Mathematical methods and its applications

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Lecture – 24
Laplace Transforms of some standard functions

So, in the last class, we have seen what are Laplace transforms, and how to find Laplace transform of some simple functions like 1 t or e k power a t.

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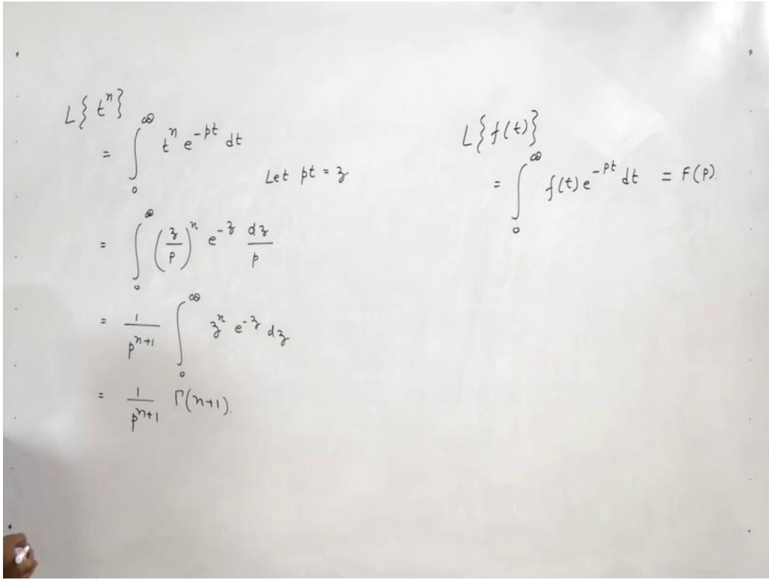

So, what is Laplace transform any function f t? Laplace transform any function f t is nothing, but 0 to infinity f t e k power minus p t dt, and we are calling it F p, function of p. So, also we have seen that Laplace transform satisfy linearity property. So, now, in this lecture, we will find Laplace transform of some standard functions. So, first what is Laplace transform of t k power n? So, Laplace transform of t k power n.

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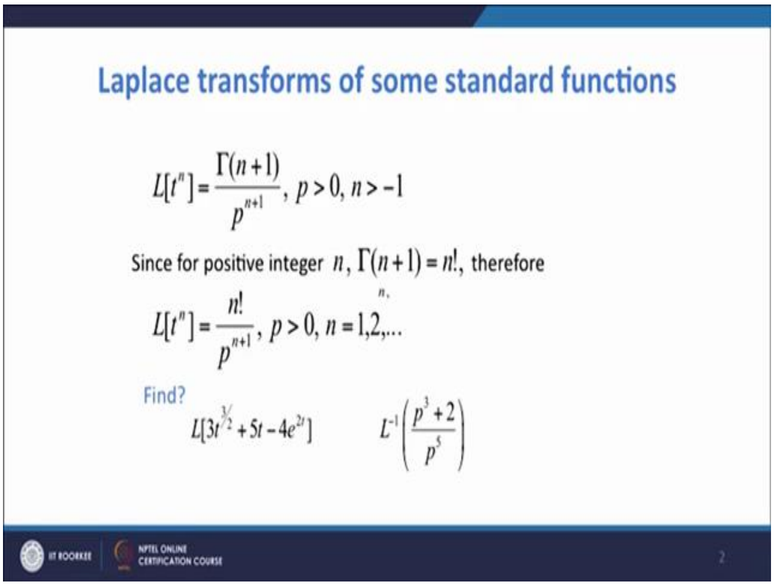

Let us try to find it Laplace or t k power n. So, using the definition of Laplace transform, Laplace t k power n nothing, but 0 to infinity, t k power n e k power minus p t dt. Now let p t is equals to some variable, say z. So, it will be nothing, but 0 to infinity z by p whole power n e k power minus z, and dt is nothing, but d z upon p. So, it is nothing, but 1 upon p t k power n plus 1 integral 0 to infinity z k power n e k power minus z into d z, and this function is nothing, but gamma function. So, it is 1 upon p n plus 1 gamma of n plus 1. Now, as gamma n plus 1 is factorial n, when n is a positive integer. Therefore, Laplace transform of t k power n will be factorial n upon p k power n plus 1, when n is a positive integer.

