Mathematical methods and its applications
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Lecture - 22
Solution of First Order Non Linear Equation – IV

Hello friends. Welcome to my last lecture on solution of first order non-linear equations. In my previous lecture we discussed about how to solve first order non-linear equations which cannot be reduced to any of the 4 standard forms. The method applied for solving such non-linear differential equations of first order is Charpit’s method.

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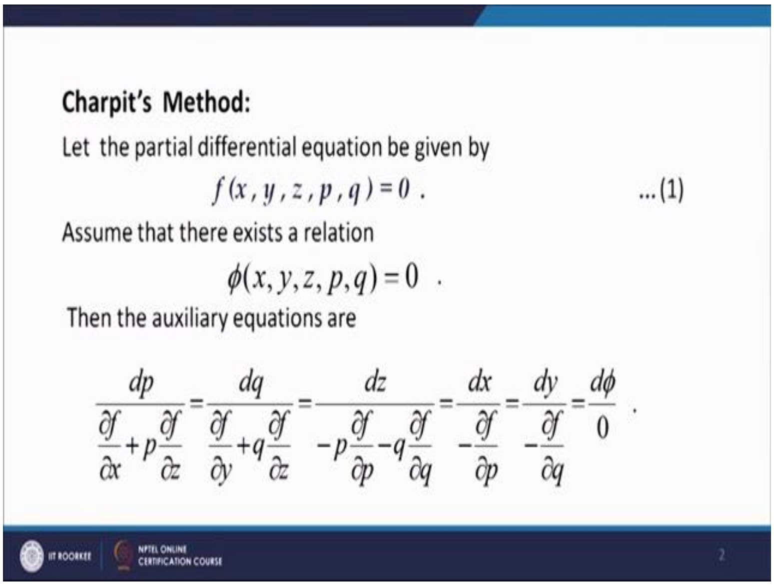

In the Charpit’s methods we have seen that if the differential equation is given by f x y z p q equal to 0. Then we try to find relation in the variables x y z and p q equal to 0, which we denote by phi x y z p q equal to 0. Such that by the 2 equations f x y z p q equal to 0 and phi x y x p q equal to 0, we determine the values of p and q. And then we put them into d z equal p d x plus q d by, if it is in integrable we get the solution that is the complete integral of the given partial differential equation.

So, let us assume that the partial differential equation is f x y z p q equal to 0, and their exist a relation in the variables phi x y z p q equal to 0. Then the auxiliary equations we have seen are given by d p over delta f over delta x, plus p delta f over delta z d q over this and d z over minus p delta for delta p, minus q delta for delta q d x over minus delta f delta p d y over minus delta f delta q, equal to d phi over 0. Now by from this question we tried to find a simplest relation, which will give us the value of p and q, there y by using the equation f x y z p q equal to 0. We shall be determining the value of the other variable that is q r p. Now once n p n q are known we shall be able to determine the complete integral.

So, let us see how we apply this method. So, we should consider the simplest relation involving at least one of p r q for phi equal to 0.

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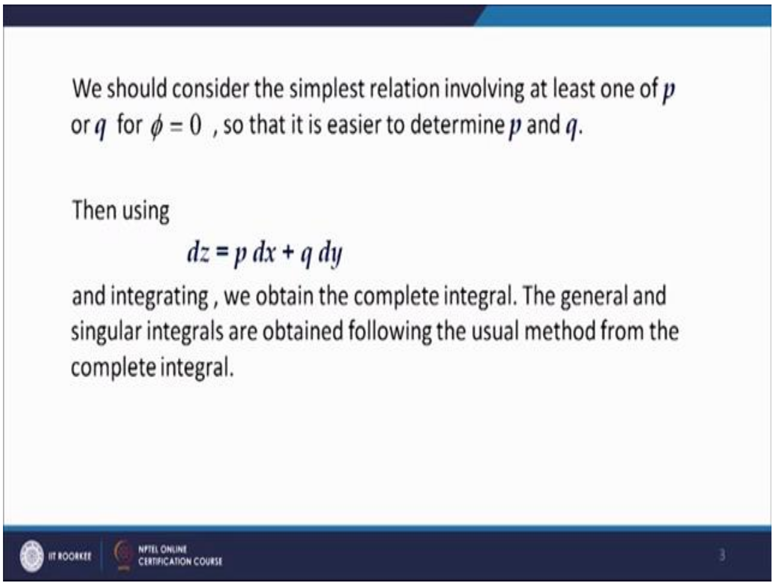

We have to look for the simplest relation here, involving one of p r q, so that we can determine the values of p and q. So, it is easier to determine p and q. Then using d z equal to p d x plus q d y and integrating we obtain the complete integral. The general and singular integrals are obtained as usual from the complete integral.

