Mathematical methods and its applications
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Lecture – 20
Solution of first order non – linear equation – II

Hello friends, welcome to my second lecture on solution of first order non-linear equations. In this lecture we shall discuss equations of the form z p q equal to 0 which come under the category type 2.

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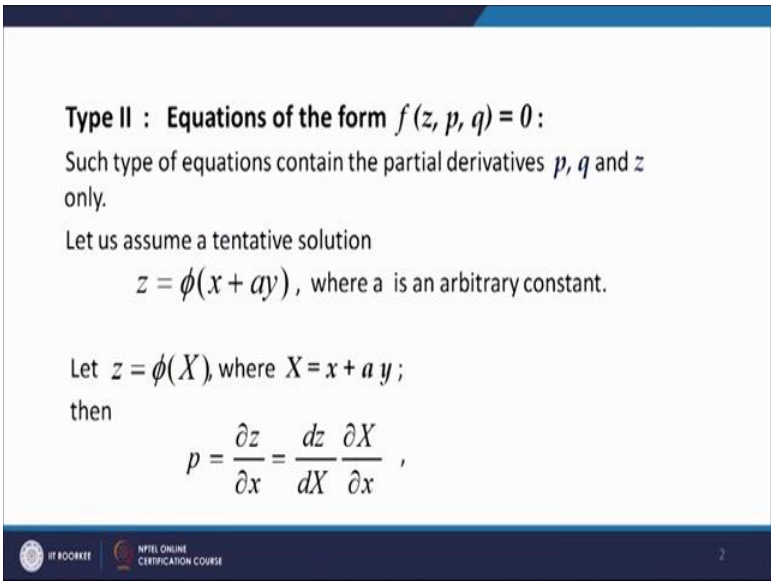

And then we will also discuss equations of the form f x, f 1 x p equal to f 2 y q, which where the variables x and p and the variables y and q can be separated. So, that those equations come under the category of type 3. So, we begin with the equations of the form f z p q equal to 0, which come under the category type 2. Such type of equations contains only the partial derivatives p and q, and the dependent variables z. What we do is let us assume a tentative solution of this partial differential equation, f z equal to phi x plus a y where a is an arbitrary constant.

What we do further is that let us assume x plus a y to be equal to capital X. So, that z becomes a function of x only. Now we know that p is the partial derivative of z with respect to x. So, this partial derivative of z with respect to x can be expressed as, d z by d X, into delta X by delta x by which enroll. And we see that delta x by if you differentiate capital X with respect to small x what you get is 1. So, delta x by delta x is equal to 1 and therefore, p is equal to d z by d X. Here q is q similarly q is delta z over delta y. So, delta z over delta y can be written as d z over d x into delta x over delta y. And this is equal to a d z over d X. Because delta x over delta y if you find then delta x over delta y is equal to a.

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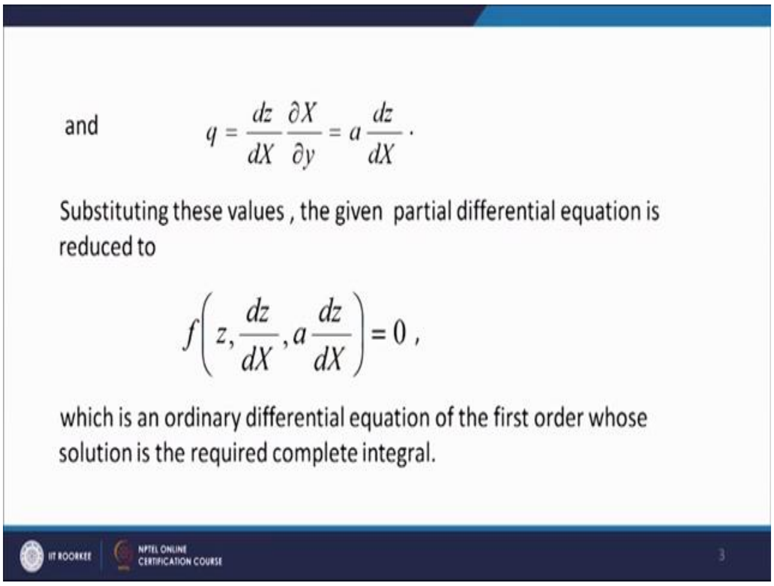