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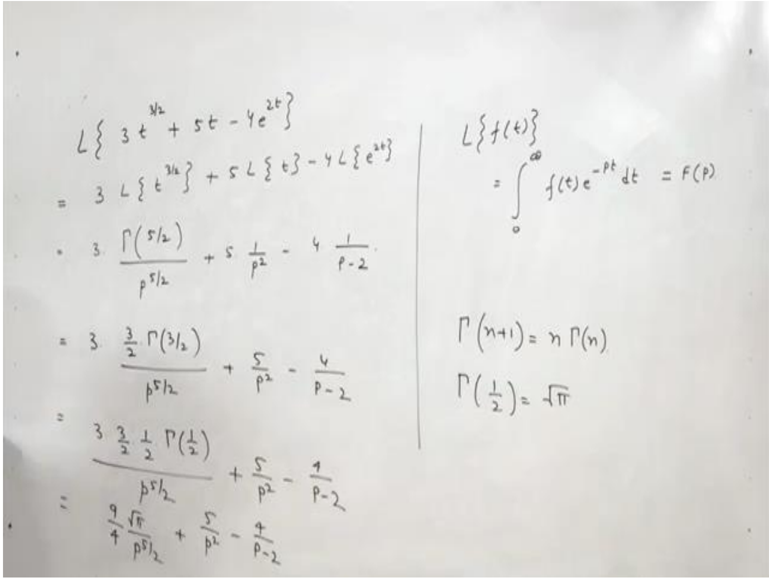

Now, let us try to solve the two problems given below, Laplace transform of 3 t k power 3 by 2, plus 5 into t minus 4 e k power 2 t. Now again by the linearity property of Laplace transform, this is nothing but 3 times Laplace of 3 by 2 t k power 3 by 2 plus 5 Laplace of t minus 4 Laplace of e k power 2. So, this is 3 into. Now Laplace of t k power n as in the first expression we are having, it is gamma n plus 1 upon p k power n plus 1.

So, here n is not a integer, here n is positive integer. So, we have to use the first definition of t k power n, it is gamma plus 1 upon t k power n plus 1. So, it is gamma. Here n is 3 by 2, so it is 5 by 2 upon p k power 5 by 2 plus 5 into. Laplace of t we have already seen that it is 1 by p square, also we can obtain here from here, also when we put n equal to 1 in the, sorry in the second expression. So, it is 1 by p square minus 4 into Laplace of e k power a t is 1 upon p minus a. So, it will be nothing, but 1 upon p minus 2.

So, now gamma n plus 1, we already know it is n gamma n gamma n plus 1 nothing, but n gamma n, and gamma is half under root pi, this we already know, so 3 into. Now gamma 5 by 2, so here it is 3 by 2 plus 1, so it is 3 by 2 gamma 3 by 2 upon p k power 5 by 2 plus 5 p square 5 by p square 4 upon p minus 2 is equal to 3 into 3 by 2. Again we will use a same expression; instead of n we have 1 by 2, because 1 by 2 plus is c by 2. So, 1 by 2 plus 1 is c by 2 it is 1 by 2 gamma 1 by 2. So, 1 by 2 gamma 1 by 2 p k power 5 by 2 plus 5 upon p square minus 4 upon p minus 2. So, it is nothing, but 9 upon 4 and gamma half is under root pi. So, it is under root pi upon p k power to 5 by 2 plus 5 upon p square minus 4 upon p minus 2. So, this is the Laplace transform of this function.

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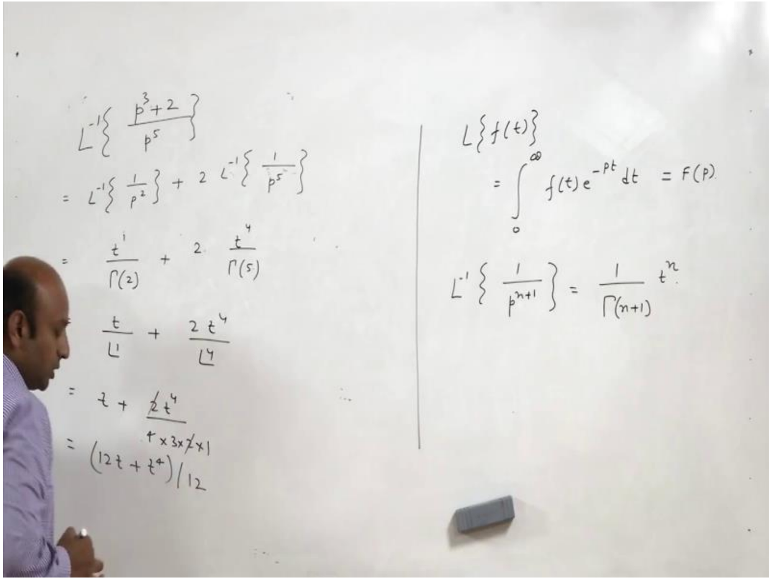

Now, how to find Laplace inverse of this expression p q plus 2 upon p k power 5? So, Laplace inverse of p cube plus 2 upon p k power 5. So, it is Laplace inverse of 1 by p square plus 2 into Laplace inverse of 1 by p k power 5. Now recall the definition, Laplace transform t k power n is given as gamma n plus 1 upon p k power n plus 1. So, what will be the Laplace inverse of 1 upon p k power n plus 1 from here? It is nothing, but gamma n plus 1 is a constant quantity. So, it will put, it will go to the right hand side. So, it will be noting, but 1 upon gamma n plus 1 into t k power n by that definition.

