Mathematical methods and its applications
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Lecture – 21
Solution of first order non – linear equation – III

Hello friends. Welcome to my third lecture on solution of first order non-linear equations. In previous 2 lectures we have discussed first order non-linear equations of type one 2 and 3. In this lecture we shall discuss the forth type of standard type of first order non-linear equations. And then we shall discuss the general method of solving such non-linear equations first of first order, which cannot be reduced to any those 4 standard forms. So, let us begin with first order non-linear equations of type 4.

So, in the in this category we consider those kind of equations which are of the form z equal to p x plus q y plus f p q and as you remember p and q denotes the partial derivatives of z with respect to x and y.

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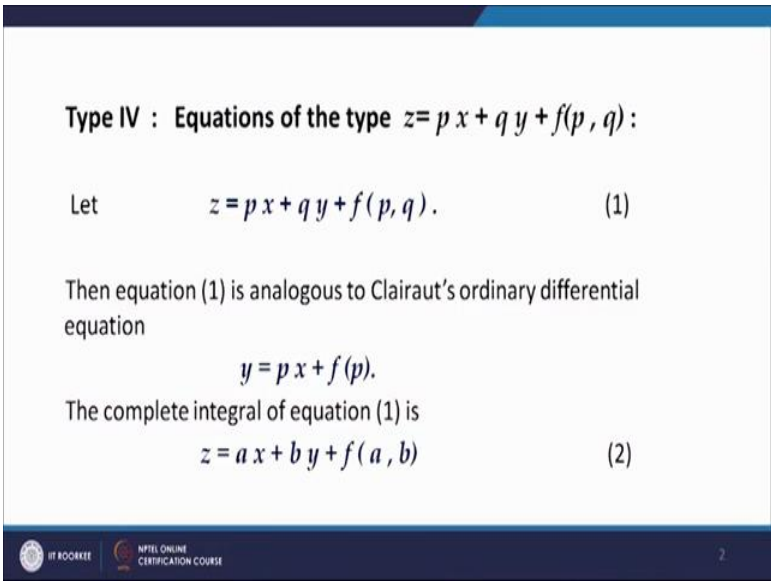

So, if z is p x plus q y plus f p q, then this equation you can see analogous to clairauts ordinary differential equation. The clalelout ordinary differential equation is y plus p x plus f p, where p is d z by dx. In the ordinary differential equation clairauts form is p x plus f p, where p is d y by d x.

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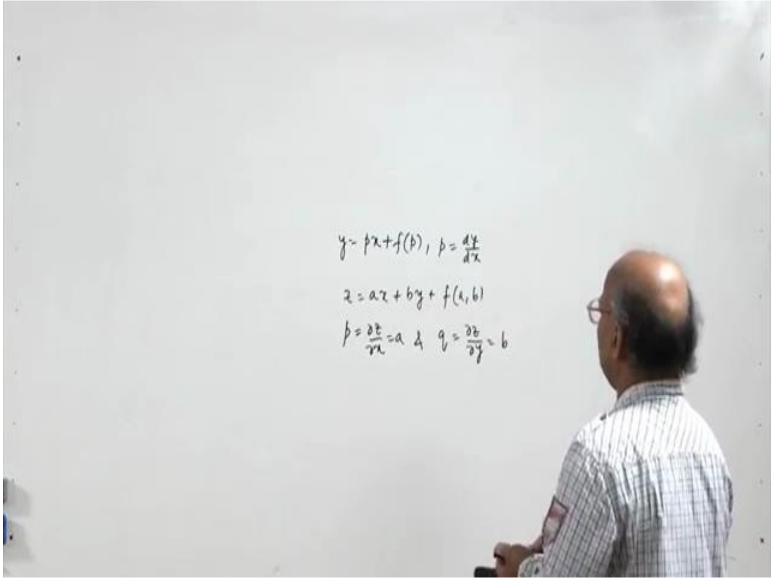

So, the partial differential equation z equal to p x plus q y plus f p q, is analogous to the clairauts ordinary differential equation y equal to p x plus f p. The complete integral of equation 1 is then given by z equal to a x plus b y plus f a b. Because we can you can see that, when z equal to a x plus b y plus f a b, then your partial derivative with of z with respect to x which we have denoted by p, it will be equal to a and q, which is the partial derivative of z with respect to y, it will be equal to b and therefore, z equal to a x plus b y plus f a b. This is a solution of equation 1. Because when you put z equal to a x plus b y plus f a b into 1, it is satisfied due to the fact that the value of p is a and the value of q is b.

So, this complete solution you can see z equal to p a x plus b y plus f a b. This solution it represents plane and it is at 2 plane family of 2 parameter family of planes. Because there are 2 parameters a and b. So, it is at 2 parameters family of planes. Now in order to obtain the general integral as we have earlier discussed.