So, this will be giving you a d z by d x. Now substituting these values, the given partial differential equation, is then reduced to f z p is d z by d x. As we have seen p is d z d x and q is a d z by d x. So, f z d z by d x a d z by d x equal to 0 which is an ordinary differential equation of the first order, whose solution will give us the required complete integral. And once we have the complete integral as we have seen earlier, the general and singular integrals are then found in the usual manner.

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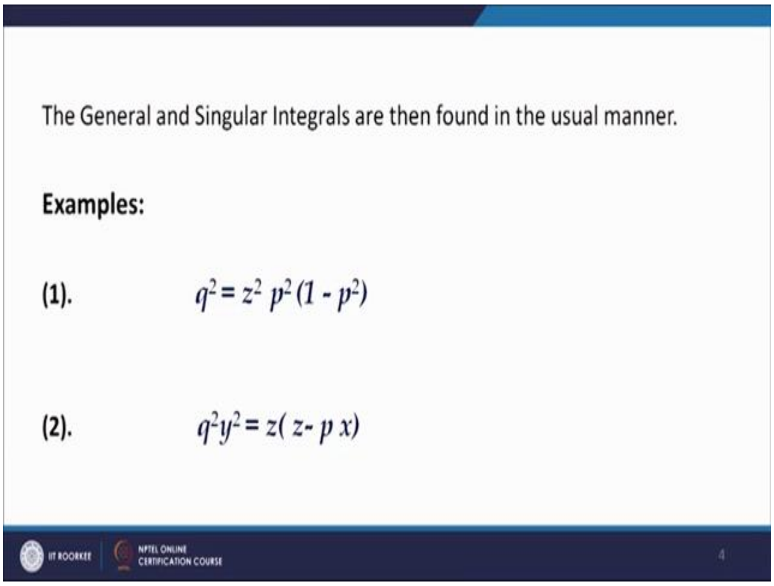

Let us take the non-linear equation q square equal to z square p square into 1 minus p square. So, this, the differential equation of type 2 because here you can see we have only the partial derivatives p and q and the dependent variable z.

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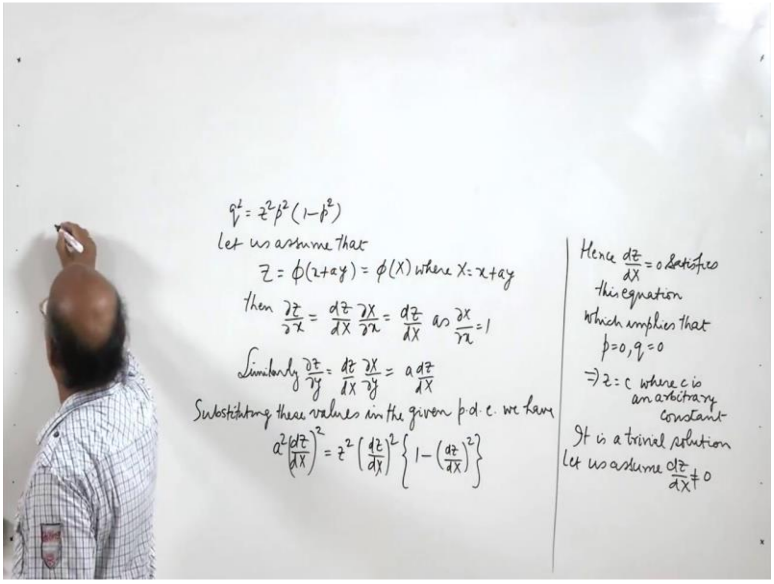

So, let us see how we solve this equation. So, q square is equal to z square into p square 1 minus p square. Let us assume that z equal to phi of x plus a y, which is equal to phi of x, where x is equal to x plus a y. Then as we have seen in the article delta z by delta x will be equal to d z by d x, into d x by delta x by delta x and delta x. When we differentiate x with respect to x, delta x by delta x is equal to 1. So, we will get d z by d x. Similarly, delta z by delta y is equal to d z by d x, into delta x by delta y. And when we differentiate x with respect to y what we get is a, so a d z by d x.