So, Laplace inverse of 1 by p square, so 1 by p square means n is 1, when n is 1, so it is nothing, but t k power 1 upon gamma 2 plus 2 into. Now for this expression n is 4, when n is 4. So, it will be nothing, but t k power 4 upon gamma 5, it is t. Now gamma n plus 1 is factorial n, when n is the positive integer. So, it is gamma 2 to this factorial 1 plus 2 t k power 4 upon factorial 4. So, it is t plus 2 t k power 4 upon 4 into 3 into 2 into 1. So, 2 2 cancels out. So, it nothing, but 12 t plus t k power 4 divided by 12. So, this will be the value of Laplace inverse of this function.

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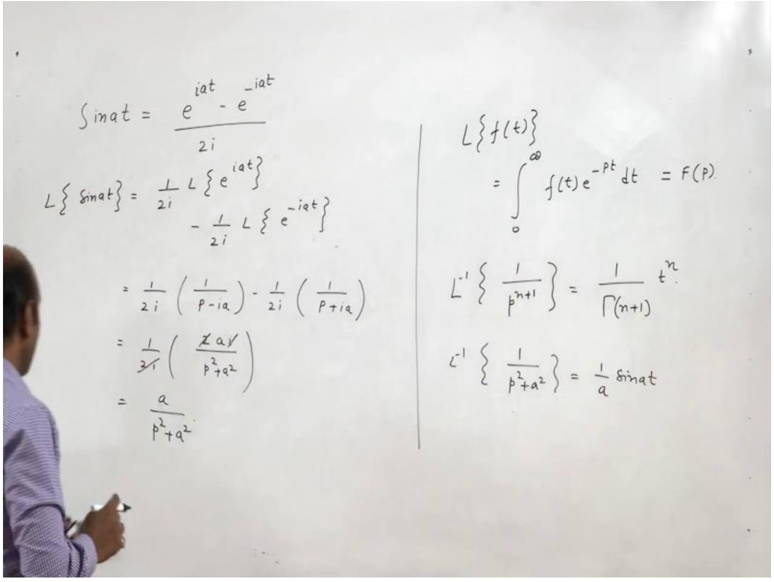

Now, how to find Laplace of sin a t? So, now, we will see how to find Laplace of sin a t. So, sin a t, as we already know is nothing, but e k power iota a t minus e k power minus iota a t upon 2 iota. This is sin a t. Now Laplace of sin a t, using linearity property of Laplace transform will be nothing, but 1 upon 2 iota, Laplace of e k power iota a t minus 1 upon 2 iota Laplace of e k power minus iota a t. So, it is nothing, but 1 upon 2 iota. As we already know let Laplace of e k power a t is 1 upon p minus a. Here instead of a, we have iota a. So, we will replace a by iota a 1 upon p minus iota a minus 1 upon 2 iota.

Again here instead of a, we are having minus iota a. So, we will replace a by minus iota a. So, we will get 1 upon p plus iota a. So, we will take 1 upon 2 iota a is common. So, and the numerator will get 2 a upon p square minus a square 2 iota a, sorry. So, 2 iota will cancels out. So, it is nothing, but it is p square plus a square when we multiply this 2. So, it is a upon p square plus a square. So, this is Laplace transform of sin a t it is a upon p square plus a square.

Now, so therefore, if you want to find out Laplace inverse of this function, this F p, this will be nothing, but sin at. So, what will be Laplace inverse of 1 upon p square plus a square, it will be nothing, but 1 by a sin a t. Now Laplace transform of cos a t, in the similar way we can find out Laplace transform of cos a t also. So, what is called cos a t in terms of t?

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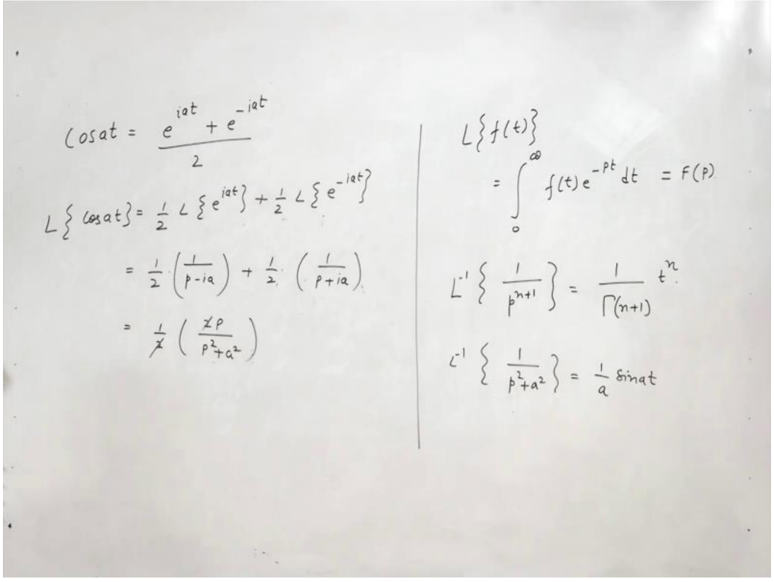

