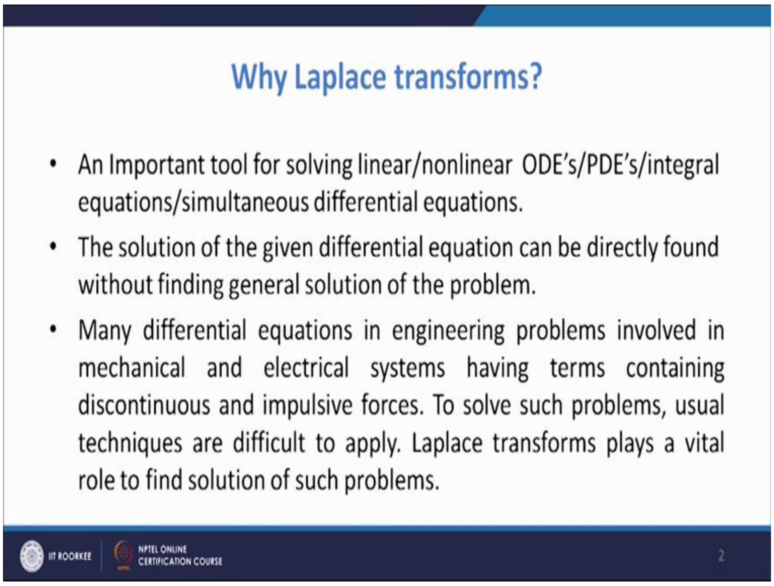
Mathematical methods and its applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 23
Introduction to Laplace Transforms

So, in this topic, mathematical methods and it is application. Here you are already studied various methods of ordinary differential equations and partial equations. Now the next topic is Laplace transforms. In Laplace transforms first we will deal that what Laplace transforms basically are, and why they are important. In what way what type of problems we can solve using Laplace transforms, and how can we solve. What are various properties of Laplace transforms and what happens if we have continues functions in our some real problems. How can we solve those type of problem? We will see in this unit.

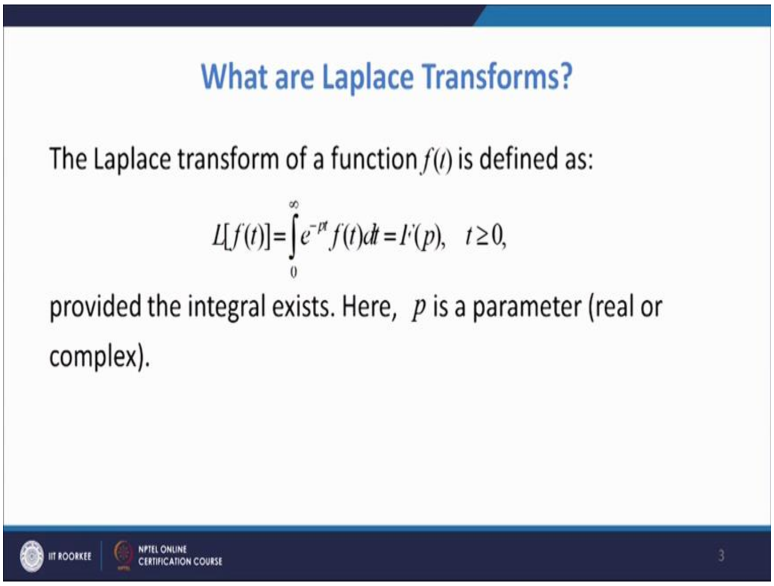
(Refer Slide Time: 01:11)



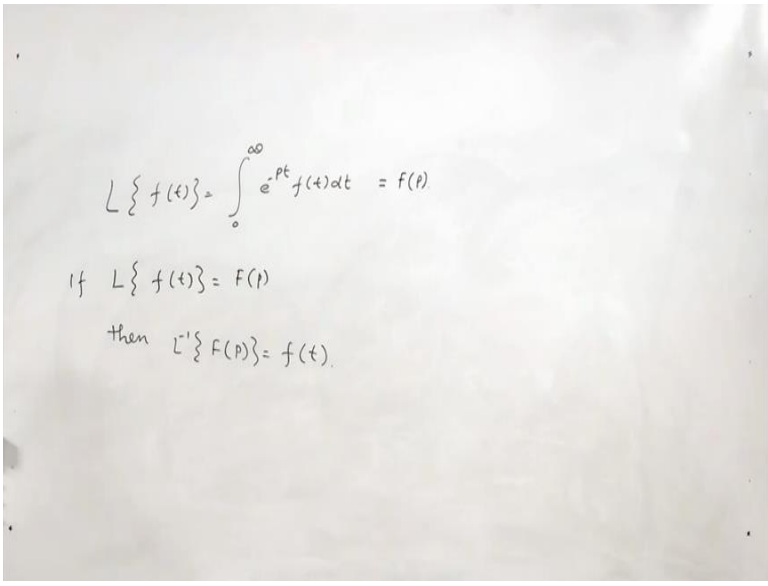
So, first is introduction to Laplace transforms. Now what, why Laplace transforms? So, Laplace transform is an important tool to solve any non-linear od or pd integral equations or difference equations. We already know that if we solve a Odeon differential equation or a partial differential equation first we have to find the compliment function. And then the particular integral, the sum of these 2 will give you the complete solution. While when we use Laplace transforms, we do not require to find out complimentary function of particular integral. We can directly find out the solution of the given system.