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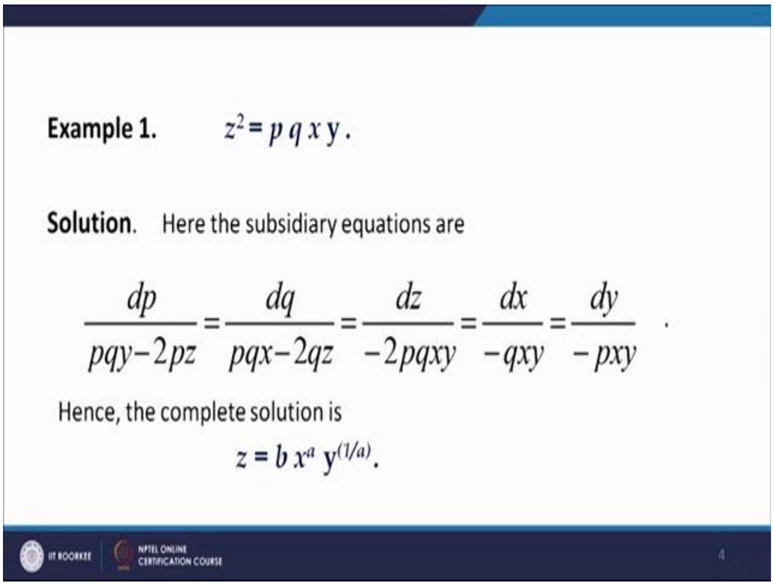

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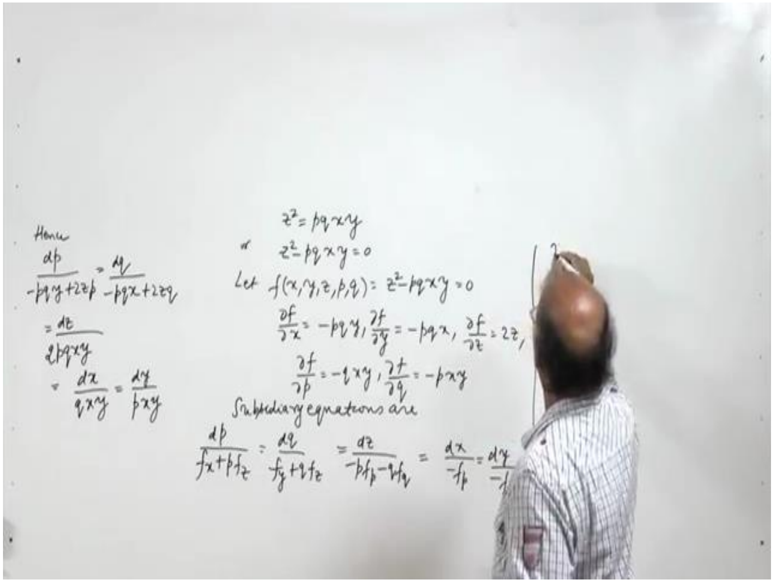

Let us look at the first partial differential equation of first order which is non-linear, z square equal to p q x y. So, here we may write the equation as z square minus, let us say let f x y z p q b equal to z square minus p q x y equal to 0. Then we can find the partial derivative of f with respect to f x y p q and z, so partial derivative with respect to x. So, let us find first. So, this is minus p q y partial derivative with respect to y we can find. So, which is minus p q x, now partial derivative with respect to z is 2 z, partial derivative with respect to p comes out to be minus q x y, and lastly partial derivative with respect to q comes out to be minus p x y.

So, once we have the values of this 5 partial derivate of f, we can then write the subsidiary equations the subsidiary equations are, the subsidiary equations let us write how the subsidiary equations are this one, so d p over f x plus p f z. We can write in short f x plus p f z, and then d q over f y plus q f z we have done d z, over minus p f p, minus q f q, and this equal to d x over, minus f p equal to d y over minus f q. And which is equal to of course, d phi over 0. So, we do not need that. So, let us put the value of the derivatives here. And we will see that what we get is this.