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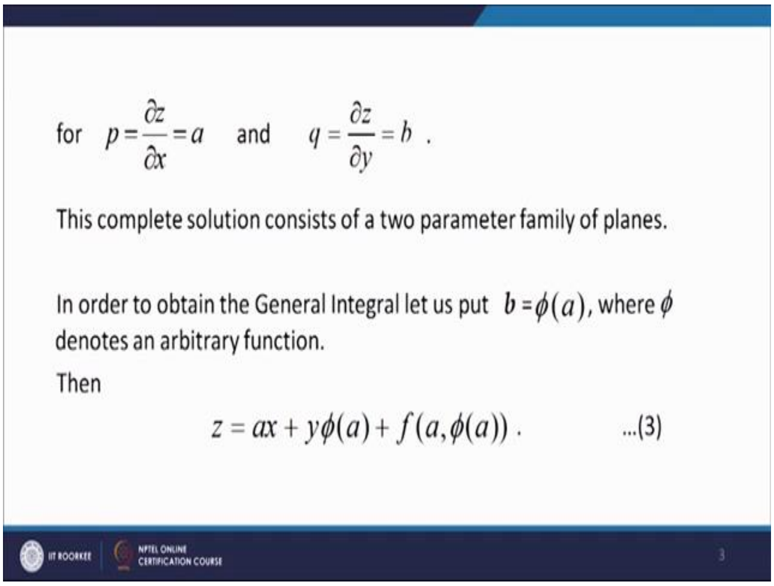

Let us put b equal to phi a in this solution and where phi denotes an arbitrary function. Then z will be equal to a z equal to a x plus y into phi a plus y into phi a plus f a phi a.

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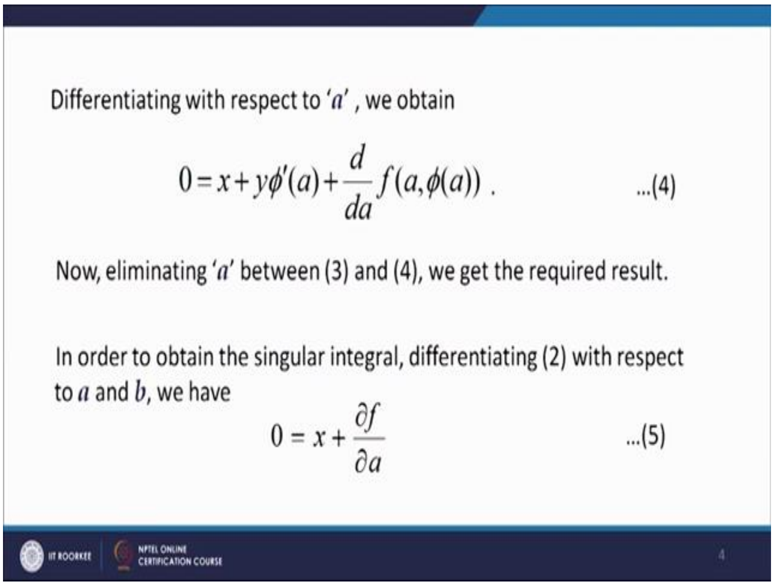

We differentiate this equation with respect to a to get z d 0 equal to x plus y phi dash a plus d over d a of f a phi a. And then eliminate a between equation 3, this equation and the equation 4, to get the general integral.

Now, in order to obtain the singular integral, we differentiate equation 2. This one z equal to a x plus b y plus f a b which is the complete integral with respect v we differentiate this with respect to a.

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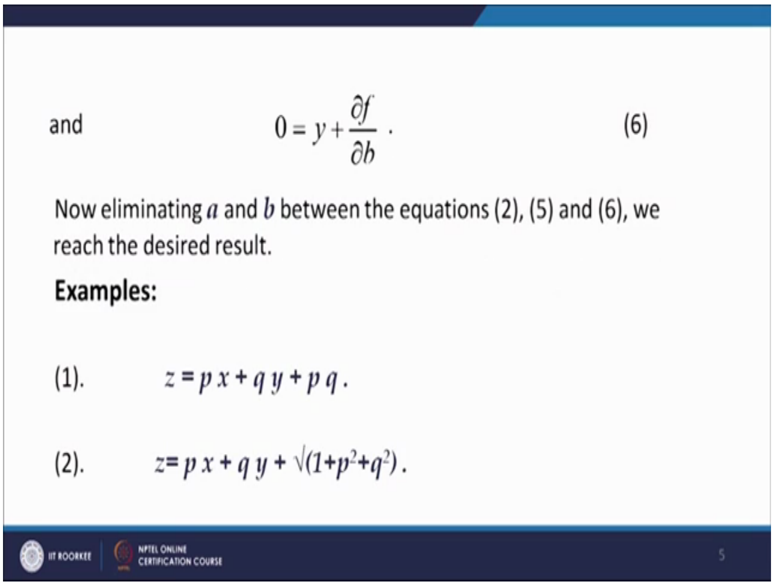