The next important point, why we use Laplace transform is there are some equations which use continuous or impulsive forces. In many engineering problems we have we can encounter such types of problems. To solve such problems usual techniques are not applicable or difficult to apply. So, we use Laplace transform to solve such type of problems. Now what Laplace transforms are.

(Refer Slide Time: 02:19)



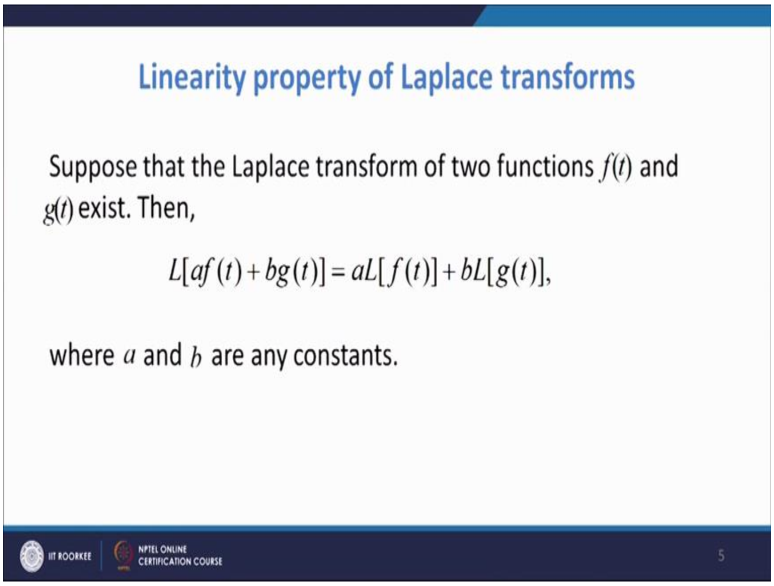
(Refer Slide Time: 02:31)



Now, Laplace transform function f t is defined as Laplace transform is simply given by Laplace transform of some function of f t is nothing but 0 to infinity e power minus p t f t d t. So, it holds for t greater than equal to 0. Now of course, we are integrating with respect to t. So, whatever integral we will obtain from here will be function of p, and here p is a parameter. So, we are calling it is some function of p. So, we are calling it F p capital F p we are calling it F p it will be a function of p.

Now, now next is inverse Laplace transform. Now if Laplace transform of f t is F p, which is a function of p. Then if this happens then Laplace inverse of F p is nothing but f t. Laplace inverse F p is nothing but f t.

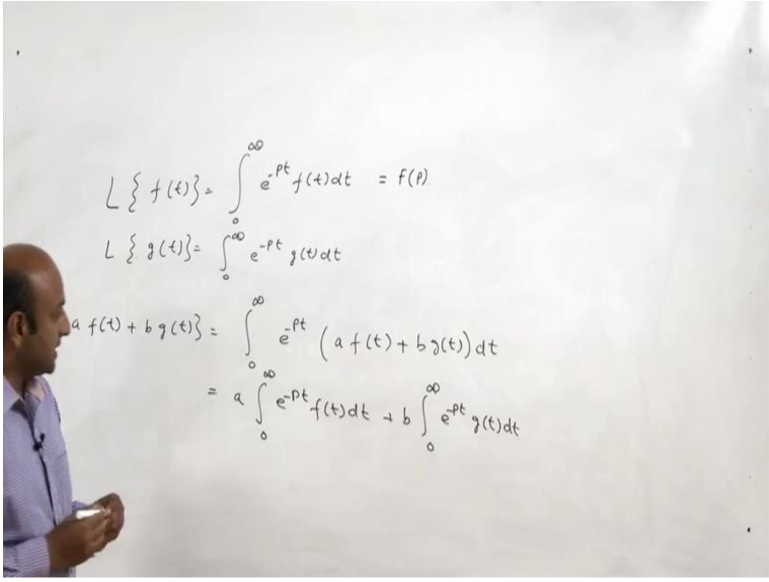
(Refer Slide Time: 03:41)



Now, the first property of Laplace transform is Laplace transform is a linear operator. It satisfies linearity property. Now what linearity property is Laplace transform of a f t plus b g t is nothing but a times Laplace of f t plus b times Laplace of g t and the result hold for any for any a and b constants for any constants a and b.

Now, let us try to this prove proof is very simple. As we already know that Laplace transform of f t is given by this expression.