So, d p over f x plus p f z, f x is equal to minus p q y. So, we get hence d p over f x, f x is minus p q y, plus p f z, p f z is 2 z p, then d q, d q over, f y, f y is equal to minus p q x and then we have q times f z. So, q times 2 z q, 2 z q then d z divided by d x by minus p f p. So, minus p q minus p into f p, so p q x y and then minus q f q, minus q f q, will also give you p q x y. So, we get double 2 times and then we get d x over, minus f p. So, we get q x y, and this is equal to d y over, minus f q minus f q means minus p x y.

Now, from these auxiliary equations are subsidiary equations, we try to find the simplest relation involving one of p r q.

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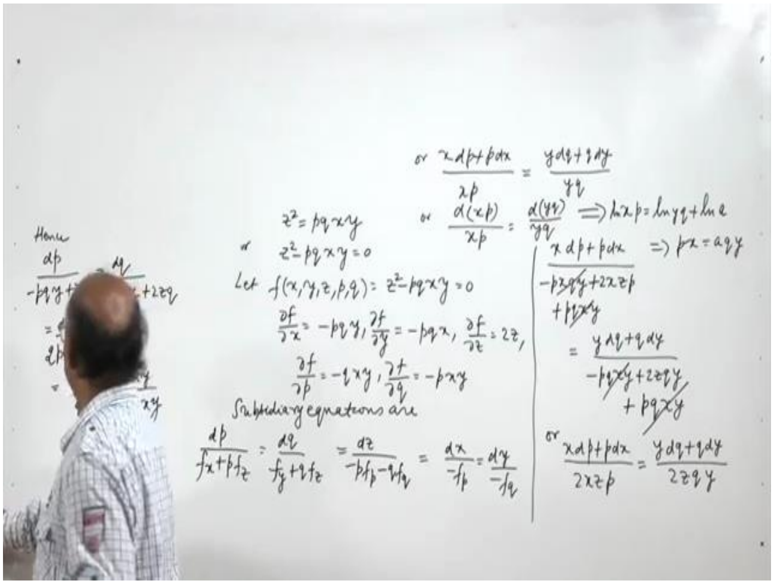

Now here it is if you just if you see this equations, directly we do not find any such relation, but we notice that if we multiply d p y x x d p, plus p d x, we get minus p q p x q y and then 2 x z p plus p d x. So, p q x y equal to y d q plus q d y, so y into d q will give you minus p q x y, plus 2 z q y, 2 z, q y and then we have q times d y. So, q times d y means p q x y. So, this p q x y will cancel with this, this p q x y will cancel with this, and we have x d p plus p d x divided by 2 x z p equal to y d q plus q d y divided by 2 z q y, or x d p, plus p d x, divided by x p equal to y d q plus q d y, divided by y, q 2 z we can cancel.

So, now this, this we can write further as, or this is d of x into p, numerator is d of x into p, and then x p equal to d of y q, over y q. Now when we integrate this, this will give you x p ln x p equal to ln x y q plus, ln let us say b or we can take ln a, ln a. So, we shall get p x equal to a q y. So, we get a relation between p and q that is p x equal to a q y.

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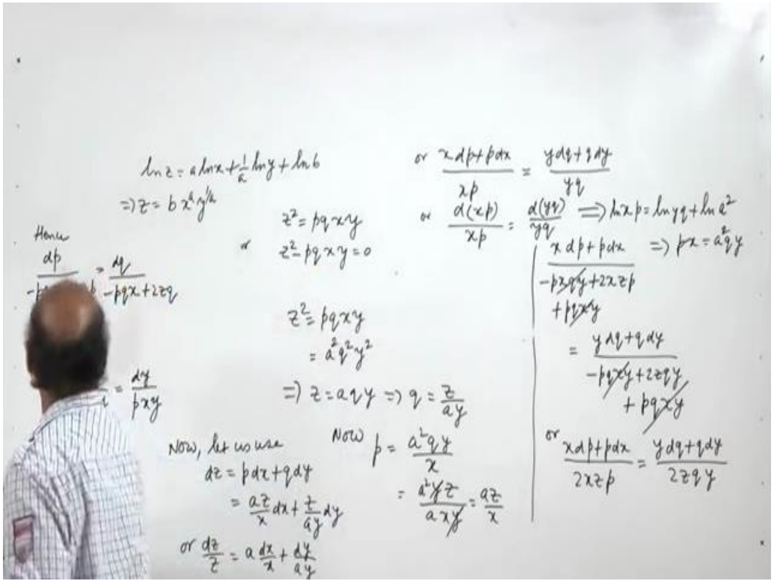

Now, let us use that relation and determine the values of p and q with the help of the given equation. So, z square is p q x y, since z square is p q. So, let us put the value of q p x equal to p x equal to a q y a times q square y square we have.