So, we will get 0 equal to x plus d over delta over delta a of f a b, 0 equal to x plus delta f over delta a. And the second one will be when we differentiate with respect to b we will get 0 equal to y plus delta f over delta v. So, we have 3 equations now. Equation 2 which is the complete integral this one and then the equations 5 z 0 equal to x plus delta f over delta a; and 6 that is 0 equal to y plus delta f over delta v. So, we eliminate a and b between these 3 equations 2 reach the singular integral.

Let us see how we apply this method to some differential equations. Let us begin with the first example z equal to p x plus q y plus f p q.

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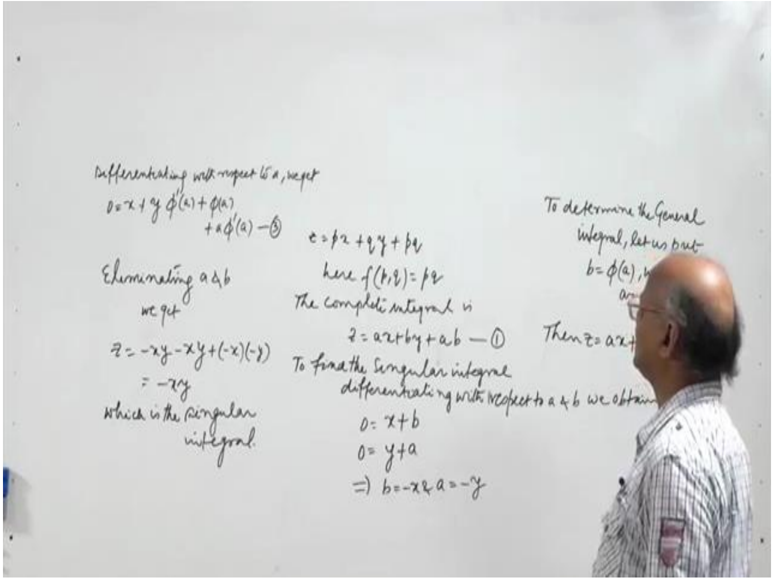

So, let us consider z equal to p x plus q y, plus f p q. And we can see of the clairauts form, z equal to p x plus sorry here it is p q not f p q. So, let us consider z equal to p x plus q y plus p q is here, f p q is equal to p q. So, it is a equation of type 4. Now let us. So, the complete integral is, z equal to a x plus b y plus a b. So, this is the complete integral to find the singular integral let us differentiate 1 with respect to a differentiating with respect to a and b. We obtain 0 equal to x plus b and when we differentiate with respect to v we will get 0 equal to x plus b, and when we differentiate with respect to b, we will get 0 plus y plus a. So, which imply x is equal to minus b, b equal to minus x.

Let us I will write like this, b equal to minus x and a equal to minus y. So, now, eliminating a and b between equation 1 and these 2 equations we have, eliminating a and b we obtain z equal to a is minus y. So, minus x y b is minus x, so minus x y plus minus x into minus y. So, we get minus x y. See this is the, which is the singular integral in this case. To determine the general integral, let us put b equal to phi a where phi is an arbitrary function. So, then z will be equal to, a x plus y into 5 a plus a into phi a. Now we differentiate with respect to a.

So, differentiating, we get 0 equal to x plus y times phi dash a, plus phi a, into a phi dash a. Let me call it as equation number 2 and this as equation number 3. So, then we eliminate a between equations 2 and 3 to determine the general integral.

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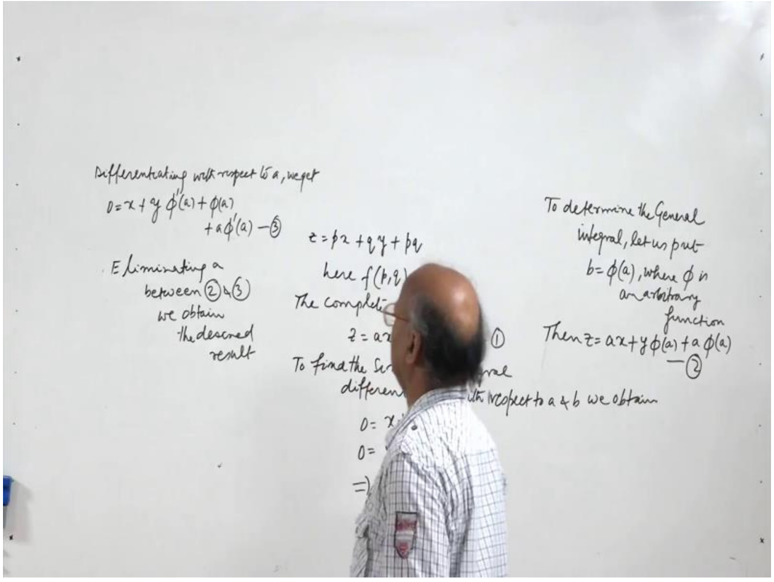