(Refer Slide Time: 04:19)

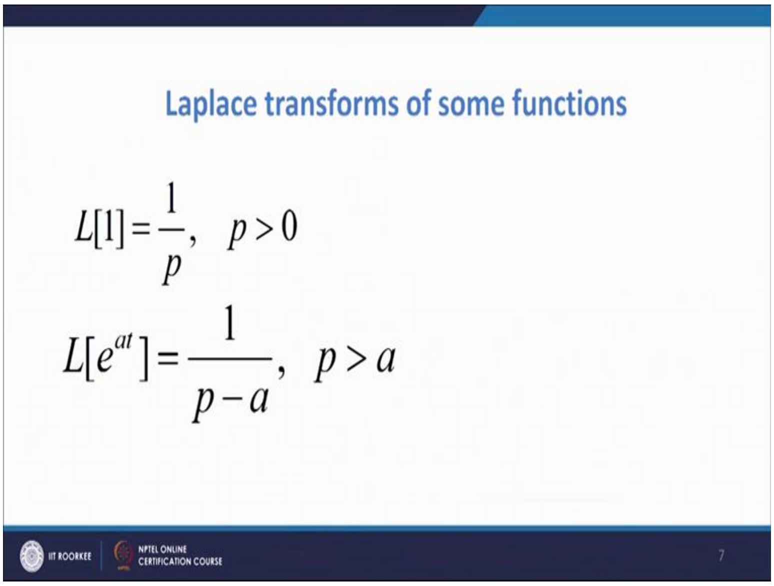


Similarly, the Laplace transform of some function g t will be nothing but 0 to infinity e power minus p t g t d t. So, Laplace transform of a f t plus b g t, we are a b are or any arbitrary constants. So, this is nothing but 0 to infinity, though here the function is a f t plus b g t. So, this will be given by e power minus p t, in here we have a f t plus b g t, b g t d t. So, this is nothing but when we apply the property of integration it is nothing but a times 0 to infinity e power minus p t f t d t, plus b times 0 to infinity e power minus p t g t d t.

So, this is nothing but Laplace of f t, and this is nothing but Laplace of g t. So we have this have linearity property that Laplace of a f t plus b g t is nothing but a times Laplace of f t plus b times Laplace of g t. Now the same holds for inverse Laplace transform also. Inverse Laplace transform is also satisfy linearity property. How that we can easily derive because Laplace inverse of this is by itself when we take Laplace inverse, Laplace inverse of a times f t, plus b times g t will be nothing but a f t plus b g t, where f t is nothing but Laplace inverse of F p, and g t is nothing but Laplace inverse of g p. So we have this result linearity property of inverse Laplace transforms.

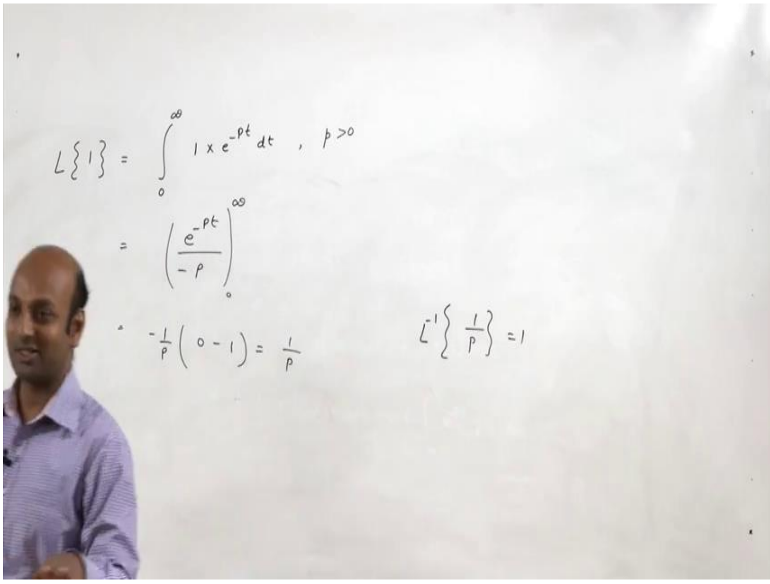
Now next is Laplace inverse; Laplace transforms of some functions.

(Refer Slide Time: 06:31)



Now, first we will see how we find Laplace transform of some functions. We will try to see some properties of Laplace forms. Then we will see how we can solve a given differential equation using Laplace transforms by applying those properties which we will study. Now first is let us find Laplace transform 1.

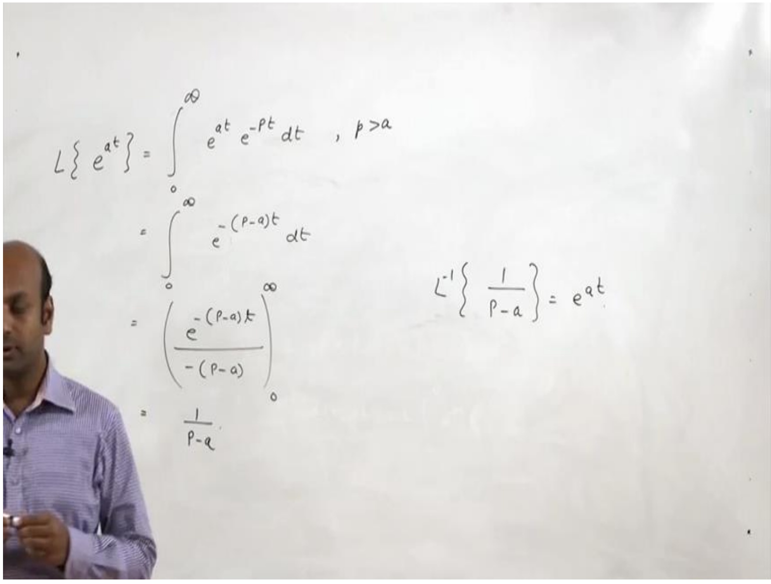
(Refer Slide Time: 07:05)