So, cos a t will be nothing, but e k power iota a t plus e k power minus iota a t upon 2, this we already known. So, to find Laplace transform of cos a t, again we will apply linearity property of Laplace transform, Laplace transforms of cos a t will be nothing, but 1 upon 2 Laplace transform of e k power iota a t plus 1 by 2 Laplace transform of e k power minus iota a t.

So, this is nothing, but 1 upon 2, Laplace transform of e k power a t is 1 upon p minus a. here instead of a we have iota a it is 1 upon p minus iota a plus 1 by 2 1 upon p plus iota a. So, when we simplify it, so we will get 1 by 2 into 2 p upon p square plus a square 2 2 cancel out. So, Laplace transforms of cos a t is nothing, but p upon p square plus a square. Similarly Laplace inverse of this F p, p upon p square plus a square is nothing, but cos a t. Now what is Laplace transform of sin hyperbolic a t. Now we will try to find out this expression, to prove this expression in fact, so what is cos hyperbolic a t, sorry sin hyperbolic a t.

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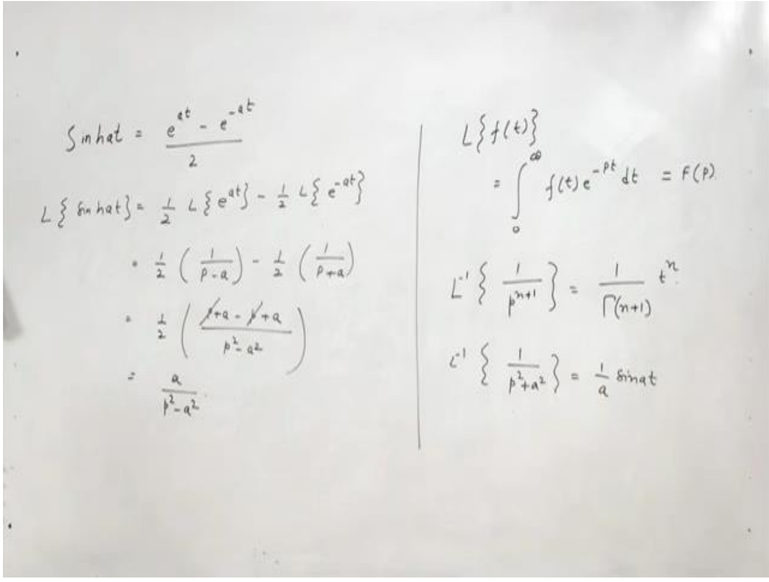

Sin hyperbolic a t is nothing but e k power a t minus e k power minus a t upon 2. This we already know. So, how to find Laplace transform of sin hyperbolic a t? Again we will use linearity property of Laplace transforms. So, Laplace transform of sin hyperbolic a t will be nothing, but 1 by 2 Laplace transform of e k power a t minus 1 by 2 Laplace form of e k power minus a t. So, it is 1 by 2 Laplace transform of e k power a t is 1 upon p minus a minus 1 upon 2 1 upon p plus a. So, when we simplify this. So, we will get 1 by 2, it is p plus a minus p plus a upon p square minus a square. This p p cancels out. So, it is nothing, but a upon p square minus a square.

So, this would be the Laplace of sin hyperbolic a t. Similarly, if you want to find out Laplace inverse of this expression this F p, so it will be nothing but sin hyperbolic a t. Now what is Laplace transform of cos hyperbolic a t. in the similar way, on the same lines, we can obtain the second expression this expression for cos hyperbolic a t.

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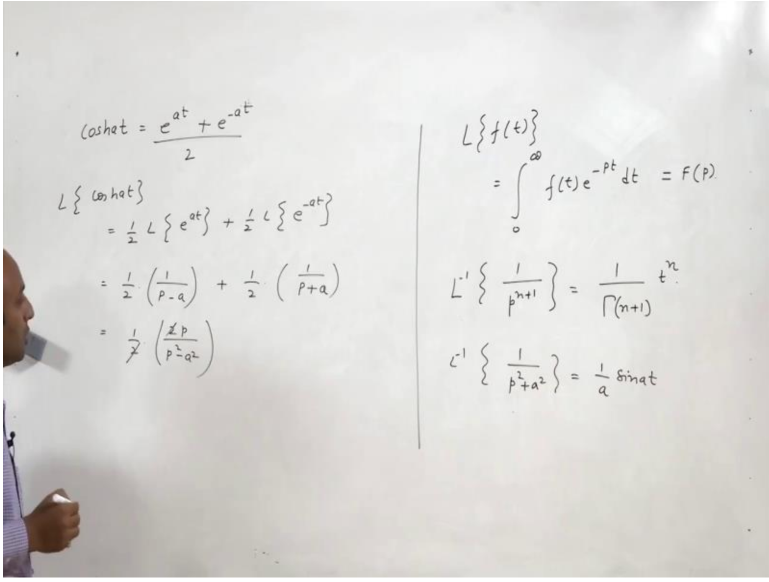