So, eliminating a between 2 and 3, we obtain the desired result. So, we will get the general integral. So, that is how we solve the problem given in example 1. Now let us take up the problem number 2. So, we have z equal to p x plus q y, plus under root 1 plus p square plus q square.

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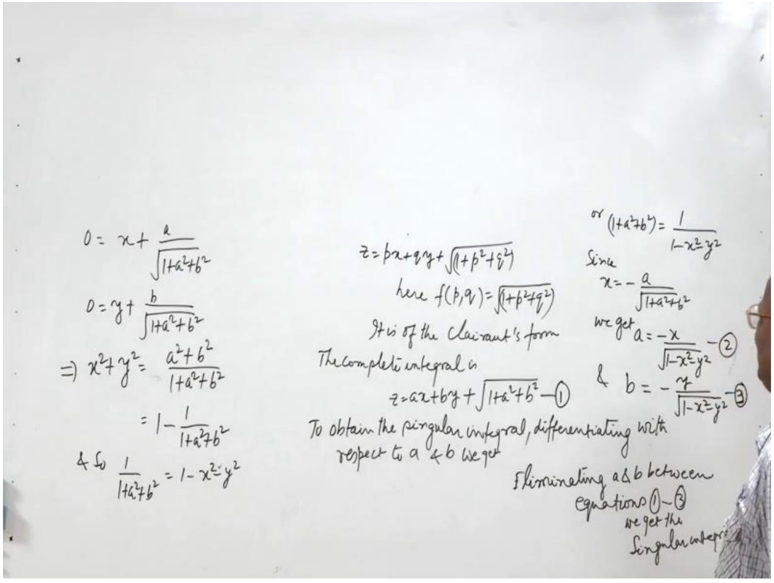

Now, we can see that here 1 plus f p q is equal to under root 1 plus p square plus q square. So, it is of the clairauts form. So, this, the complete integral is therefore, a z equal to a x plus b y plus under root 1 plus a square plus b square. To determine the complete integral to determine the; this is the complete detail to obtain the singular integral, we differentiate with respect to a and b. Differentiating we get 0 equal to x plus a upon under root 1 plus a square plus b square. And when we differentiate with respect to b we get 0 equal to y plus b upon under root 1 plus a square plus b square.

Now, from these 2 equations we get, x equal minus a upon under root 1 plus a square plus b square. And y equal to minus b upon under root a square plus 1 plus a square plus b square. So, x square plus y square is equal to a square plus b square divided by 1 plus a square plus b square, which can be written as 1 minus 1, upon 1, plus a square plus b square. And so 1 upon, 1 plus a square plus b square is equal to 1 minus x square minus y square, or we can say 1 plus a square plus b square is equal to 1 over 1 minus x square minus y square.

Now, x is equal to minus a upon under root 1 plus a square plus b square. So, x is equal to minus a upon under root 1 plus a square plus b square. We will get a as a equal to minus x into under root 1 plus a square plus b square. So, minus x times, So, a is equal to minus x into under root 1 plus a square, which is this equal to this and similarly b is equal to b is equal to minus y minus y upon under root 1 minus x square minus b square yeah. So, now, eliminating a and b between equation 1. And so eliminating a and b between equations 1 2 3, we get the singular integral. We get the singular integral and for general integral as we have discussed earlier.

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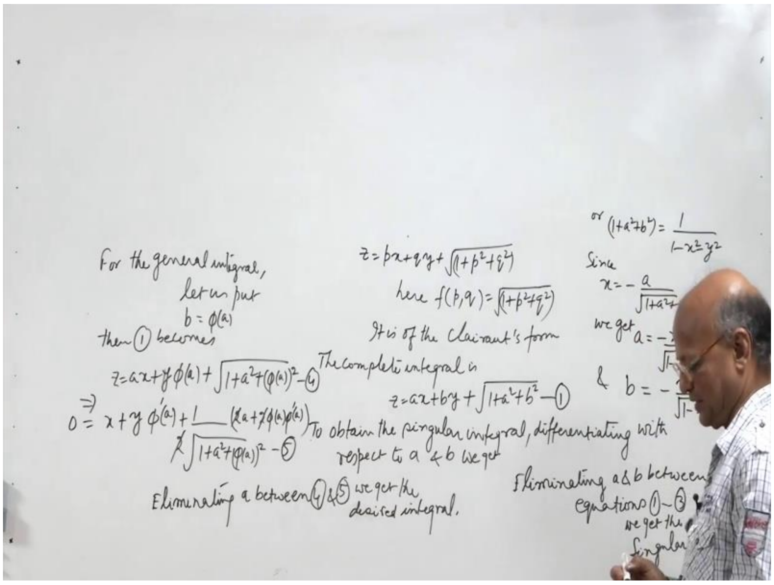