So, how we will find Laplace transform 1? So, Laplace transform 1 is nothing but we apply the definition. What is Laplace transform f t? Laplace transform f t is nothing but 0 to infinity e power p t f t d t what we have studied. So, Laplace transform 1 is nothing but 1 into e power minus p t into d t because here function whose Laplace is to find is 1 is unity. So, Laplace transform 1 is nothing but 0 to infinity 1 into e power minus p t d t. So, it is nothing but what is the integration of e power minus p t, it is e power p t into upon minus p it is from 0 to infinity. So, it is minus 1 by p upper limit minus lower limit will give value as this. So, it is nothing but 1 by p provided p is positive. So, we are assuming p greater than 0, p greater than 0 then only the limit of this quantity will tends to 0 as t tends to infinity.

So Laplace of 1 is 1 by p. So, hence we can also see from here let Laplace inverse of 1 by p is nothing but 1. Because if Laplace of f t is say f p. So, Laplace inverse of F p is nothing but f t. So, Laplace inverse of 1 by p is nothing but 1, by the same logic, by the same concept.

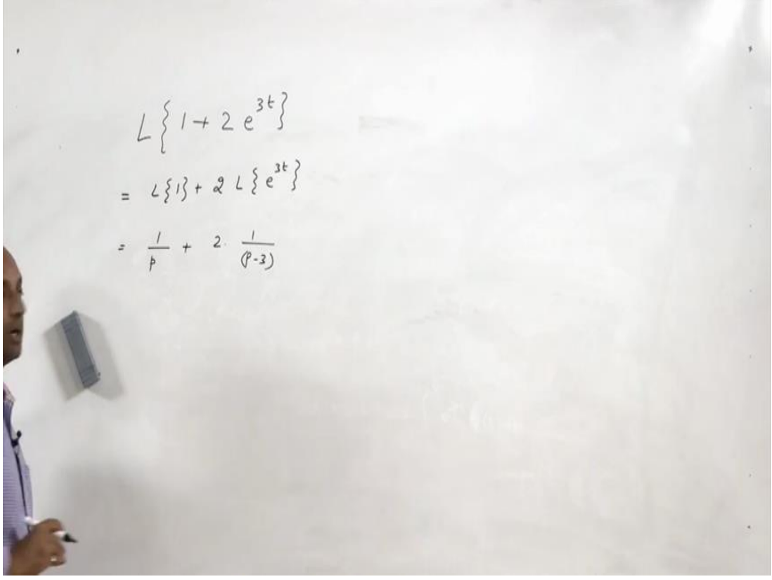
(Refer Slide Time: 08:57)



Now, now let us find Laplace of e power a t. So, again we apply the definition of Laplace transform. Laplace of f t nothing but 0 to infinity f t e power minus p t d t, it is nothing but 0 to infinity e power a t e power minus p t d t. It is nothing but 0 to infinity e power minus p minus a times t into d t, we integrate it. So, it is nothing but minus p minus a times t upon minus p minus a 0 to infinity. So, again in order that this limit exit is this integral have some value this much tend to 0 as t tends to infinity. So, we assume that p minus a is positive, because then only e power negative quantity will tend to 0, when t tends to infinity.

So, we assume that p greater than a. So, it is nothing but 1 upon p minus a. So, Laplace transform of Laplace transform of e power a t is nothing but 1 upon p minus a. So, again if Laplace transform of e power of a t is 1 upon p minus a. So, Laplace inverse of 1 upon p minus a is nothing but e power a t. Now let us try to find out Laplace transform of some functions.

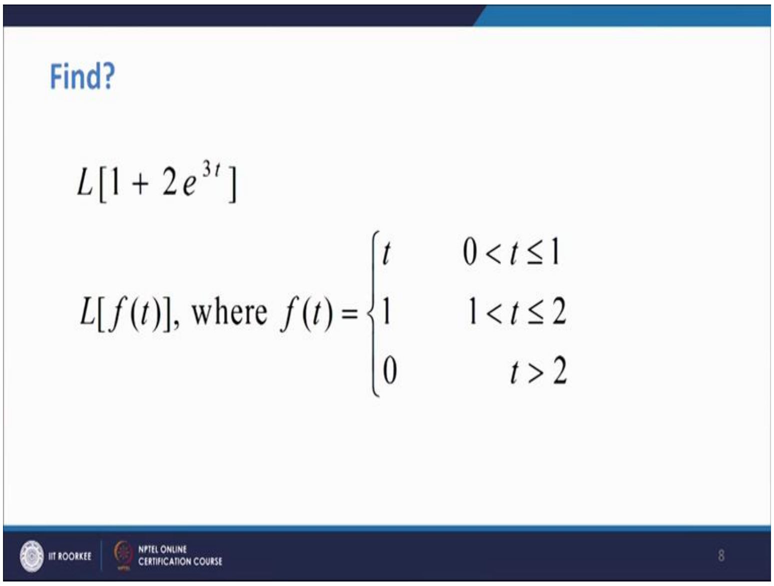
(Refer Slide Time: 10:54)