So, what is cos hyperbolic a t? Cos hyperbolic a t is nothing, but e k power a t plus e k power minus a t upon 2, this we already know. Now Laplace transform of cos hyperbolic a t, will be 1 by 2, Laplace transform of e k power a t plus 1 by 2, Laplace transform of e k power minus a t. So, it is nothing, but 1 by 2, 1 upon p minus a Laplace of e k power a t we already know, it is 1 upon p minus a plus 1 by 2, it is 1 upon p plus a. So, it is 1 by 2, numerator we get 2 p upon p square minus a square, 2 2 cancels out. So, this will be the Laplace of cos hyperbolic a t. Now Laplace inverse of p upon p square minus a square is similarly cos hyperbolic a t, because this Laplace of f t is p f, then Laplace inverse of F p will be nothing, but f t, that we already know. So, this we already seen, Laplace inverse of these 3 expressions we already discussed.

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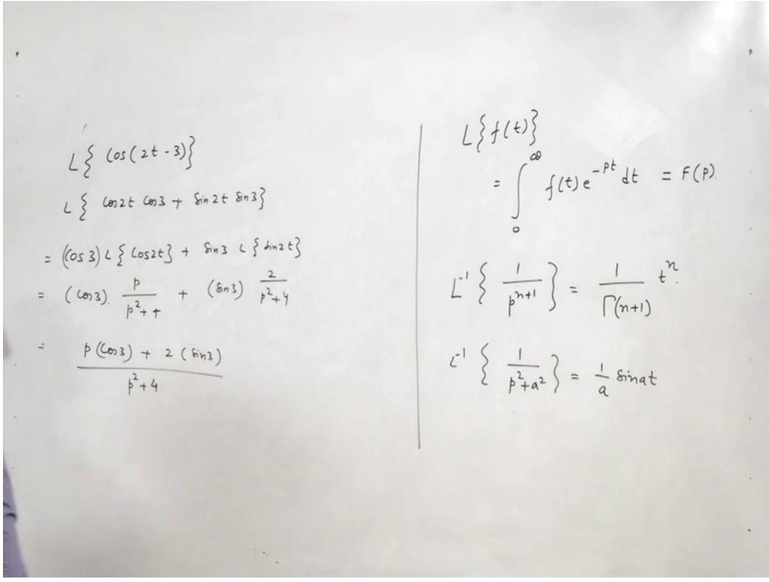

Now, how to find Laplace of this two function, this two simple function, let us see how to find Laplace of these functions Laplace of cos 2 t minus 3. So, we know the Laplace of Laplace transformation of cos a t, Laplace transform of cos a t is, p upon p square plus a square, but how to find Laplace transform of cos 2 t minus 3; that is a t plus b types how to find out the Laplace of this. So, it is cos a minus b. So, cos a minus b is nothing, but cos a cos b plus sin a sin b.

Now, cos 3 and sin 3 are the constants free from t. So, can be taken out, because of the linearity property of Laplace transforms, so it is nothing, but cos 3, Laplace transforms of cos 2 t plus sin 3 Laplace transform of sin 3 t sin 2 t. So, this is nothing, but cos 3, and Laplace transform of cos 2 t is, p upon p square plus 4 and plus sin 3 Laplace transform of sin 2 t is 2 upon p square plus 4, p square plus a square a is 2. So, this is nothing, but p into cos 3 plus 2 into sin 3 upon p square plus 4. So, this will be the Laplace transformation of cos 2 t minus 3.