So, for the general integral, let us put b equal to phi a, then 1 becomes z equal to a x plus b phi plus by phi a, plus under root 1 plus a square plus phi a whole square. We differentiate this equation with respect to a, and get 0 equal to this implies 0 equal to x plus y phi dash a, plus 1 over 2 times under root 1 plus a square plus phi a whole square. And then we have 2 a plus 2 times phi a into phi dash a. This 2 can be cancelled. So, we have this equation. So, eliminating a, between this equation and this equation this. We call it as this as 4 and this as 5. So, eliminating a between 4 and 5 we get the desired integral.

This is how we solve these 2 equations. Now we go to the general method of solving first order non-linear equations which we call as the Charpit's method.

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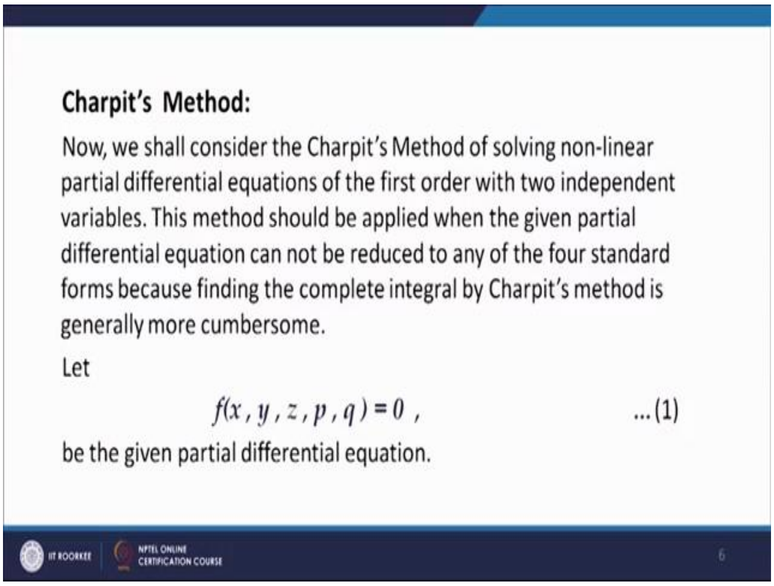

So, here in this method, we consider non-linear partial differential equation of first order which cannot be reduced to the 4 standard forms, which we have already discussed. We will apply this method only in the cases where the first order non-linear partial differential equations cannot be reduced to the 4 standard form because to find the complete integral by applying the Charpit's method, it is rather more cumbersome.

So, we prefer to solve the partial differential equations which are of first order and non-linear by trying to reduce them to the 4 standard forms. If it is not possible to do that then we apply the Charpit's method. So, here what we do is let us consider first order non-linear partial differential equation, as a given by f x y z p q equal to 0. We know that since there are 2 independent variables x and y and z is dependent on x and y. So, we can write the d z equal to p d x plus q d y.

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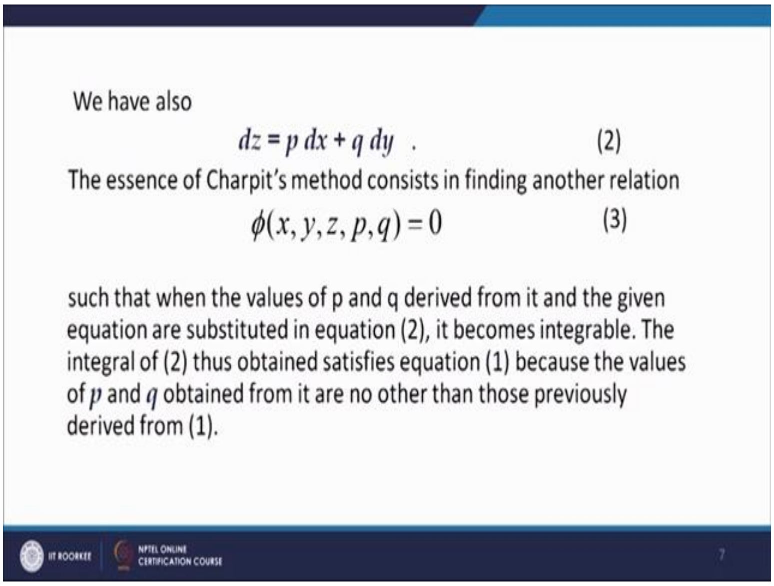