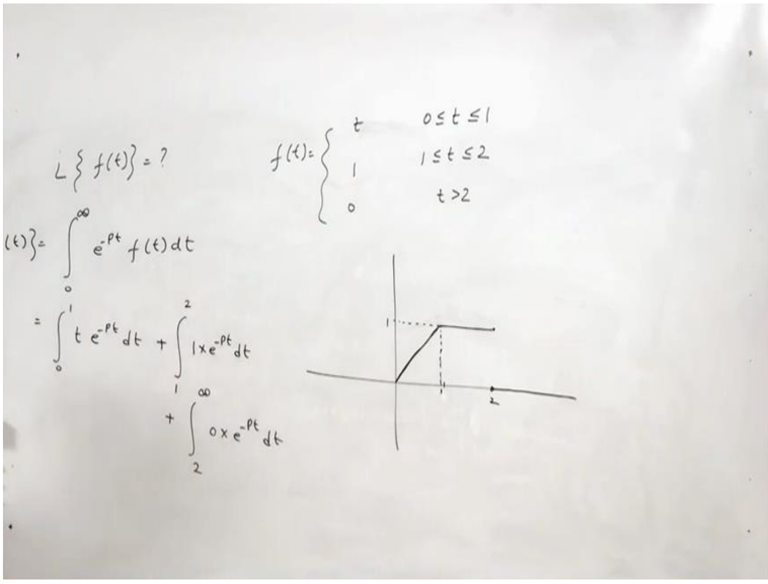
So, let us try to find out Laplace transform of one plus 2 into e power 3 t. So, we know that Laplace satisfy linearity property. So, by the linearity property we can simply say that Laplace of f t plus a times g t is nothing but Laplace transform of f t plus a times Laplace transform of g t.

Now, Laplace transform of 1 which we have already derived is 1 by p. So, it is nothing but 1 by p, plus, now what is Laplace transform of e power 3 t? It is nothing but 1 upon p minus 3. So, it is 2 into 1 upon p minus 3. So, this will be 1 upon p plus we can take l cm and simplify. So, we can get Laplace transform of the first problem 1 plus 2 e power 3 t.

(Refer Slide Time: 12:02)



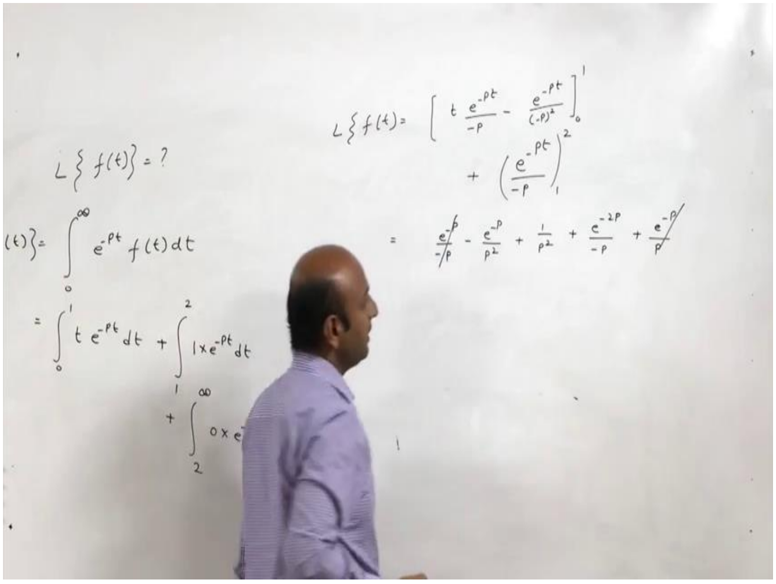
(Refer Slide Time: 12:11)



Now, we have second problem. Laplace transform of f t is to find out where f t is nothing but f t is nothing but t when 0 less than equal to t less than equals to 1, it is 1 when 1 less than equals to t less than equals to 2 and 0 when t greater than 2. So, for this function we have to find Laplace transform of f t. So, what this f t is? If we visualize graphically, so from 0 to 1 suppose this is 1 from 0 to 1 f t is t, that is the straight line, and from 1 to 2 it is 1 from 1 to 2 it is 1, this is 1 and after 2 it is 0 at 2 it is 1 and after 2 it is 0. So, basically this is f t. So, we have to find out Laplace of this f t. So, it is clear that this function is discontinuous at t equal to 2, because from the left side the value is 1, and while from the right side the value is 0.

