 limit p tend to infinity t to infinity e k power minus p t f t dt. So, now, this expression, now when we take limit t tending to infinity this expression, so this will be less than equals to this quantity as p tending to infinity this quantity tend into 0 so that means, limit of this expression is less than equal to 0 that means, limit of quantity is equal to 0. As p tend to infinity since this quantity having some finite value, so as p tending to infinity since we have a e k power minus p t here and f is a piecewise continuous function. So, this value tending to 0. So, we can say that this quantity tending to 0 this quantity also tending to 0. So, this quantity will be equal to 0 or tends to 0.

So, this is the first result that limit p tend into infinity F p will be 0. Because if we take limit p tending to infinity here, so this quantity will tend to 0, and this quantity is also tend to 0 by in quality as p tend into infinity this tending to 0. So, this quantity will tend to 0, so hence 0 plus 0 is 0. So, what does it mean, it mean that whenever function is piecewise continuous and it is of exponential order alpha then limit p tending to infinity F p is always 0.

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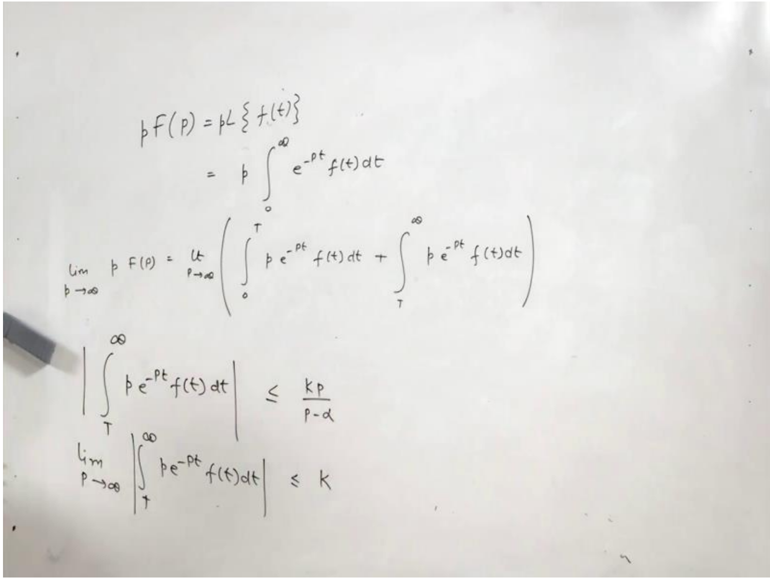

Now, the second result that limit p tending to infinity p F p is bounded. So, again we can show using the same definition p into p F p into Laplace transform of f t. So, p into it is 0 to infinity e k power minus p t f t dt, it is nothing but again 0 to t p into e k power minus p t f t dt plus t to infinity p into e k power minus p t f t dt. Now, limit p tending to infinity of p F p will be nothing but limit p tend into infinity of this whole expression. Now, this quantity we have already seen that mod of this quantity p into e k power minus p t f t dt this quantity that we have already seen that if it is not p here, so this expression is less than equals to k upon p minus alpha. This we have just shown using the property of exponential order because f is of exponential order alpha. So, we can say that since p is here, so this quantity will be less than equals to k p upon p minus alpha. So, as limit p tending to infinity, so this expression will be less than equals to k because this is tending to 1.

Now, this expression - the second expression is p tending to infinity is always less than equals to k as p tending to infinity. And the first expression, where you take p tend into infinity, so this is if you take here limit p tend into infinity, so this is tending to 0. So, we can say that this limit is always bounded because what we have finally coming to 0 plus less than equal to k that means, this expression is always less than equal to k that means, this limit is always bounded. So, the second result that limit p tend to infinity p F p is always bounded. So, again if p F p is not bounded, so we can say that they does not exist any piecewise continuous function, whose Laplace transform is that F p.

Thank you very much.