So, let us put these values in the given differential equation. So, substituting in the given partial differential equation, we shall we have q square this is q, q square means a square d z by d x whole square equal to z square, z is z is z. So, z square and p square p is d z by d x whole square and then 1 minus d z by d x whole square. So, we can see that, hence d z by d x equal to 0, d z by d x equal to 0 satisfies this equation. Now d z by d x equal to 0 means p equal to 0, and also q equal to 0 which implies that p is equal to 0, q equal to 0 which means that the partial derivatives of z with respect to x and y both are 0. So, it will mean that z is an arbitrary constant, some constant c, where c is an arbitrary constant.

Now, it since it is a trivial. So, this is the trivial solution. Let us assume d z by d x to be not equal to 0. So, let us assume d z by d x not equal to 0.

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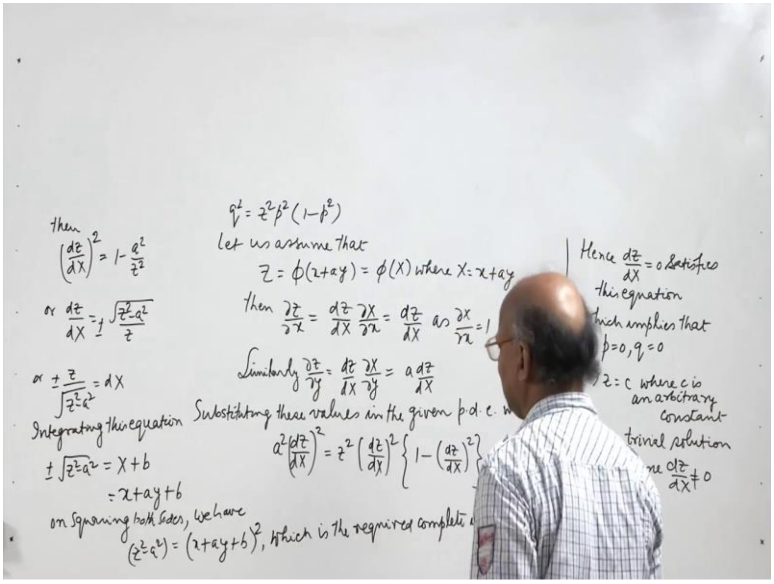

Then we shall have a square by z square equal to 1 minus d z by d x whole square. So, then d z by d x whole square, will be equal to 1 minus a square y z square. Or we can say d z by d x will be equal to z square minus a square divided by z plus minus, d z by d x is equal to plus minus under root z square minus a square divided by z. So, or we can say plus minus z over under root z square minus a square is equal to d x.

Now, this when we integrating this equation, what we get is under root z square minus a square, equal to x plus some constant b and this gives you then a x, sorry x plus a y plus b. Now squaring both sides on squaring, we have z, z square minus a square equal to x plus a y plus b whole square which is the required complete integral, because it contains 2 arbitrary constants. Now as we know to find the general integral we put y b equal to some arbitrary functions says psi of a and then differentiate this equation with respect to a, a, then we eliminate a from this equation. And the equation that we get by differentiating this equation with respect to a after putting b equal to phi a.

And in order to find the complete integral, we differentiate this equation with respect to a and a put it equal to 0 and then differentiate this equation, and then differentiate this equation with respect to b, and then solve the 3 equations. This equation and the 2 equations that we obtained by differentiating with respect to a and b and eliminate the arbitrary constants a and b to get the similar integral. So, these general and singular integrals are obtained in the usual method manner.