However, it is Laplace transform, this we can find out it is a same definition the Laplace transform f t will be nothing but 0 to infinity e power minus p t f t d t. So, it is nothing but from 0 to 1, it is t into e power minus p t, because 0 to 1 f t is t. It is in the definition, 0 to 1 f t is t. So, 0 to 1 f t is t. And 1 to 2 1 to 2 f t is 1, 1 into e power minus p t. And from 2 to infinity it is 0.

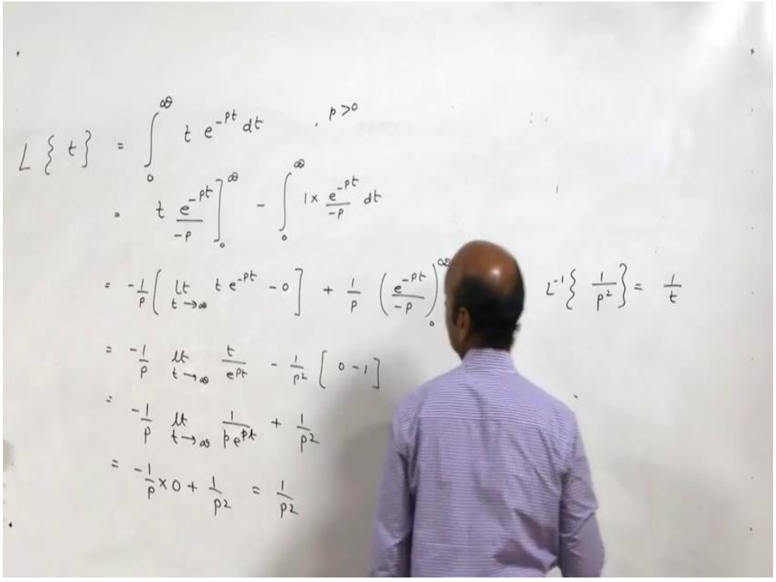
(Refer Slide Time: 14:51)



So, what will be the Laplace transform of this function f t? So, this will be nothing but now we can simplify it. So, Laplace transform f t will be nothing but now to integrate this this part, the first part, we will use integration by parts. So, when you use integration by part, when you use integration by part we will get t into e power minus p t upon minus p, minus e power minus p t upon minus p square, from 0 to 1. Then plus you can use integration by parts here to simplify this expression. Now plus it is e power minus p t. So, 1 to 2 it is e power minus p t upon minus p from 1 to 2 and rest is 0.

So, no need to find out the third expression rest is 0. Now here we will get when t is 1, it is e power minus p upon minus p minus e power minus p upon p square when t is 0, this is 0, when t is 0 it is minus minus plus 1 upon p square. And it is nothing but plus e power minus 2 p upon minus p. And at minus at 1 minus, minus plus e power minus p upon p, so we simplify it. So, these 2 ratios cancel out, we simplify it and we will get the Laplace transform of the required function f t.

(Refer Slide Time: 16:42)

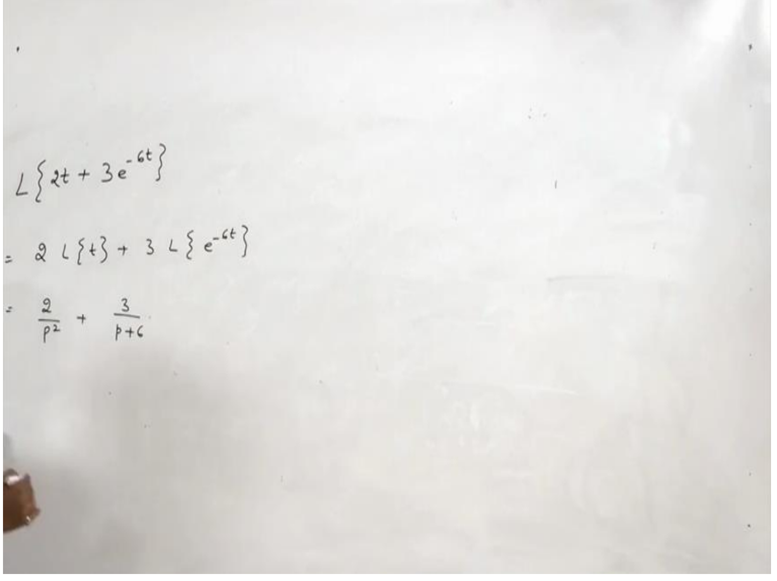


Now, let us try to find out Laplace transform of t. The third function Laplace transform of t. So, Laplace transform of 1, is 1 by p Laplace transform of e power a p is 1 upon p minus a. Now let u see what is the Laplace transform of function t. So, it is nothing but again apply definition of Laplace transform 0 to infinity t, into e power minus p t d t. So, it is nothing but again you will apply integration by parts. So, when you apply integration by part it is nothing but e power minus p t upon minus p, from 0 to infinity and then negative integration 0 to infinity, derivative of first integration of second.