So, if we can put here instead of a a square, it will be better. So, we will get z square equal to p x into q y p x is a square q y. So, a square q square y square and z is equal to a q y. So, this means that q is equal to z upon a y, we have got the value of q and it is z over a y. Now from p x is equal to a square q y now p p is equal to a square q y divided by x. So, let us put the value of q here.

So, a square y, square y q is z over a y. So, z over we have a square y, a square y is z q z over a y z, over a y into x. So, this will give you we can cancel this, and we can cancel one a. So, we can get a z divided by x. So, we have got the values of both p and q. Now let us use d z equal to p d x plus q d y. So, p is equal to a square p is equal to a z divided by x, d x and q is z over a y d y.

Now, we can divide this equation by z. So, or d z by z equal to a times d x over x, plus d y over a y, when we integrate this equation, we get l and z equal to a ln x plus 1 by a ln y plus let us say the constant ln b. So, this will give you z equal to x b times x to the power a, and y to the power 1 by a, which is the complete integral; so which is the complete integral. We know that the general integral.

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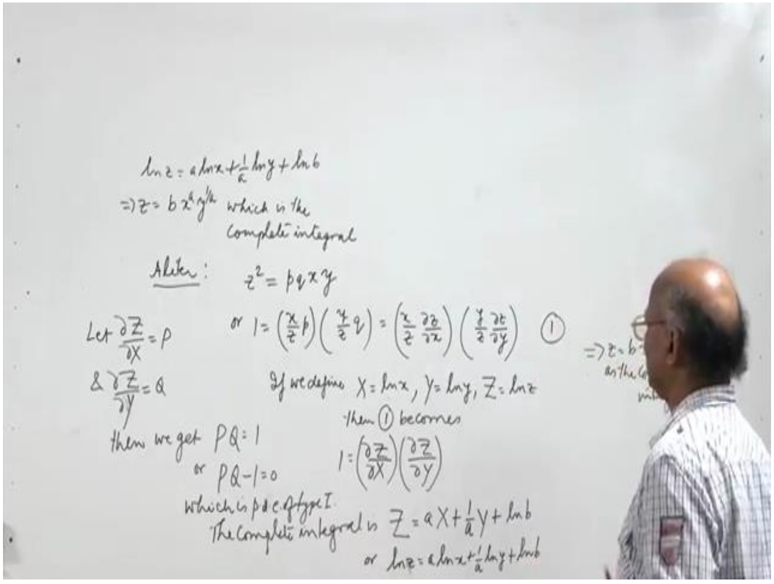

And the similar integral can be obtained from the complete integral for the general integral. We put b equal to phi a then differentiate that this equation with respect to a, and then eliminate a between the equation, which we get from this equation by putting b equal to phi a and the equation which we obtained by differentiating this equation with respect to, so, we know that method and then for the similar integral, we differentiate the complete integral with respect to a and b and then eliminate a and b between the 3 equations, that is this one and the 2 equations which we obtained by differentiating with respect to a and b, so that we can easily do once the complete integral is known. Now if you closely note observe this given differential equation, it can be solved by substitution also, we will be able to convert this equation into the equation of this standard form that is type one, let us see we have. So, we have alternate method aliter.

So, let us notice that z square is equal to p q x y. And this can be written as or one equal to x into x by z into p, and y by z into q. Or this is also equal to x by z delta z by delta x, and this is y by z and delta z by delta y. So, if we define capital x equal to ln x, and capital y equal to ln y, and capital z equal to ln z. Then this equation 1, 1 will become then one becomes one equal to delta z by delta x, into delta z y delta y, we can see this easily. So, now, this if you regard delta z by delta x, let us say let delta z capital z by delta capital x equal to p, and delta z capital over delta x v by y equal to q then we get p q equal to one or p q minus 1 equal to 0. So, which is of which is p d e of type 1. And the complete integral in this case is z equal to a capital x, plus b is equal to one over a. So, one over a by plus some constant b, or we can say now let us values of x y and z. So, ln z equal to a ln x plus 1 by a ln y.