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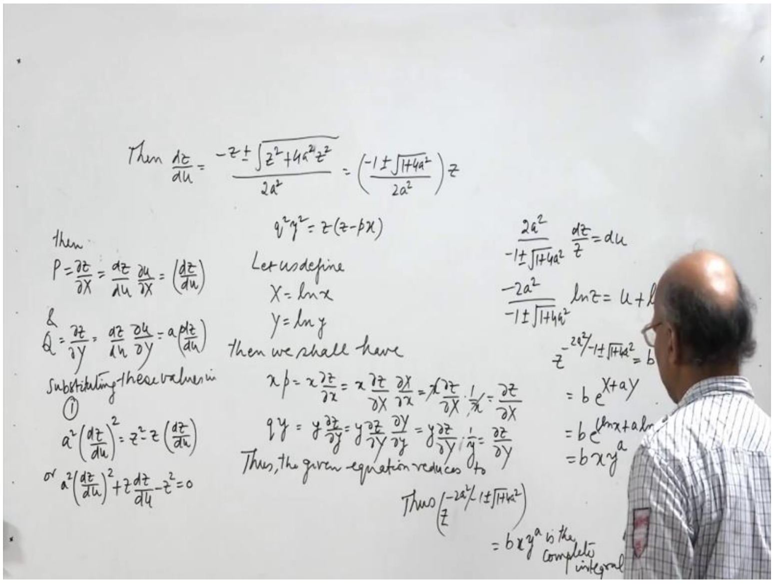

Now, let us go to next problem. So, the next problem is q square y square equal to, q square y square equal to z into z minus p x. Now this equation is not in the standard form. That is type 2 because here a independent variables x and y are also occurring. So, we will have to make a substitution in order to reduce this equation to the standard form. Suppose I define z equal to capital Z equal to ln z. Suppose we define let us define capital X equal to small ln x. Capital Y equal to ln y, then we shall have x into p, x into p equal to x delta z by delta x.

And delta z by delta x can then be written as x times delta z by delta capital X into delta x by delta x. Now this from here x equal to ln x, we have delta x by delta x says 1 by x. So, we have x this x, this x cancel. And we get it delta z by here I have capital X. This and similarly q into y, is equal to q into delta z over delta y, which can be written as q times sorry y times delta z over delta y because q is delta z over delta y. So, y times delta z over delta y which you can be written as y times delta z by delta x and then delta x by delta y. Now when we differentiate sorry this can be written as delta z by delta y. And this is delta z y over delta y let us write like this. So, y times delta z over delta y, into delta y over delta y is 1 over y, so this delta z over delta y.

So, q y is equal to delta z with thus the given equation, reduces to delta z over delta y whole square, equal to z into z minus x into p x into p is delta z by delta x. So, if we designate or if we denote delta z by delta x, equal to capital P and delta z over delta y, equal to capital Q then we have q square plus equal to z into z minus capital P. So, this equation contains only the partial derivatives p and q and the dependent variable results. So, it is of therefore, this equation is of type 2.

Now, let us try to find the complete integral of this. So, let us define z equal to some function of x plus a y because here the independent variables now are capital X and capital Y. So, let us call it as phi of u, where u is equal to capital X plus a y. And a is an arbitrary constant. So, then delta z over delta x will be equal to, then p equal to delta z by delta x, will be equal to d z by d u, into delta u by delta x. And delta u by delta x is equal to 1. So, we have d z by d u and capital Q will be equal to delta z by delta y, which will be equal to d z by d u into d u y, delta u by delta y, delta u over delta y is equal to a. So, we have a d z by d u.

Let us replace these values of p and q in the equation, let us call it as 1. So, substituting these values in equation 1 we will get a square d z by d u, whole square, equal to z square minus z into p. So, z into p is d z by d u. So, a square d z by d u whole square equal to z square minus z into p we have this. Now or a square d z by d u whole square plus z, d z by d u minus z square equal to 0, now it is a quadratic equation in d z by d u.