So, at infinity to see the value of this expression, we will simply find limit at t tending to infinity of t into e power minus p t, and at t equal to 0 value is of course, 0 because of this expression t. When you put t equal to 0, this is value is 0 minus minus plus it is 1 by p the integration of this is e power minus p t upon minus p again from 0 to infinity. So, it is minus 1 by p limit t tend infinity, t upon e power p t minus 1 by p square. So, we assume that p is positive p, is greater than 0 we assume that p is greater than 0. So, p is greater than 0, this value will tend to 0 tend to infinity again. So, it is 0 minus 1, and as t tend to infinity, this is infinity upon infinity form because p is positive. So, we will apply we can apply hospital rule to simplify this expression to find this limit.

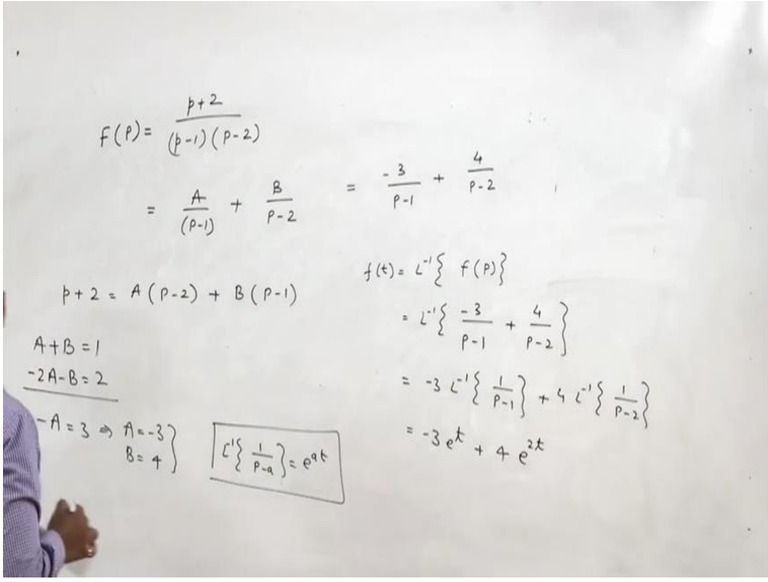
This is minus 1 by p, where we apply hospital rule it is nothing but derivative of the numerator 1, derivative of the denominator is p e power p t, minus minus plus 1 by p square. So, when t tends to infinity this is infinity and this value tend to 0. So minus 1 by p into 0 plus 1 by p square, this is nothing but 1 by p square. So, the Laplace transform of t is nothing but 1 by p square. In this expression for to find this Laplace, Laplace of f t to t we assume that p is greater than 0, what we will Laplace of inverse of 1 by 1 by p square? Laplace inverse of 1 by p square is nothing but 1 by t, because Laplace of t is 1 by p square. So, Laplace inverse of 1 by p square will be nothing but t.

(Refer Slide Time: 20:19)



Now, suppose we want to find out Laplace of t plus 2 t plus 3 e power minus 6 t. So, again we can use linearity property of Laplace transform. So, by the linearity property, Laplace of this f t will be nothing but 2 times Laplace of t plus 3 times Laplace of e power minus 6 t. So, it is nothing but 2 upon p square plus 3 upon p plus 6. This will be the Laplace of this function.

(Refer Slide Time: 21:23)

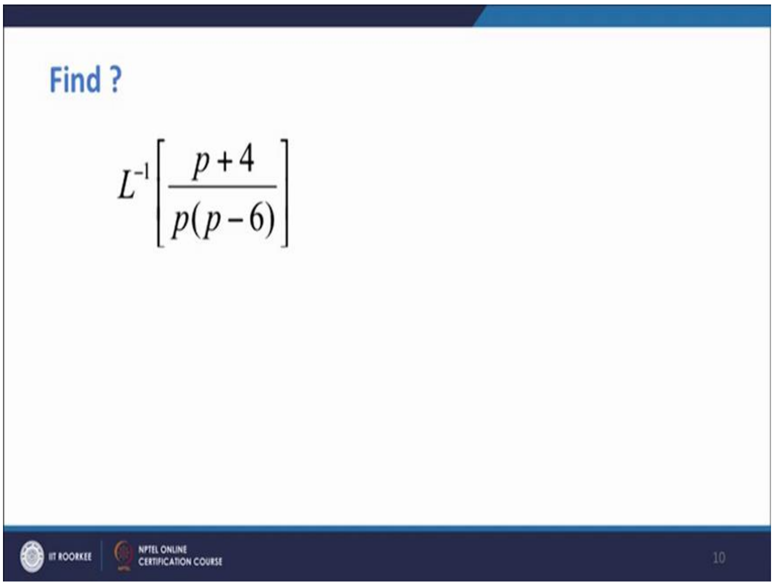


Now, let us find Laplace inverse of this function. So, Laplace inverse of this function, so what is F p for this problem? For this problem the function of p is p plus 2 upon p minus 1 into p minus 2. We have to find the Laplace transform of this function. So, first we will use partial fractions. So, what it will be nothing but equal to a upon p minus 1, plus b upon p minus 2. So, we can find a and b by comparing the coefficient from both the sides. So, here p plus 2 will be nothing but a into p minus 2 plus b into p minus 1. So, a plus b is 1. The coefficient of p here is a, here is b. The total coefficient of p in the right hand side is a plus b in the left hand side plus 1. So, a plus b is 1 and comparing the constants, it is minus 2 a minus b is equal to 2.