Now, here I can write ln b I think it will better if you write ln b. So, b we have and then we have z equal to b times x to the power a, y to the power 1 by a, as the this is the complete integral. So, if we closely observe the given non-linear first order function equation by making the substitution and x equals to ln n y equal to z equal ln z, we are able to reduce it to the partial differential equation of type one, and for type one partial differential equation, we know the complete integral it is written like this. So, we get the complete integral for the given differential equation.

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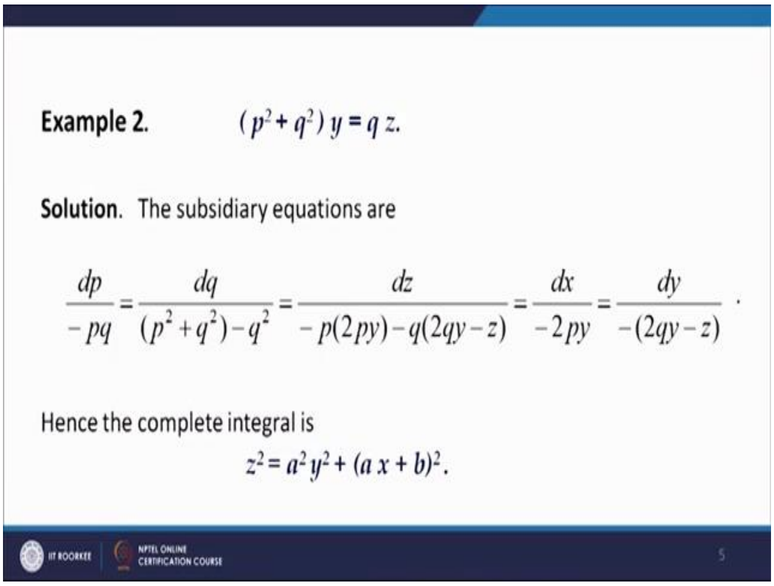

Now, let us go to another case. So, here we notice that the subsidiary equation are such that it is not easy to find the simplest relation involving p and q. We have to consider p d x plus q p q p d plus x d p and q d y plus y d q. So, that also we have to do sometimes.

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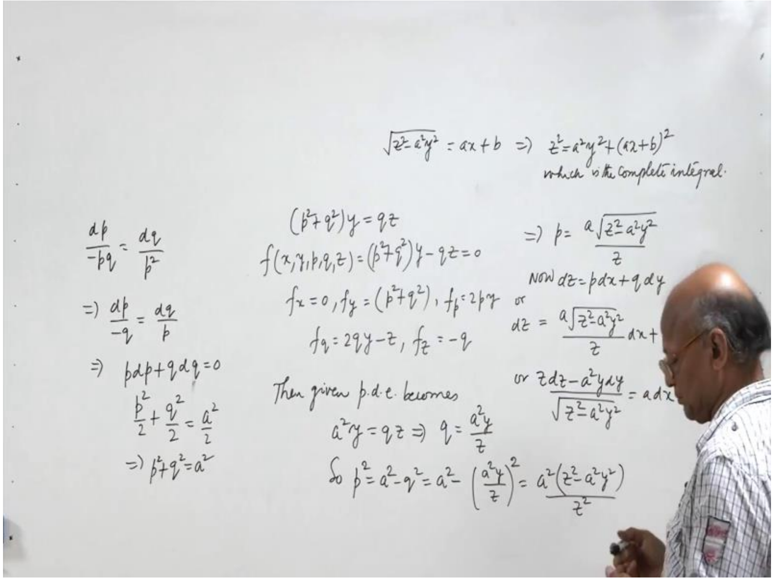

Now, here we have the partial differential equation p square plus q square into y equal to q z. So, here p f x y p q z equal to p square, plus q square into y minus q z equal to 0. We can find the partial derivatives partial derivative of f with respect to x is 0. Because the right hand side there is no x, partial derivative of f with respect to y is p square plus q square partial derivative of f with respect to p is 2 p y, partial derivative of f with respect to q is 2 q y minus z, and the partial derivative of f with respect to z is equal to minus q. So, once we have these derivatives, we can then write the subsidiary equations. The subsidiary equation turn out to be d p over minus p q d q over p square, plus q square minus q square, d z over this d x over this and d y over this.