So, we can solve it. So, then d z by d u can be obtained from here, which will be minus z plus minus under root z square plus 4 a square z square divided by 2 a square.

So, this is equal to, minus 1 plus minus under root 1 plus 4 a square, by 2 a square, into z. Now we can differentiate easily we can integrate it easily. So, integrating, we can write it as 2 a square upon minus 1 plus minus under root 1 plus 4 a square into d z by z, equal to d u. Now integrating this we will have minus 2 a square upon minus 1 plus minus under root 1 plus 4 a square ln z equal to say u plus let us write ln b. And this will give you then, z to the power minus 2 a square divided by minus 1 plus minus under root 1 plus 4 a square equal to b times e to the power u which is equal to b times e to the power u is equal to x plus a y.

So, which is equal to e to the power now x is ln x and y is ln y. So, this is ln x plus ln y to the power a, and which will give us ln x y to the power a. So, we have b times x y to the power a. And thus z to the power minus 2 a square divided by minus 1 plus minus under root 1 plus 4 a square, equal to b times x into y to the power a is the complete integral.

The general integral and the singular integrals are then found in the usual manner as; we have discussed earlier. So, this, the solution for the second example.

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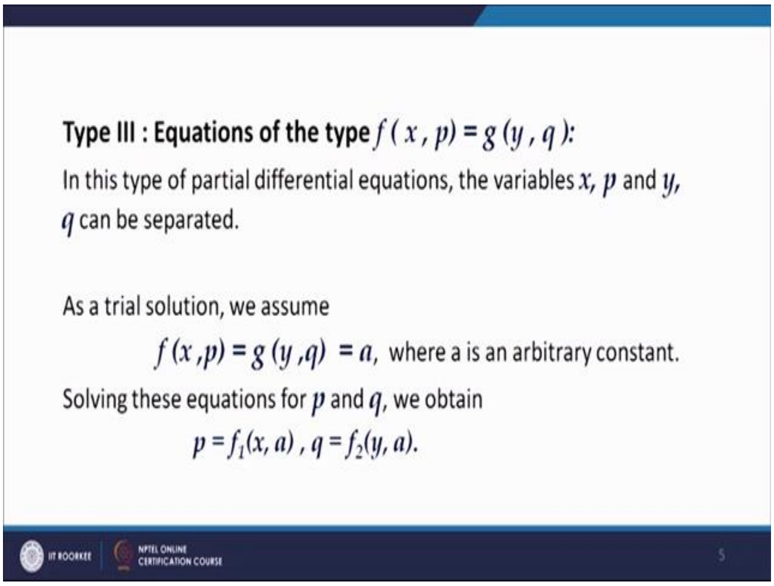

Now, let us go to equations of the type 3. So, here we consider those types of equations partial differential equations, such that the variables x p and y q are separable they can be separated. So, in this type of partial differentiation equations the variables x p y and q can be separated. Let us; that means, on one side we will have the expression involving x and p on the right side, we will have expression involving y and q, which we can write as f function of x in p and equal to function some function of y and q.

Now, as trail solution we will assume that f x p is equal to g y q is equal to some arbitrary constant let us say a. And then we will have f x p equal to a, and g y q equal to a. So, let us assume that these equations can be solved for the values of p and q. And when we solve of x p equal to a we get say p equal to some function f f 1 of x and a and q as some function f 2 of y and a, then recalling that we have d z equal to p d x plus q d y because d z is the total differential.