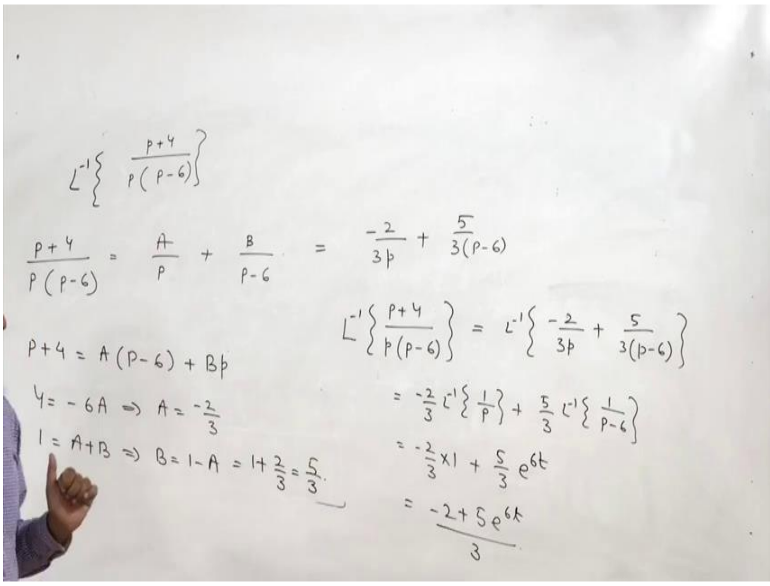
So, when you find a and b from these 2. So, minus a will be 3 when you add this 2. So, implies a is minus 3. And when you put a equals to minus 3. So, b will be nothing but 4. So, this expression is nothing but minus 3 upon p minus 1, plus 4 upon p minus 2, so this f p. So, what will be f t? f t will be Laplace inverse of F p. Laplace inverse of this to find out. So, Laplace inverse of this we are calling as f t. So, Laplace inverse of F p which is equal to this and this is equal to this, this is nothing but minus 3 upon p minus 1 plus 4 upon p minus 2. So, this is nothing but minus 3 times Laplace inverse of 1 upon p minus 1, plus 4 times Laplace inverse of 1 upon p minus 2. This is again by the linearity property. By the linearity property of inverse Laplace transforms we can write we can write f t as minus 3 times Laplace inverse of upon p minus 1, and 4 times Laplace inverse of 1 upon p minus 2. So, what is Laplace inverse of 1 upon p minus 1 it is nothing but e power t because we know that Laplace inverse of 1 upon p minus a is nothing but e power a t. This we have already discussed.

So, here a is 1. So, Laplace inverse of 1 upon p minus 1 will be e power t plus 4 times again for the second term a is 2. So, the Laplace inverse of 1 upon p minus 2 is e power 2 t. So, this will be the Laplace inverse of this problem. So, this is the answer.

(Refer Slide Time: 25:15)



(Refer Slide Time: 25:42)



Now, suppose you want to find out Laplace inverse of p plus 4 upon p into p minus 6, some expression on this type. So, again we will make use of partial fractions. So, what is p plus 4 upon p into p minus 6? It is nothing but a upon p plus b upon p minus 6. So, what will be a and b? You can use you can compare coefficients from both the sides. So, a will be a nothing but when you compare the coefficients. So, it is p plus 4, will be equals to a times p minus 6 plus b times p. So, 4 is nothing but minus 6 a. So, this implies a is minus 2 by 3. And the coefficient of p from both the sides is here it is 1, and here it is a plus b. So, this implies b is nothing but 1 minus a and a is minus 2 by 3. So, it is 1 plus 2 by 3 that is 5 by 3.

So, this ratio is nothing but a is minus 2 by 3, minus 2 by 3 times p plus b is 5 by 3, so 5 by 3 times 1 upon p minus 6. So, the Laplace inverse of p plus 4 upon p into p minus 6 is nothing but Laplace inverse of minus 2 by 3 p, plus 5 by 3 times p minus 6 because this expression is equal to this expression. So, again by the linearity property of inverse Laplace transforms this is nothing but minus 2 by 3 Laplace inverse of 1 by p plus 5 by 3 Laplace inverse of 1 upon p minus 6. So, this is nothing but minus 2 by 3. So, Laplace inverse 1 by p is 1. So, into 1 plus 5 by 3 Laplace inverse of 1 upon p minus 6 is e power 6 t. So, this will be the. So, this is equals to minus 2 plus 5 e power 6 t upon 3. So, this will be the Laplace transform of Laplace inverse of this expression.

So, in this lecture we have seen that what that first of all what is significance of Laplace transforms. The basic definition of Laplace transform is Laplace transform any function is simply 0 to infinity e power minus p t ft d t that we have already seen. Laplace and it is inverse both satisfy linearity property and we have also seen that Laplace transform 1 is 1 by p, that we have discussed. Laplace transform e power a t is 1 upon minus a and Laplace transform t is 1 upon p square. So, rest we will discuss in the next class, that whatever Laplace transform of some standard functions.