Now, in the Charpit's method we try to find relation between x y z and p q. Say we can when from which and the given partial differential equation which when we when we obtain the values of p and q, and put them in the equation d z equal to p d x plus q d y, if it is integrable, we get the complete integral of the given partial differential equation. So, in this method we will try to find a relation involving x y z p q equal to 0. And then saw which this equation 3 and the equation 1. The given partial differential equation be then solving to find the values of p and q, which when we put in d z equal to p d x plus q d y, if it is integrable we get the complete integral of the given partial differential equation.

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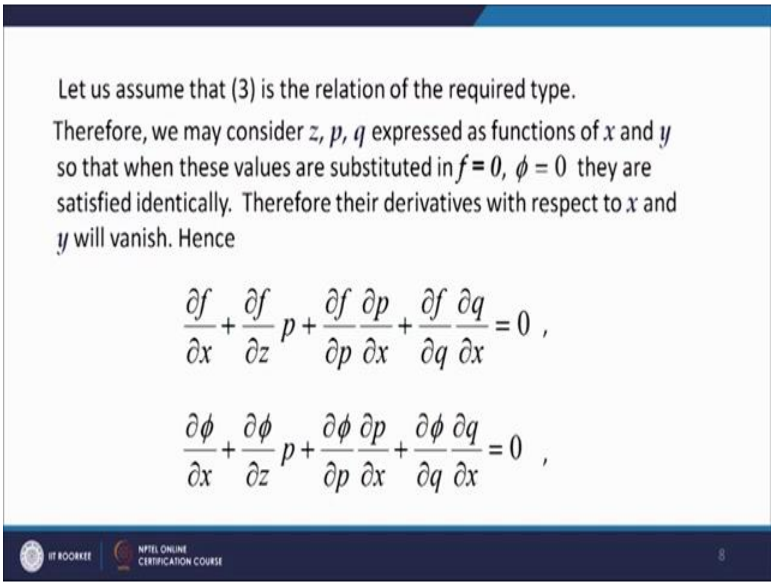

So, let us assume that the 3 this equation, z phi z phi x y z p q equal to 0 is the required relation, then z p and q. So, p is delta z by delta x q is delta z by delta y, they are expressive they are functions of x and y. So, that when they are substituted in the equations f equal to 0, this equation f equal to 0, and phi equal to 0 they are satisfied identically.

So, now, when we differentiate f equal to 0 with respect to x, what we get is delta f by delta x plus delta f by delta z into delta z, by delta x which is equal to p, plus delta f by delta p, delta p by delta x, delta f by delta q, delta q by delta x equal to 0. This is obtained by solving f equal to 0 with respect to x. Similarly, when we solve differentiate phi with respect to x, we will get delta phi by delta x plus delta phi by delta z into p, delta phi by delta p, delta p by delta x plus delta phi by delta q delta q by delta x is equal to 0.

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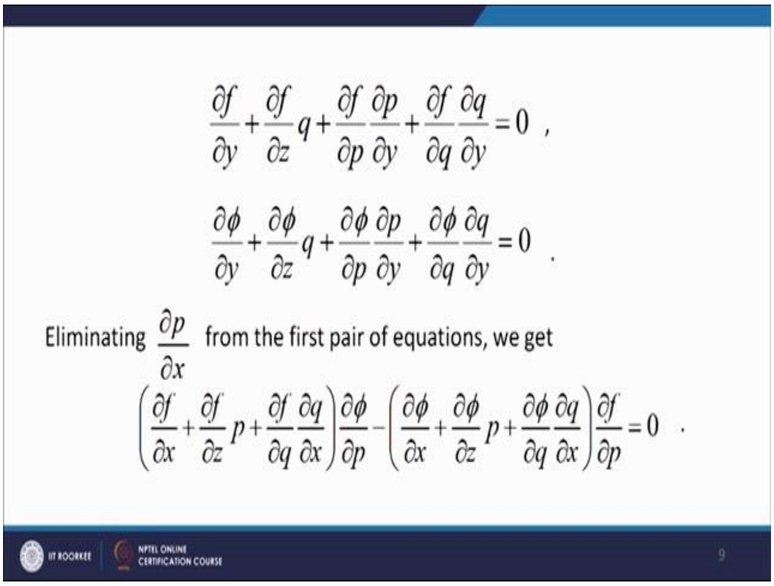

And 2 similar equations are obtained from f equal to 0 phi equal to 0 by differentiating them with respect to y. And they are delta f over delta y, plus delta f over delta z, delta z by delta y which is q delta f over delta p delta p by delta y, delta f over delta q, delta q by delta y equal to 0. And delta phi over delta y plus delta phi over delta z, delta z by delta y, is q delta phi over delta phi, delta phi over delta p, delta p over delta y, delta phi over delta q and delta q over delta y equal to 0.