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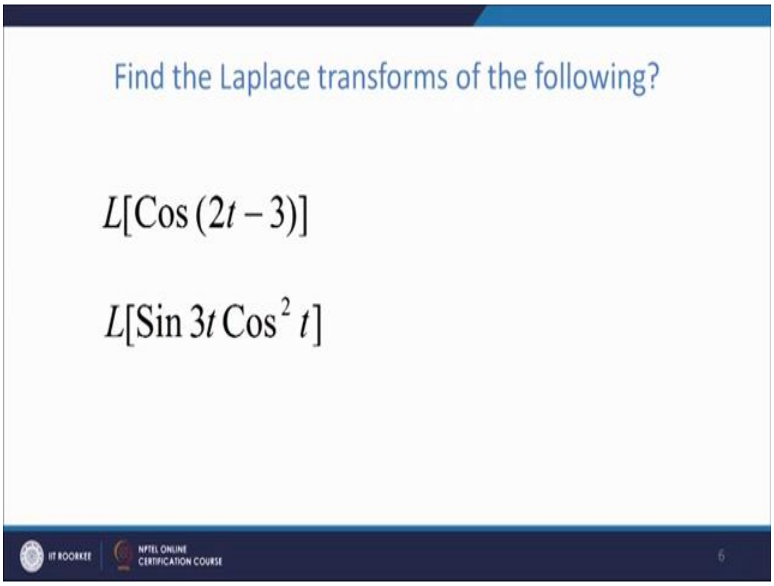

Now, the second problem, and now the next problem Laplace transform of sin 3 t into cos square t. So, Laplace transform of sin 3 t into cos square t.

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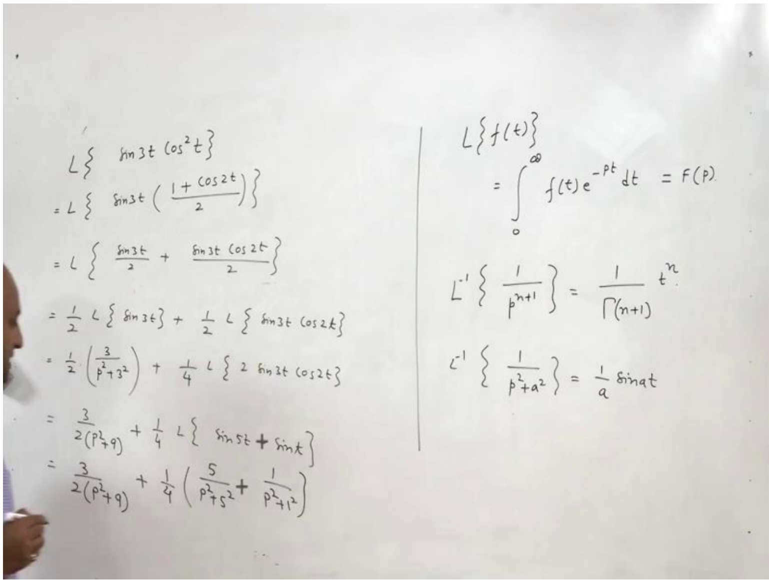

Now, we know how to find Laplace transform of sin 3 t, but we do not know how to find out Laplace transform of product of 2 functions, when both involves t. So, first we try to break this function. Now sin 3 t, what is cos square t? So, cos square t can be written as 1 plus cos 2 t by 2. Now this is nothing, but Laplace transform of sin 3 t upon 2 plus sin 3 t into cos 2 t upon 2. This can be written as 1 by 2 Laplace transform of sin 3 t plus 1 by 2 Laplace transform of sin 3 t into cos 2 t.

So, Laplace of transform of sin 3 t we already know, this is this is 3 upon p square plus 3 square plus, but again we have a problem here, how to find Laplace transform of this. So, we multiply and divide by 2 in this expression. Now 2 sin a cos b, we know that it is equal to sin a plus b plus sin a minus b. So, it is 3 by 2 p square plus 9 plus 1 by 4 Laplace of, it is sin 5 t a plus b 2 t plus 3 t 5 t plus this 3 t minus 2 t is t.

So, now we can find out the Laplace of this expression. So, this is 3 by 2 p square plus 9 plus 1 by 4. Laplace of sin 5 t is a upon p square plus 5 square plus sin t is 1 upon p square plus 1 square. So, this will be the Laplace transform of this function sin 3 t into cos square t. So, now how to find Laplace inverse of this simple F p, the simple function? So, Laplace inverse of, it is p minus 3 upon p square plus 4.

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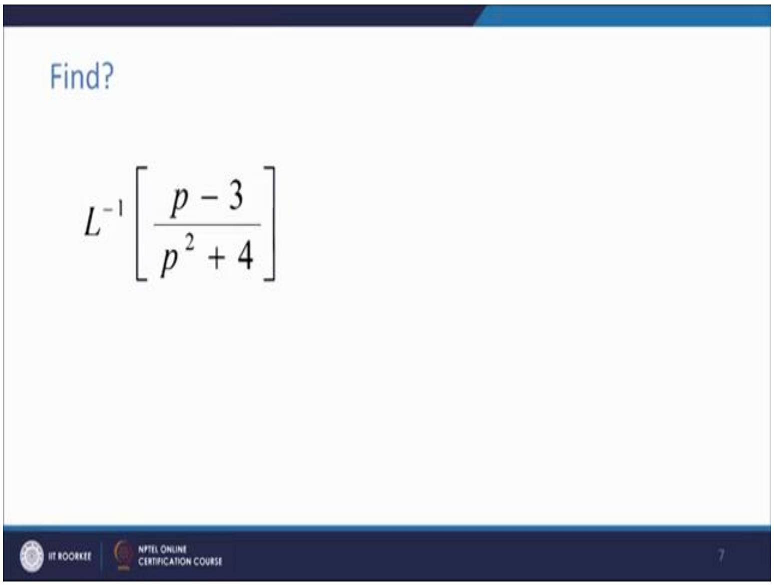