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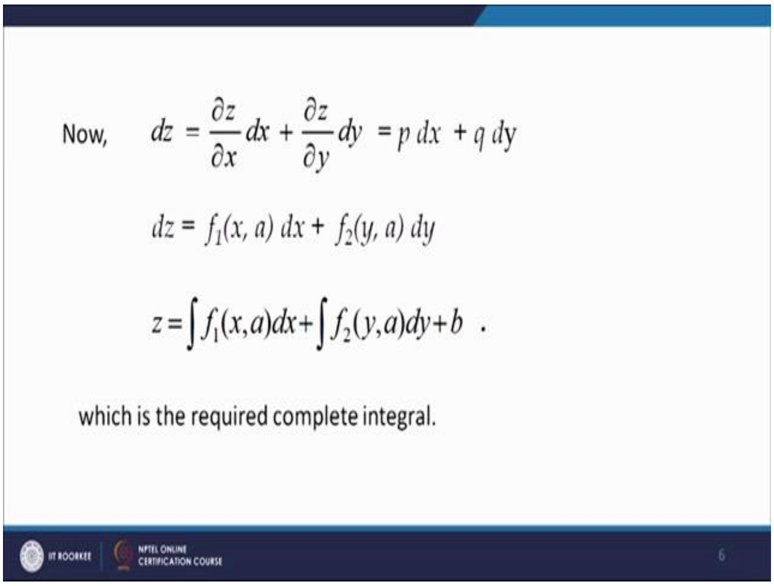

So, p z p d x plus q d y putting the values of p and q, so obtained we have d z equal to f 1 x a d x plus f 2 y a d y. And then we integrate on both sides to get z equal to integral of f 1, x a d x plus integral of f 2 y a d y plus some arbitrary constant b.

And since this solution involves 2 arbitrary constants a and b, corresponding to the 2 independent variables x and y. So, it is the required complete integral. And once we have the complete integral as we know the general integral and the singular integrals can be found in the usual way.

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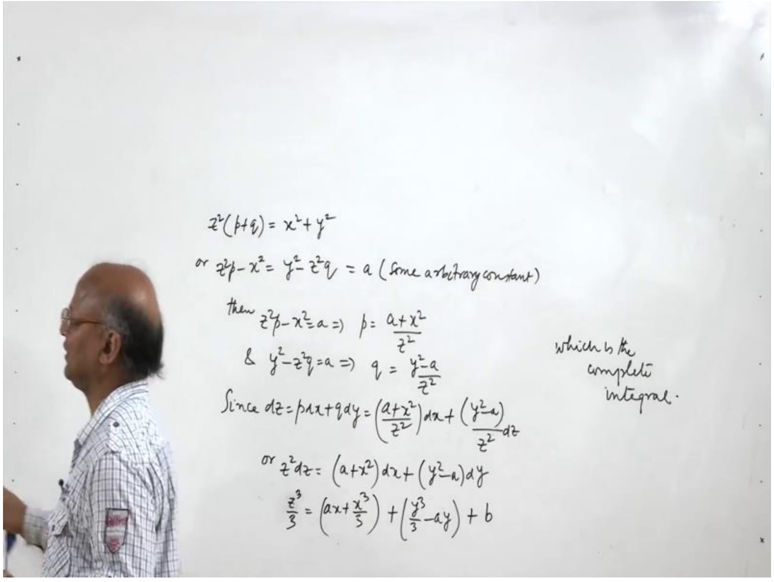

So, let us study some examples on partial differential equation of type 3. So, z square into p plus q, equal to x square plus y square. Let us write it as r z square p minus x square equal to y square minus z square into q.

Let us put it equal to some constant a, some arbitrary constant. Then z square p minus x square equal to a gives you p equal to a plus x square divided by z square. And similarly y square minus z square q equal to a gives you q equal to y square minus a, divided by z square. Now let us recall that since d z equal to p d x plus q d y. We have a plus x square upon z square d z d x, plus y square minus a divided by z square d z. Now let us multiply by z square. So, or z square d z equal to a plus x square into d x, plus y square minus a into d y. And it can be easily integrated. So, z cube by 3 equal to a x plus x cube by 3, y cube by 3 minus a y plus some constant let us say b. So, this is the complete integral. This is the solution of the given partial differential equation which involves 2 arbitrary constant a and b. So, it is the completely integral which is the completely integral.