Now, what we do is, let us eliminate delta p over delta x, from the first pair there are 4 equations, so from the first 2 equations first pair of equations. Let us eliminate this derivative. Derivate of p with respect to x, so delta p over delta x let us eliminate from these 2 equations, and which is not difficult because it is a linear equation. So, this will give you delta f over delta x, delta f over delta z into p, delta f over delta q delta. So, this we can obtain this equation is very simple. And then what we do is we can write it as we can write it in the form.

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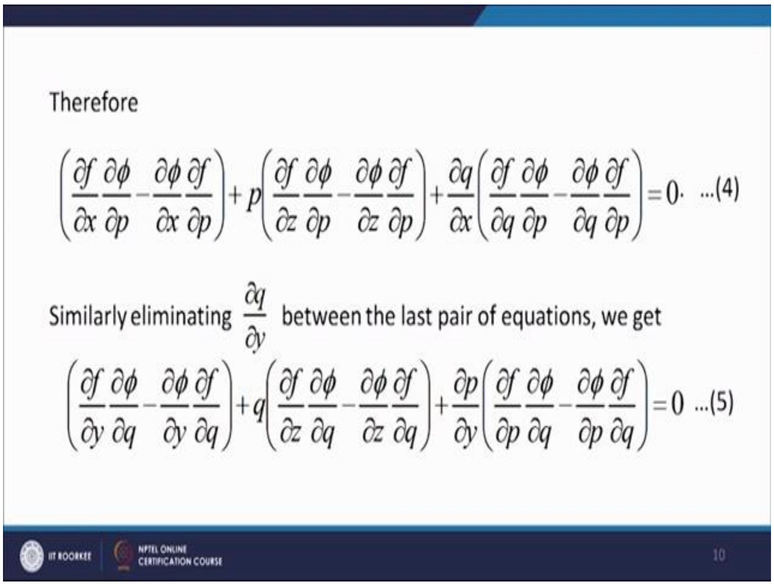

The we collect the terms which contain p and the terms, which contain delta q by delta x, and this constant this term, which is free from p and delta q by delta x.

So, we can write those that this equation. We can write this equation in the form this one. So, we similarly let us eliminate delta q over delta y, to this delta q over delta y in let us eliminate from this pair of equations. So, for that you need to multiply this equation by delta phi over delta q, and this 4th equation by delta f over delta q and then subtract. So, delta q over delta y term will vanish, so eliminating delta q over delta y between the last pair of equations. We shall have this one, delta f over delta y delta phi by delta q minus delta phi by delta y, delta f by delta q plus q, times this, plus delta v over delta y times this equal to 0.

Now, let us notice one thing, delta q is delta z by delta y. So, delta q by delta x is delta square z by delta x delta y, and p is delta z by delta x. So, delta p by delta y is delta square z, by delta x delta y. And assuming that the second order partial derivatives are continuous, delta square z by delta x delta y is same as delta square z by delta y delta x. We know is we know this result. So, delta q over delta x and delta p over delta y are same. And this they are coefficient you can see is also same, except for the negative sign. This is negative of this. So, when we add these 2 equations, the last term this term here and this term here they will cancel out.

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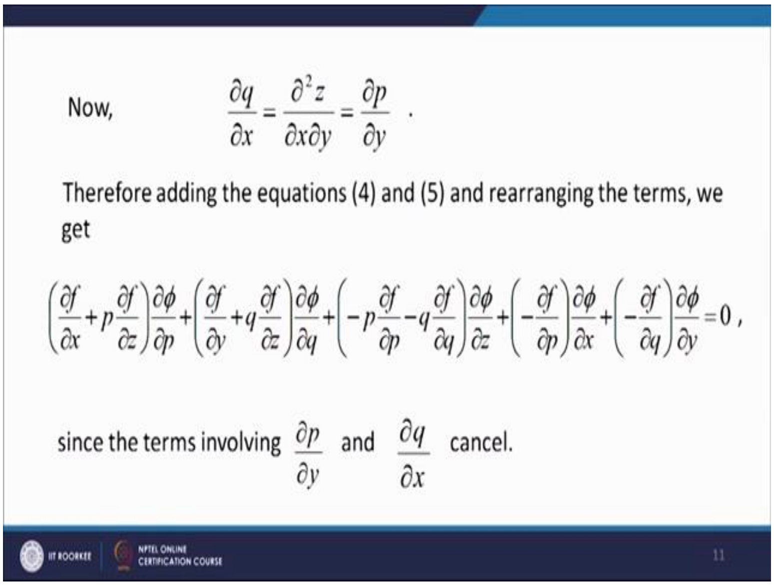