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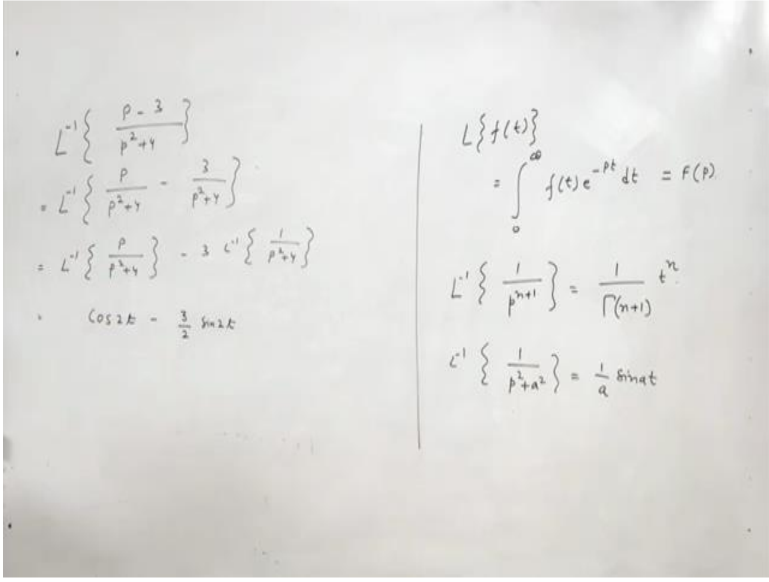

So, this is simple to find out, you just split this into 2 parts p into p square plus 4 minus 3 upon p square plus 4. Now, by the linearity property of inverse Laplace transforms, it is this into p upon p square plus 4 minus 3 times Laplace inverse of 1 upon p square plus 4. So, this Laplace inverse we already know. This Laplace inverse of this expression is cos 2 t, because it is 4 2 square. So, it is cos 2 t, because Laplace of cos 2 t is p upon p square plus 2 square; that is p upon p square plus 4. Now it is minus. Now Laplace inverse of 1 upon p square plus a square is 1 upon sin a t. So, here a is 2. So, it is minus 3 by 2 sin 2 t. So, this will be the Laplace inverse of this F p.

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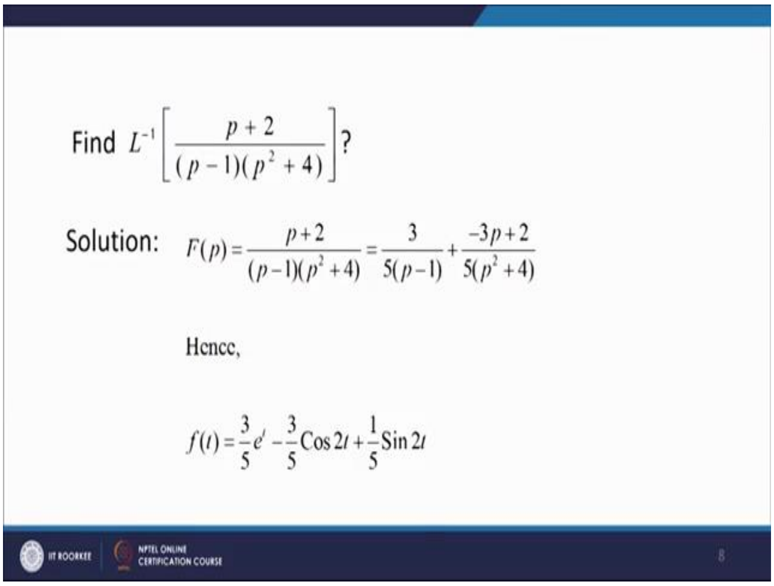

Now, how to find Laplace inverse of this problem? Now here F p is what? F p is p plus 2 upon p minus 1 into p square plus 4.