So, we can notice here one thing that, as such the given partial differential equation is such that the variables x p and y q are separable. So, it is of the type 3. And here we noted that it can be then expressed as the values of p and q can be found as a plus x square by z square and q equal to y square minus a over z square. So, then put them in the expression d z equal to p d x plus q d y. And integrate in the usual way, and obtain the complete integral. General and singular integrals can then be found as we have discussed already.

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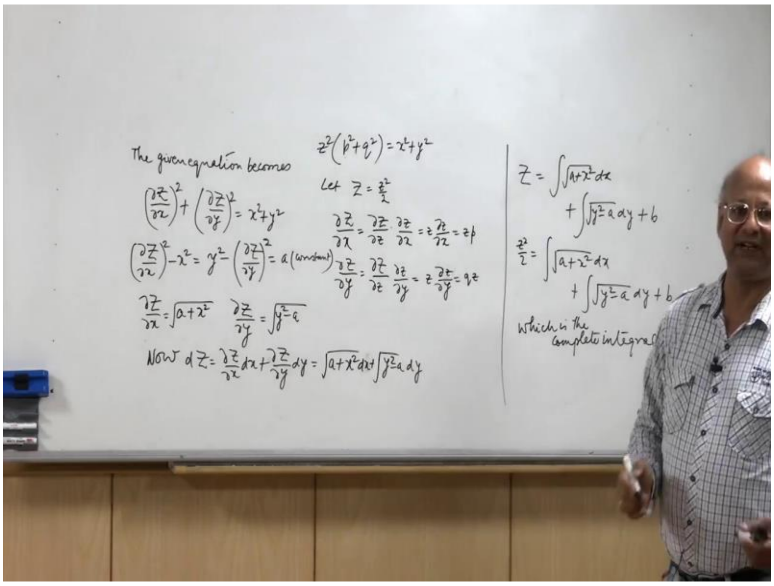

Let us now take up another one partial differential equation to make it more clear. So, let us consider z square into p square plus q square, and then this is equal to x square plus y square. So, here I think we have to define let capital Z be equal to because this can be this is z p whole square. And this is z q whole square. So, we have to define it as z square by 2. Then what will happen delta z by delta x will be equal to delta z by delta z small z, into delta z by delta x and delta z by delta z, will be from here. This will be equal to z. So, z into delta z by delta x which is z into p and delta z over delta y, will be equal to delta z over delta z small z, into delta small z by delta y delta z over delta z is z here.

So, z into delta z over delta small z, so delta small z over delta y, which is equal to q z, so z p is equal to delta z over delta x, and q z is delta z over delta y. So, we have the given equation becomes delta z, this is capital Z, by delta small x whole square plus delta z, by delta y whole square equal to x square plus y square. Now we can separate the variables delta z by delta x whole square minus x square equal to y square minus delta z over delta y whole square. We can put it as some constant a and then solve for the values of delta z over delta z. So, delta z over delta x will be equal to a plus x square under root. And similarly delta z over delta y will be equal to y square minus a under root. Now d capital Z equal to delta capital Z over delta x d x plus delta capital Z over delta y d y and which is equal to under root a plus x square plus under root into d x plus under root y square minus a d y.

Now, we can integrate this. So, integrating we get capital Z equal to integral under root a plus x square d x, plus integral y square minus a d y, plus some constant b. And then we put the capital value of the capital Z which is z square by 2. So, z square by 2 equal to integral under root a plus x square d x, plus integral under root y square minus a d y plus b. Now you know the formula for the integral of under root a plus x square, and the integral of under root y square minus a. So, apply those formulas to value add these integrals and then we will have the complete integrals, which involves 2 arbitrary constants. And the general and singular integrals are then found in the usual manner. So, this is the complete integral with that I would like to conclude my lecture.

Thank you very much for your attention.