Now, we can look at this first relation, d p over minus p q equal to d q over, p square plus q square minus q square. So, this is d p over minus p q equal to d q over p square, d p over minus p q equal to d q over p square, and from this we can see that d p over d p over minus q equal to d q over p, or we can say p d p plus q d q, equal to q d equal to 0. So, we will integrate this we will get integrate this, we can write this is p square y 2 plus q square y 2, equal to a constant. Let us write a square y 2. So, that we will have p square plus q square equal to a square where a is arbitrary constant. And then the given then given p d becomes a square into y equal to q z.

So, this gives us the value q, q equal to a square y divided by z, q equal to a square over z and. So, p square is equal to a square minus q square, gives you a square minus a square y divided by z whole square. So, this is a square times z square minus a square y square, divided by z square. And this implies p is equal to a times under root z square minus a square y square divided by z.

Now, p d x, d z equal to p d x plus q d y, so this is d z equal to p d x means a under root z square minus a square y square, divided by z d x plus q is a square y over z. So, z are z d z minus a square y d y divided by under root z square minus, a square y square, so z d z minus a square minus d over under root, this equal to a d x. Now it is easy to integrate. So, integrating this we get under root z square minus a square y square equal to a x plus some constant b which implies that z square equal to a square y square plus a x plus b whole square which involves 2 arbitrary constants a and b. And so, it is the complete integral which is we can easily find the general integral, and the similar integral from the complete integral.

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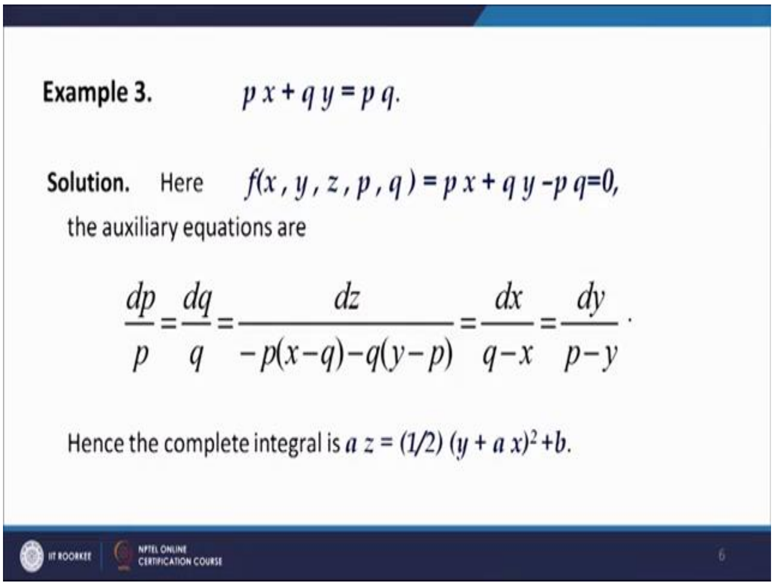

This, another example p x plus q y equal to p q, so here f x by z p q equal to p x plus q y minus p q equal to 0, the auxiliary equations are d p over p, equal to d q over q equal to d x, over the d z over this d x over this, d y over p minus y. Here it is easy we can see that there is a exist a relation between p and q, which is given by the first equation d p over p equal to q over q, so d p over p.

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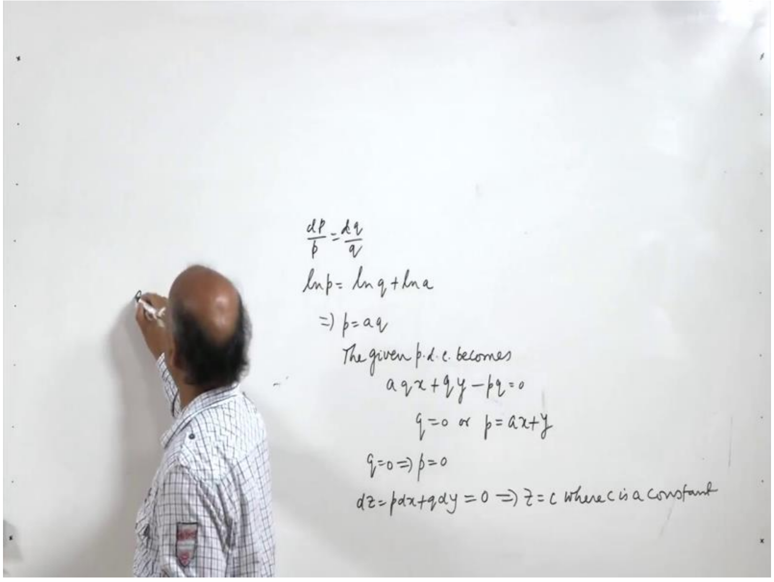