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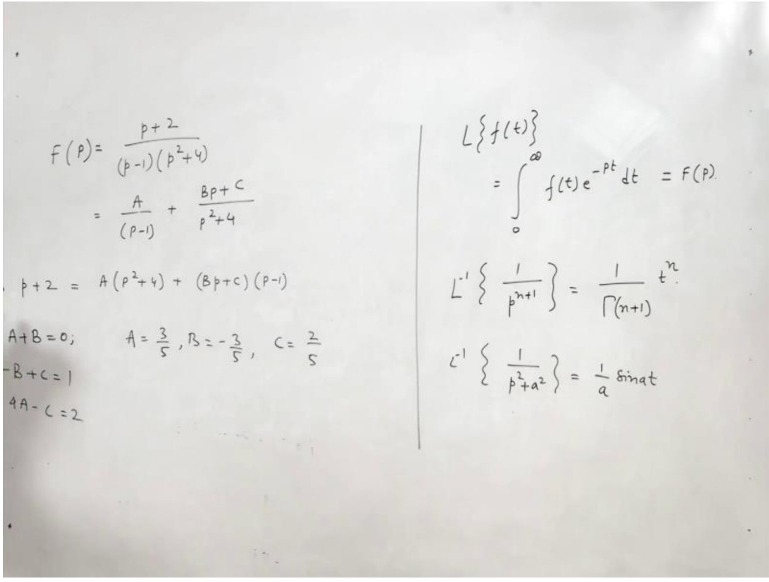

So, this is a linear part, and this is a quadratic part. So, how to find out Laplace inverse of this F p? So, this can be written as. Now we will make use of partial fractions. First we will try to reduce into partial fractions, and then we will try to find out its Laplace inverse. So, it can written as a upon p minus 1 plus.

Now, since it is a quadratic part. So, in the numerator we will be having a linear term, linear in p. So, it will be something b p plus c upon p square plus 4. So, compare the coefficients from the both side, this implies p plus 2 will equals to this into this; that is a into p square plus 4 and plus b p plus c into p minus 1. Now we have to compute the values of a b and c first. So, we can make use of; compare the coefficients of the both sides. Here the coefficient of p square is a, from here it is b and here there is no term involving p square, so a plus b will be 0.

And the coefficient of p here is 0, and the coefficient of p here is minus b plus c, and here it is 1, so it will be 1. And the constant here is 4 a minus c that here constant is 2. So, we are having 3 equations with 3 unknowns. So, we can find the values of a b and c. So, a here we have already obtained actually. So, a is 3 by 5, b is minus 3 by 5 and c is 2 by 5. This we can easily compute by solving these 3 equations.

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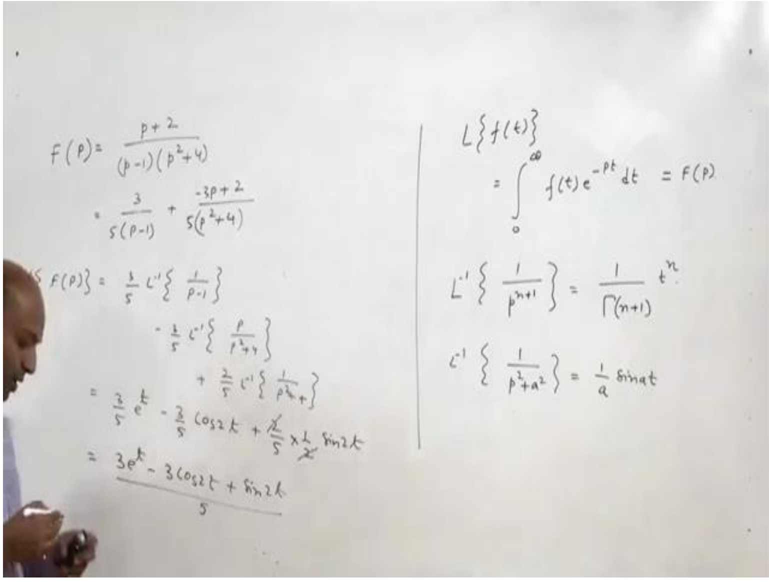

So, once we obtained the values of a b and c, then it is nothing, but, sorry it is nothing, but 3 by 5 times this, it is minus 3 plus 2 upon 5. Now the Laplace inverse of this F p will be nothing, but 3 by 5 times Laplace inverse of 1 upon p minus 1 minus 3 by 5 times Laplace inverse of p upon p square plus 4 plus 2 by 5 times Laplace inverse of 1 upon p square plus 4.

So, this is nothing, but 3 upon 5. Now Laplace inverse of 1 upon t minus a e k power a t here a is 1. So, Laplace inverse of 1 upon p minus a t is e k power t minus 3 by 5 Laplace inverse of this is cos 2 t. So, it will be cos 2 t a is 2, and here for this, it is plus 2 by 5 Laplace inverse of this expression a is 2. So, it will be nothing, but 1 by 2 times sin 2 t. So, the final answer is e k power 3 t e into 3 e k power t minus 3 cos 2 t this 2 cancels out plus sign 2 t upon 5.

So, we have seen that how we can Laplace transform of some standard functions; like sin a t cos a t sin hyperbolic a t or cos hyperbolic a t, and using these Laplace transforms, how we can find out Laplace transform of some functions or inverse Laplace transforms.

Thanks.