So, now let us notice that delta q over delta x is equal to delta p over delta y. So, we have this one, delta q over delta x equal to delta square z by delta x, delta y and which is equal to delta p over delta y, because we are assuming that second order partial derivatives are continuous. So, when we add the equations 4 and 5, as we have discussed just now. This is negative of this quantity. And this delta q over delta x and delta p over delta y are same.

So, when we add the equations 4 and 5, and rearrange the terms, what happens is this term and this term they cancel out. And what we get is this. So, delta f over delta x plus p times delta f over delta z into delta phi over p, plus delta f over delta y plus q times delta f over delta z, delta phi by delta q plus, minus p delta f f by delta p, minus q delta f by delta q, delta phi by delta z plus minus delta f by delta p, into delta phi by delta x minus delta f, by delta q into delta phi over delta y equal to 0, as since the terms involving are the partial derivative of p with respect to y, and partial derivative of q with respect to x cancel.

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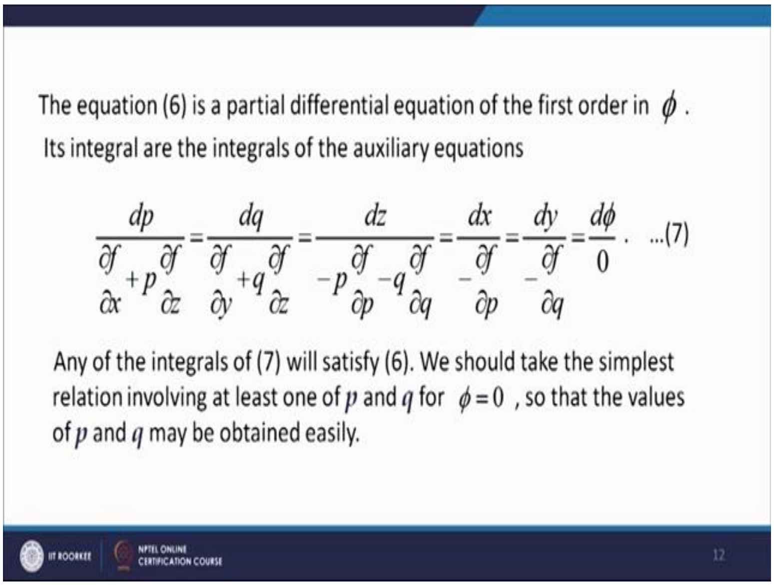

Now, the equation 6 is a partial differential equation of the first order in q, no phi let us see. This is a partial differential equation of the first order in phi. Because phi is a function of x y z p and q, and you can see here we have in the first term partial derivative of phi with respect to p, partial derivative of phi with respect to q, partial derivative of phi with respect to z, partial derivative of phi with respect to x, partial derivative of phi with respect to y which are of first order. So, it is a first order linear differential equation in phi. And therefore, it is integrals are the integrals of the auxiliary equations, as we have discussed in the case of Lagrange's linear equation in p and q it is same it is of the same.

So, we have d p over this quantity and d q over this d z over this quantity, and d x over minus delta f by delta p d y over minus delta f by delta q, equal to d phi over 0. We are actually solving this equation like we have solved the Lagrange's linear equation in p and q. So, it is a linear equation in the derivatives of phi with respect to p q z x and y. So, we solve this auxiliary system of equations. Now any of the integrals of this equation 7 will satisfy 6, and therefore, we take the simplest relation involving at least one of p and q, for phi equal to 0, so that the values of p and q may be obtained easily. Once the values of p and q are obtained easily, we can take the help of the given equation f x y z p q equal to 0, and determine the value of the other q r p. And once p and q are known, we can use d z is equal to p d x plus q d y and determine the complete integral.

So, we shall be able to determine the complete integral by writing the auxiliary system of equations. And this auxiliary system of equations will then, we will find the shortest simplest relation involving at least one of p and q for phi equal to 0. And then we may use these value, this relation for p with in p and q and determine the values of p and q. So, when we put them in the equation d z equal to p d x plus q d y, and integrate we shall obtain the complete integral which will be the complete integral of the given partial differential equation. And the similar and a general integral are then obtained in the usual manner. So, with that I would like to conclude my lecture.

Thank you for your attention.