So, let us use that relation d p over p, equal to d q over q then this ln p, when we integrate ln p equal to ln q plus ln a. So, this gives p equal to a q, when p is equal to a q the given question reduces to then give me the equation becomes, p x p x means a q x plus q y, minus p q equal to 0. So, this will give you q equal to 0, q equal to 0, or p is equal to a x plus y.

Now, when let us look at the case q equal to 0, when q equal to 0, p is also 0, q equal to 0 p equal to 0 means d z is equal to 0 because d z is equal to p d x. So, q equal to 0 and p equal to 0 implies d z equal to 0, which implies d z equal to an arbitrary constants c where c is a constant, and we see that z is equal to c means p 0, q 0 and so, it is satisfies the given partial differential equation.

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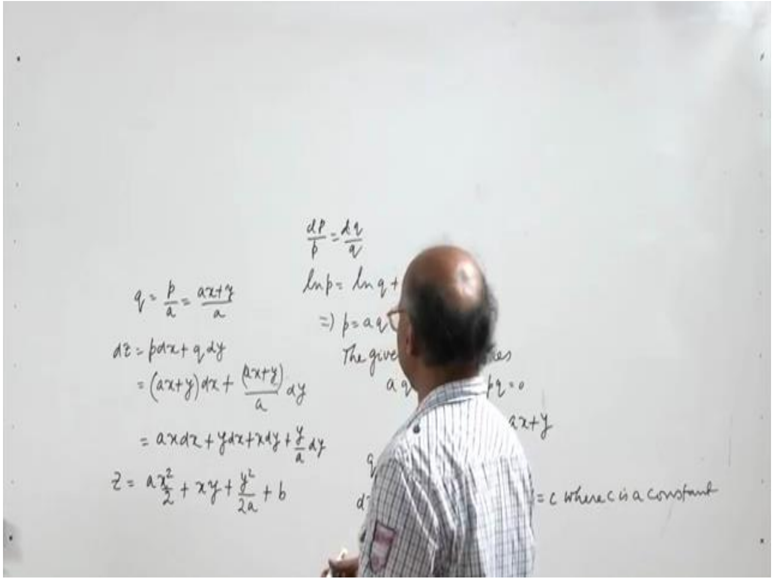

Now, the other non-trivial solution is obtained when we take p equal to a x plus y. So, when we p is equal to a x plus y, q will be equal to p upon a. So, a x plus y divided by a. Now d z is equal to p d x plus q d y. So, p means a x plus y, d x and q means a x plus y divided by a d y. So, this is equal to a x d z, a x d x plus y d x plus x d y, plus y over a d y and so, we can integrate it easily. So, z is equal to integrating z equal to a times x square y 2 plus y d x plus x d y x gives you x into y plus, y square y 2 into a plus some constant. We can take as b. So, z is equal to x square by 2 plus x y plus y square by 2 plus b, this is complete integral.

As usual the general and similar integrals can be obtained from this complete integral. Now let us go to the last problem, q equal to minus x p plus p square. So, here we can write f x y, z p q plus p square minus x p minus q equal to 0. The auxiliary equation when we find they come out to be d p, over minus p equal to d q over 0, d x over minus 2 p plus, d x over one d z over minus, 2 p, square plus x p plus q.

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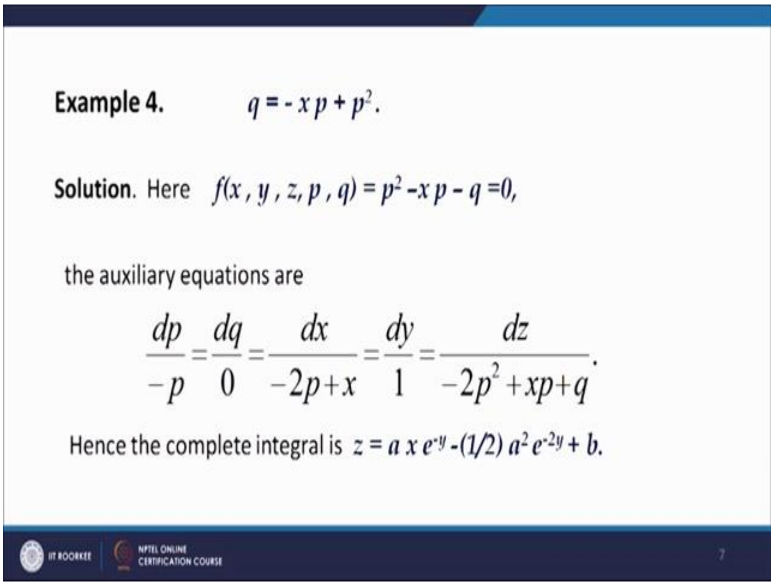

Now, here let us consider d p over minus p equal to d q d y over 1. So, because if d q is equal to 0, what we will have here, d q equal to 0 means q will be some constant and when q is some constant, p square minus x p minus some constant we shall have, so d q over it because we need a relation between p and q.

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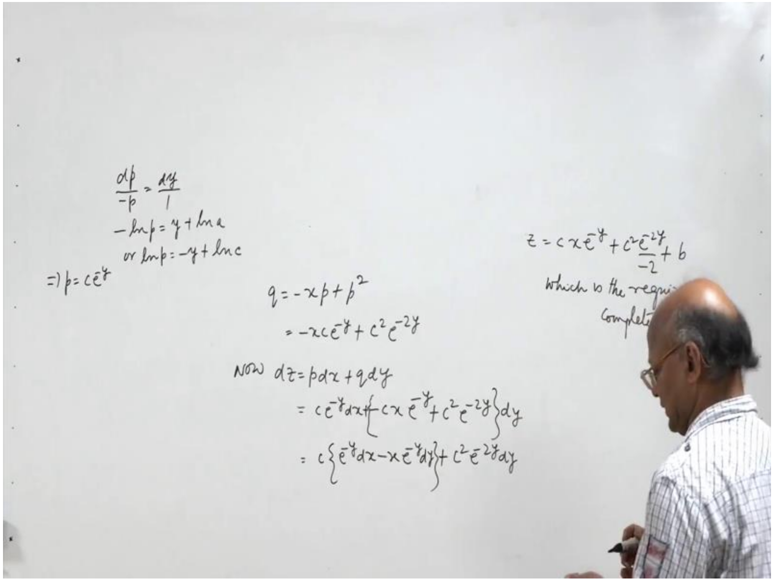

So, what we do is we consider d p over minus p equal to d y over 1. D p over minus p equal to d y over 1, so this will give you minus ln p when we integrate, y plus ln or I can write it as ln p equal to minus y plus some constant ln c let us say where ln c is minus ln

a. So, p is equal to c times e to the power minus y. And once we have p we can find q from the given equation. So, p q is equal to minus x p plus p square. So, let us put equal to c e to the power minus y minus x e e, to the power y plus p x square. So, x square e to the power minus 2 y this the value of q.

Now, d z is equal to p d x plus q d y. So, here what we have p is equal to c e to the power minus y d x. And then we have minus let me replace minus c x e to the power minus y, plus c square e to the power minus 2 y. I think we have to write d y everywhere. So, let me write it like this. See this is p c e to the power minus d x plus q, is minus c x e to the power minus y plus c square e to the power minus 2 y d y.

Now, here we notice that c times e to the power minus y d x, and then minus x times e to the power minus y, we have taken this pair and then, we have c square, this is d y also here and then plus c square e to the power minus 2 y d y. So, integrating we get z equal to c x times e to the power minus y, from here and then c square e to the power minus 2 y divided by minus 2 plus some constant, which we can take as b. So, now, this, the solution involves 2 arbitrary constants b and c, and so, which is the required complete integral.

So, once we have complete integral, the general and similar integrals can be obtained from the complete integral. So, we have written z equal to a x e to the power minus y. Here in place of y we have written c there, and then minus half a square e to the power 2 y plus b. So, this is the complete integral and as I said general integral and the similar integral can b e obtained from this complete integral, why applying the usual methods. So, with that I would like to conclude this lecture.

Thank you very much for your attention.